Problem **1**:

First of all, note that in QED the counterterm vertex is $ie\delta_1\gamma^\mu$, so it's $e(E)\times\delta_1(E)$ which acts as the counterterm coupling $\delta^g(E)$, that's why in eq. (2) the derivative WRT $\log E$ acts on the product $e(E)\times\delta_1(E)$ rather than on just the $\delta_1(E)$. Consequently, that derivative amounts to

$$\frac{d(e\times\delta_1)}{d\log E} = e\times\frac{d\delta_1}{d\log e} + \frac{de}{d\log E}\times\delta_1 = e\times\frac{d\delta_1}{d\log e} + \beta_e\times\delta_1\,. \tag{S.1}$$

Plugging this formula into eq. (2) gives us (after a bit of algebra)

$$\beta_e = (2\gamma_e+\gamma_\gamma)\times e(1+\delta_1) - e\times\frac{d\delta_1}{d\log e} - \beta_1\times\delta_1 \tag{S.2}$$

$$\Downarrow$$

$$\beta_e\times(1+\delta_1) = (2\gamma_e+\gamma_\gamma)\times e(1+\delta_1) - e\times\frac{d\delta_1}{d\log e} \tag{S.3}$$

$$\Downarrow$$

$$\beta_e = e\times(2\gamma_e+\gamma_\gamma) - \frac{e}{1+\delta_1}\times\frac{d\delta_1}{d\log E}\,. \tag{S.4}$$

Now, in the last term here

$$\frac{1}{1+\delta_1}\times\frac{d\delta_1}{d\log E} = \frac{d\log(1+\delta_1)}{d\log E} = \frac{d\log Z_1}{d\log E}\,. \tag{S.5}$$

But thanks to the Ward identity $\delta_1(E)=\delta_2(E)$ at all energies, hence $\log Z_1(E)=\log Z_2(E)$ at all energies and therefore

$$\frac{d\log Z_1}{d\log E} = \frac{d\log Z_2}{d\log E} = 2\gamma_e\,. \tag{S.6}$$

Plugging the last two formulae into eq. (S.4), we end up with

$$\beta_e = e\times(\cancel{2\gamma_e}+\gamma_\gamma) - e\times\cancel{2\gamma_e} = e\times\gamma_\gamma\,. \tag{3}$$

Quod erat demonstrandum.

Problem **2**(a):

By the power-counting analysis, the electron's two-point function $\Sigma_e(\not{p})$ has superficial degree of divergence $\mathcal{D} = +1$, so naively one expects δ_m to be linearly divergent while δ_2 is logarithmically divergent. However, in the $m_e \to 0$ limit, QED gets an additional axial symmetry $\Psi(x) \to \exp(i\theta\gamma^5)\Psi(x)$, and this symmetry forbids the δ_m counterterm altogether. Consequently, for a small but non-zero electron mass, we end up with

$$\delta_m \;\sim\; m_e \times \log\Lambda \quad \text{rather than} \quad \delta_m \;\sim\; \Lambda. \tag{S.7}$$

To see how this works, consider the chiral structure of the electron's two-point function $\Sigma_e^{\rm loops}(\not{p})$ for a massless electron. The massless electron's propagator $i\,\not{q}/(q^2+i\epsilon)$ anticommutes with the γ^5 matrix, and so does the QED vertex $ie\gamma^\mu$. In a general 1PI diagram for the $\Sigma_e^{\rm loops}(\not{p})$, the electron line connecting the incoming and the outgoing lines has n propagators and $n+1$ vertices, which altogether makes for an odd number $2n+1$ of matrices anticommuting with the γ^5. Consequently, as a Dirac matrix, the $\Sigma_e^{\text{this diagram}}(\not{p})$ anticommutes with the γ^5, and this is true for every diagram contributing to the $\Sigma_e^{\rm loops}(\not{p})$, so altogether

$$\gamma^5\Sigma_e^{\rm loops}(\not{p}) \;=\; -\Sigma_e^{\rm loops}(\not{p})\gamma^5. \tag{S.8}$$

At the same time, by the Lorentz invariance we have

$$\Sigma_e^{\rm loops}(\not{p}) \;=\; A^{\rm loops}(p^2) \;+\; B^{\rm loops}(p^2)\times\not{p}. \tag{S.9}$$

In terms of $A(p^2)$ and $B(p^2)$ eq. (S.8) means $A^{\rm loops} = 0$, and hence — for the off-shell conditions (6) — $\delta_m = 0$ to all loop orders.

For a small but non-zero electron mass, we no longer have axial symmetry and hence do not expect $A = 0$. But for $m_e \ll E$, $A^{\rm loops}$ should be an analytic function of m_e, hence

$$A^{\rm loops}(p^2, m_e) \;=\; A^{\rm loops}(p^2; m_e = 0) \;+\; m_e \times \frac{\partial A^{\rm loops}}{\partial m_e} \;+\; O(m_e^2), \tag{S.10}$$

where the first term on the RHS vanishes because $A(m_e = 0) = 0$. Also, the superficial degree of divergence of the derivative $\partial A/\partial m_e$ is

$$\mathcal{D}\left[\frac{\partial A}{\partial m_e}\right] \;=\; \mathcal{D}[A] \;-\; 1 \;=\; 0, \tag{S.11}$$

so this derivative is logarithmically rather than linearly divergent. Thus altogether

$$A^{\text{loops}} \;=\; m_e \times \Big(O(\log\Lambda)\ \text{constant} \;+\; \text{finite}\Big) \tag{S.12}$$

and therefore

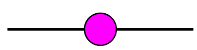
$$\delta_m \;=\; m_e \times \Big(O(\log\Lambda)\ \text{constant} \;+\; \text{finite}\Big). \tag{S.13}$$

Quod erat demonstrandum.

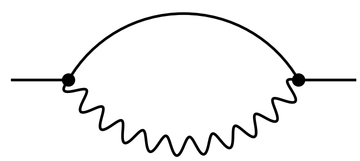
Problem $\mathbf{2}$(b):

At the one-loop level, the electron's self-energy correction is

$$-i\Sigma_e^{\text{order}\,\alpha}(\not p) \;=$$



$$+$$



$$= \; i\delta_2^{\text{order}\,\alpha}\,\not p \;-\; i\delta_m^{\text{order}\,\alpha} \;-\; i\Sigma_e^{1\,\text{loop}}(\not p) \tag{S.14}$$

where the loop diagram evaluates to

$$-i\Sigma_e^{1\,\text{loop}}(\not p) \;=\; \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,(ie\gamma_\mu)\frac{i}{\not p + \not k - m + i0}(ie\gamma_\nu) \times \frac{-i}{k^2+i0}\left[g^{\mu\nu} \;+\; (\xi-1)\frac{k^\nu k^\nu}{k^2+i0}\right]. \tag{S.15}$$

In other words,

$$\Sigma_e^{1\,\text{loop}}(\not p) \;=\; -ie^2 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.16}$$

where

$$\frac{1}{\mathcal{D}} \;=\; \frac{1}{(p+k)^2 - m^2 + i0} \times \frac{1}{(k^2+i0)^2} \tag{S.17}$$

while

$$\mathcal{N} \;=\; k^2 \times \gamma_\mu(\not{p}+\not{k}+m)\gamma^\mu \;+\; (\xi-1)\times \not{k}(\not{p}+\not{k}+m)\,\not{k}. \tag{S.18}$$

As usual, we use the Feynman parameter trick to simplify the denominator

$$\frac{1}{\mathcal{D}} \;=\int\limits_0^1 \frac{2(1-x)\,dx}{[\ell^2-\Delta+i0]^3} \tag{S.19}$$

for

$$\ell^2 \,-\, \Delta \;=\; (1-x)k^2 \;+\; x[(p+k)^2-m^2] \;=\; (k+xp)^2 \;+\; x(1-x)p^2 \;-\; xm^2 \tag{S.20}$$

and hence

$$\ell \;=\; k \;+\; xp, \quad \Delta \;=\; xm^2 \;-\; x(1-x)p^2. \tag{S.21}$$

Consequently, changing the order of $\int dx$ and momentum integration and changing the momentum integration variable from k to ℓ, we get

$$\Sigma_e^{1\,\text{loop}}(\not{p}) \;=\; -2ie^2 \int\limits_0^1 dx(1-x) \int\limits_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}(\ell)}{[\ell^2-\Delta+i0]^3} \tag{S.22}$$

where

$$\mathcal{N}(\ell) \;=\; (\ell-xp)^2\times\gamma_\mu(\not{\ell}+(1-x)\,\not{p}+m)\gamma^\mu \,+\, (\xi-1)(\not{\ell}-x\,\not{p})(\not{\ell}+(1-x)\,\not{p}+m)(\not{\ell}-x\,\not{p}). \tag{S.23}$$

The numerator (S.23) is a cubic polynomial in ℓ^μ, but in the context of the momentum integral (S.22) the odd cubic and linear terms integrate to zero. As to the even quadratic and constant terms, only the quadratic terms contribute to the UV divergence of the integral (S.22) while the constant terms affect only its finite part. For our present purposes, we care only about the UV divergent part of the self-energy (S.22), so we are going to truncate the numerator $\mathcal{N}$ to the terms which are quadratic in ℓ^μ and discard the rest of the terms, thus

$$\begin{aligned} \mathcal{N}_{\rm div} \;&\cong\; \ell^2\times\gamma_\mu((1-x)\,\not{p}+m)\gamma^\mu \;-\; 2x(\ell p)\times\gamma_\mu\,\not{\ell}\gamma^\mu \\ &\quad+\; (\xi-1)\times\Bigl[-2x\ell^2\,\not{p}+\not{\ell}((1-x)\,\not{p}+m)\,\not{\ell}\Bigr] \end{aligned} \tag{S.24}$$

Furthermore, the difference between the numerator algebras in $D=4-2\epsilon$ dimensions v. 4 dimensions is $\Delta\mathcal{N}=O(\epsilon)$, which after multiplying by the $O(1/\epsilon)$ UV divergence becomes finite rather than infinite. Consequently, for the purposes of calculating only the UV-divergent part of the momentum integral, we may use the 4D algebra to simplify the numerator (S.24) in the context of the integral (S.22). Thus:

$$\gamma_\mu\,\not{\ell}\gamma^\mu \;=\; -2\,\not{\ell}, \tag{S.25}$$

$$\gamma_\mu((1-x)\,\not{p}+m)\gamma^\mu \;=\; 4m \;-\; 2(1-x)\,\not{p}, \tag{S.26}$$

$$(\ell p)\times\not{\ell} \;=\; \ell_\mu\ell_\nu\times p^\mu\gamma^\nu \;\cong\; \frac{\ell^4}{4}g_{\mu\nu}\times p^\mu\gamma^\nu \;=\; \frac{\ell^4}{4}\times\not{p}, \tag{S.27}$$

$$\text{and}\quad \not{\ell}\,\not{p}\,\not{\ell} \;=\; \ell_\mu\ell_\nu\times\gamma^\mu\,\not{p}\gamma^\nu \;\cong\; \frac{\ell^2}{4}g_{\mu\nu}\times\gamma^\mu\,\not{p}\gamma^\nu \;=\; \frac{\ell^2}{4}\times(-2\,\not{p}). \tag{S.28}$$

Applying these formulae to eq. (S.24) for the numerator gives us

$$\begin{aligned} \mathcal{N}_{\rm div} \;&\cong\; \ell^2\bigl(4m-2(1-x)\,\not{p}\bigr) \;+\; x\ell^2\,\not{p} \;+\; (\xi-1)\Bigl[-2x\ell^2\,\not{p} \;+\; \ell^2 m \;-\; \frac{1-x}{2}\ell^2\,\not{p}\Bigr] \\ &=\; \ell^2\times\Bigl[(3+\xi)m \;+\; \tfrac{1}{2}\bigl((9x-3)\;-\;(3x+1)\xi\bigr)\,\not{p}\Bigr]. \end{aligned} \tag{S.29}$$

Plugging this divergent part of the numerator into eq. (S.22), we arrive at

$$\Sigma^{\rm 1\,loop}_{\rm div}(\not{p}) \;=\; 2e^2\int\limits_0^1 dx(1-x)\times\Bigl[(3+\xi)m+\tfrac{1}{2}\bigl((9x-3)-(3x+1)\xi\bigr)\,\not{p}\Bigr]\times\int\limits_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\ell^2}{[\ell^2-\Delta+i0]^3} \tag{S.30}$$

where the momentum integral evaluates to

$$\int\limits_{\rm reg}\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\ell^2}{[\ell^2-\Delta+i0]^3} \;=\; \frac{1}{16\pi^2}\left(\frac{1}{\epsilon}\;+\;\log\frac{\mu^2}{\Delta}\;+\;\text{const}\right). \tag{S.31}$$

For an off-shell $p^2 = -E^2$ (with $E^2 \gg m^2$), we may approximate

$$\Delta \;\approx\; x(1-x)\times E^2, \tag{S.32}$$

hence

$$\log\frac{\mu^2}{\Delta(x)} \;=\; \log\frac{\mu^2}{E^2} \;+\; \mathit{finite\_f}(x) \tag{S.33}$$

and therefore

$$\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\ell^2}{[\ell^2-\Delta+i0]^3} \;=\; \frac{1}{16\pi^2}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\,+\,\mathit{finite\_f}(x)\right). \tag{S.34}$$

Thus, the divergent part of the $\Sigma_e^{1\,\text{loop}}$ amounts to

$$\Sigma^{1\,\text{loop}}_{\text{div}}(\not{p}) \;=\; \frac{\alpha}{2\pi}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\right)\times\int\limits_0^1 dx(1-x)\Big[(3+\xi)m\,+\,\tfrac{1}{2}\big((9x-3)-(3x+1)\xi\big)\,\not{p}\Big], \tag{S.35}$$

where the remaining integral over the Feynman parameter x yields

$$\int\limits_0^1 dx(1-x)\Big[(3+\xi)m \;+\; \tfrac{1}{2}\big((9x-3)\;-\;(3x+1)\xi\big)\,\not{p}\Big] \;=\; \tfrac{1}{2}(3+\xi)m \;-\; \tfrac{1}{2}\xi\,\not{p}, \tag{S.36}$$

hence

$$\Sigma^{1\,\text{loop}}_{\text{div}}(\not{p}) \;=\; \frac{\alpha}{4\pi}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\right)\times\Big((3+\xi)m\;-\;\xi\,\not{p}\Big). \tag{S.37}$$

Finally, to cancel this divergence we need the counterterms

$$\delta_2^{\text{order}\,\alpha} \;=\; -\frac{\xi\alpha}{4\pi}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\,+\,\text{const}\right), \tag{S.38}$$

$$\delta_m^{\text{order}\,\alpha} \;=\; -\frac{(3+\xi)\alpha m}{4\pi}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\,+\,\text{const}\right), \tag{S.39}$$

in perfect agreement with eqs. (7) and (8) for

$$C_2(\xi) \;=\; -\frac{\xi}{2} \quad\text{and}\quad C_m(\xi) \;=\; -\frac{3+\xi}{2}\,. \tag{S.40}$$

Problem **2**(c):

By inspection of eq. (S.40), $C_m(\xi) - C_2(\xi) = -\frac{3}{2}$ for any ξ.

Problem **2**(d):

Taking the derivatives of eqs. (7) and (8) WRT $\log E$, we find

$$\frac{d\delta_2(E)}{d\log E} \;=\; -2\times\frac{C_2\alpha}{2\pi} \;+\; O(\alpha^2), \tag{S.41}$$

$$\frac{d\delta_m(E)}{d\log E} \;=\; -2\times\frac{C_m\alpha m}{2\pi} \;+\; O(\alpha^2). \tag{S.42}$$

Consequently, to the one-loop order $O(\alpha)$, the electron's anomalous dimension is

$$\gamma_e \;=\; \frac{1}{2}\frac{d\log Z_2}{d\log E} \;\approx\; \frac{1}{2}\frac{d\delta_2}{d\log E} \;=\; -\frac{C_2\alpha}{2\pi} \;+\; O(\alpha)^2. \tag{S.43}$$

As to the electron mass β-function, eq. (9) gives us

$$\beta_m \;=\; 2\gamma_e\times m \;+\; \big(2\gamma_e\times\delta_m \;=\; O(\alpha^2 m)\big) \;-\; \frac{d\delta_m}{d\log E} \;=\; -\frac{2C_2\alpha m}{2\pi} \;+\; \frac{2C_m\alpha m}{2\pi} \;+\; O(\alpha^2 m), \tag{S.44}$$

in perfect agreement with eq. (10). Specifically, we found in part (a) that $2(C_m - C_2) = -3$ for any ξ, hence

$$\beta_m \;=\; -\frac{3\alpha m}{2\pi} \;+\; O(\alpha^2 m). \tag{S.45}$$

Problem **2**(e):

To solve the differential equation (S.45), we start by letting the running electron mass be some unknown function of the running coupling,

$$m(E) \;=\; F(\alpha(E)), \tag{S.46}$$

hence

$$\frac{dm}{d\log E} \;=\; \frac{dF(\alpha)}{d\alpha}\times\frac{d\alpha(E)}{d\log E} \;=\; \frac{dF}{d\alpha}\times\beta_\alpha(\alpha) \;\approx\; \frac{dF}{d\alpha}\times\left(\frac{2\alpha^2}{3\pi} \;+\; O(\alpha^3)\right). \tag{S.47}$$

where the last equality follows from the known one-loop beta-function (11). Plugging this formula into eq. (S.45) gives us

$$\frac{dF}{d\alpha}\times\frac{2\alpha^2}{3\pi}\times(1+O(\alpha)) \;=\; -\frac{3\alpha F}{2\pi}\times(1+O(\alpha)) \tag{S.48}$$

and hence

$$\frac{dF}{d\alpha} \;=\; -\frac{9}{4}\times\frac{F}{\alpha}\times(1+(O\alpha)). \tag{S.49}$$

To solve this equation, we rewrite it as

$$\frac{dF}{F} \;=\; -\frac{9}{4}\frac{d\alpha}{\alpha}\times(1+O(\alpha)), \tag{S.50}$$

hence

$$d\log(F) \;=\; d\left(-\frac{9}{4}\times\log(\alpha)\;+\;O(\alpha)\right), \tag{S.51}$$

which integrates to

$$\log F(\alpha) \;=\; \text{const} \;-\; \frac{9}{4}\log(\alpha) \;+\; O(\alpha) \tag{S.52}$$

and therefore

$$F(\alpha) \;=\; \text{const}\times\alpha^{-9/4}\times(1+O(\alpha)). \tag{S.53}$$

Disregarding the two-loop and higher-order corrections on the RHS, we have

$$F(\alpha) \;=\; \text{const}\times\alpha^{-9/4} \tag{S.54}$$

and therefore

$$m(E) \;=\; \text{const}\times\big(\alpha(E)\big)^{-9/4}, \tag{S.55}$$

in perfect agreement with eq. (12) for $r=-\frac{9}{4}$.

Finally, the constant pre-factor in eq. (S.55) obtains from the boundary condition at the particle's mass m_0. Indeed, for $E \sim m_e$ eq. (S.55) should yield $m(E) \approx m_0$ (up to small $O(\alpha m)$ threshold corrections we neglect here), thus

$$m_0 \;=\; \text{const} \times (\alpha_0)^{-9/4} \tag{S.56}$$

for the same constant as in eq. (S.55). Consequently,

$$m(E) \;=\; \frac{m_0}{(\alpha_0)^{9/4}} \times \bigl(\alpha(E)\bigr)^{9/4} \;=\; m_0 \times \left(\frac{\alpha(E)}{\alpha_0}\right)^{-9/4}, \tag{S.57}$$

in perfect agreement with eq. (12). *Quod erat demonstrandum.*

PS: While the renormalization group equation (10) for the electron's mass applies at all energies, the solution (S.57) is valid only for $E \lesssim$ masses of other charged particles besides the electron; in practice, this means $E \lesssim m_{\text{muon}} \approx 100$ MeV. The reason for this limitation is that above 100 MeV, the muon — and eventually the other charge particles — start contributing to the electric charge renormalization, which changes the coefficient of the one-loop beta-function (11) for the $\alpha(E)$. This changes the numeric factor in eq. (S.49) and hence the power of $\alpha(E)$ in eq. (S.55). Thus, above 100 MeV $m(E)$ start scaling as a different power $r' \neq -\frac{9}{4}$ of the $\alpha(E)$, and at higher energies this power r' keeps changing as more and more charged particles come into play.