Poisson resummation formula ($\star$):

Poisson's resummation formula (1) amounts to

$$\sum_{\ell=-\infty}^{+\infty} e^{2\pi i\ell\nu} \;=\; \sum_{n=-\infty}^{\infty} \delta(\nu-n). \tag{S.1}$$

To prove this formula, we begin by regulating the divergent sum on the LHS as in eq. (2) for a small but positive ϵ. Then we calculate

$$\begin{aligned}
S(\nu,\epsilon) \;&\stackrel{\rm def}{=}\; \sum_{\ell=-\infty}^{+\infty} e^{2\pi i\ell\nu}\times e^{-\epsilon|\ell|}\\
&\quad\langle\!\langle \text{ using } \ell=-\ell' \text{ for } \ell<0 \,\rangle\!\rangle\\
&=\; 1\;+\;\sum_{\ell=1}^{\infty} e^{2\pi i\ell\nu-\epsilon\ell}\;+\;\sum_{\ell'=1}^{\infty} e^{-2\pi i\ell'\nu-\epsilon\ell'}\\
&=\; 1\;+\;\frac{\exp(+2\pi i\nu-\epsilon)}{1-\exp(+2\pi i\nu-\epsilon)}\;+\;\frac{\exp(-2\pi i\nu-\epsilon)}{1-\exp(-2\pi i\nu-\epsilon)}\\
&=\;\frac{1-e^{-2\epsilon}}{1+e^{-2\epsilon}-e^{-\epsilon}(e^{+2\pi i\nu}+e^{-2\pi i\nu})}\\
&=\;\frac{\sinh(\epsilon)}{\cosh(\epsilon)-\cos(2\pi\nu)}\;=\;\frac{\sinh(\epsilon)}{\sinh^2(\epsilon/2)+\sin^2(\pi\nu)}\,,
\end{aligned} \tag{S.2}$$

which for a small $\epsilon\ll 1$ becomes

$$S(\nu,\epsilon)\;\approx\;\frac{4\epsilon}{\epsilon^2+4\sin^2(\pi\nu)}\,. \tag{S.3}$$

Next, look at this S as a function of ν and notice that:

- $S(\nu)$ is periodic in ν with period 1, $S(\nu+1)=S(\nu)$.
- For a non-integer ν, $S(\nu)=O(\epsilon)\to 0$ when $\epsilon\to 0$.
- For an integer ν, $S(\nu)=(4/\epsilon)\to\infty$ when $\epsilon\to 0$.

- For $|\nu| \ll 1$,

$$S(\nu,\epsilon) \;\approx\; \frac{4\epsilon}{\epsilon^2+(2\pi\nu)^2} \;\xrightarrow[\epsilon\to+0]{}\; \delta(\nu), \tag{S.4}$$

  where the limit $\epsilon \to +0$ is taken at a fixed small ν.

Together, these 4 observations clearly imply that

$$S(\nu,\epsilon) \;\xrightarrow[\epsilon\to+0]{}\; \sum_{n=-\infty}^{\infty} \delta(\nu-n), \tag{S.5}$$

exactly as in eq. (3). *Quod erat demonstradnum.*

Problem **1**(a):

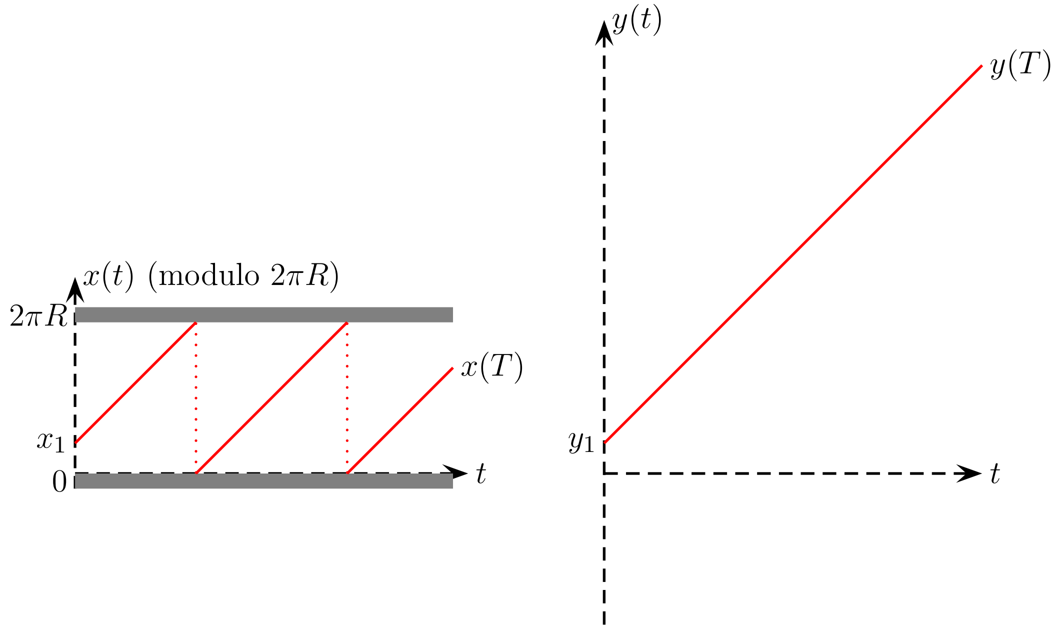
The difference between a circle and a straight line is that on a circle the path of a particle going from point x_1 to point x_2 does not need to be 'straight' but may wrap around the whole circle one or more times. Indeed, let us compare a particle moving on a circle according to $x(t)$ (modulo $2\pi R$) with a particle moving on an infinite line according to $y(t)$. If the two particles have exactly the same velocities at all times,

$$\frac{dx}{dt} \;\equiv\; \frac{dy}{dt} \tag{S.6}$$

and similar initial positions $x_1 = y_1$ (according to some coordinate systems) at time t_1, then at a later time t_2 one generally has

$$y(T) \;=\; x(T) \;+\; 2\pi R \times n \tag{S.7}$$

for some integer $n = 0, \pm1, \pm2, \pm3, \ldots$ because the $x(y)$ path may wrap around the circle n times while the $y(t)$ path may not wrap. For example, the two paths depicted below have same (constant) velocities and begin at $y_1 = x_1$ but end at $y(T) = x(T) + 2\pi R \times 2$:



It is easy to see that the paths $x(t)$ (modulo $2\pi R$) and $y(t)$ (modulo nothing) are in one-to-one correspondence with each other, provided we restrict the initial point y_1 of the particle on the infinite line to a particular interval of length $L = 2\pi R$, say $0 \leq y_0 < 2\pi R$. Consequently, in the path integral for the particle on the circle

$$\iint\limits_{x(t_1)=x_1\,(\mathrm{mod}\,L)}^{x(t_2)=x_2\,(\mathrm{mod}\,L)} \mathcal{D}'[x(t)\,(\mathrm{mod}\,L)] \;=\; \sum_{n=-\infty}^{+\infty} \iint\limits_{y(t_1)=x_1}^{y(t_2)=x_2+nL} \mathcal{D}'[y(t)]\,. \tag{S.8}$$

Furthermore, in the absence of potential energy, the circle path $x(t)\,(\mathrm{mod}\,L)$ and the corresponding ∞ line path $y(t)$ have equal actions

$$S[x(t)\,(\mathrm{mod}\,L)] \;=\; S[y(t)] \;=\int_{t_1}^{t_2} dt\,\left[\tfrac{M}{2}\dot{x}^2 \;=\; \tfrac{M}{2}\dot{y}^2\right], \tag{S.9}$$

and therefore

$$
\begin{aligned}
U_{\text{circle}}(x_2, t_2; x_1, t_1) &= \iint\limits_{x(t_1)=x_1\,(\text{mod}\,L)}^{x(t_2)=x_2\,(\text{mod}\,L)} \mathcal{D}'[x(t)\,(\text{mod}\,L)]\, e^{iS[x(t)\,(\text{mod}\,L)]/\hbar} \\
&= \sum_{n=-\infty}^{+\infty} \iint\limits_{y(t_1)=x_1}^{y(t_2)=x_2+nL} \mathcal{D}'[y(t)]\, e^{iS[y(t)]/\hbar} \\
&= \sum_{n=-\infty}^{+\infty} U_{\text{line}}(y_2 = x_2 + nL, t_2; y_1 = x_1, t_1),
\end{aligned}
\tag{S.10}
$$

precisely as in eq. (4).

Problem **1**(b):

The evolution kernel of a free particle living on an infinite line is

$$
U_{\text{line}}(y_2, t_2; y_1, t_1) = \sqrt{\frac{M}{2\pi i\hbar(t_2 - t_1)}} \times \exp\left(\frac{i}{\hbar} S_{\text{classical}} = \frac{i}{\hbar}\, \frac{M(x_2 - x_1)^2}{2(t_2 - t_1)}\right), \tag{5}
$$

hence according to eq. (4), the kernel of a particle living on a circle is

$$
U_{\text{circle}}(x_2, t_2; x_1, t_1) = \sqrt{\frac{M}{2\pi i\hbar(t_2 - t_1)}} \times \sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar(t_2 - t_1)} \times (x_2 - x_1 + nL)^2\right). \tag{S.11}
$$

To evaluate this sum, we use Poisson re-summation formula (1), which gives

$$
\sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM(x_2 - x_1 + nL)^2}{2\hbar(t_2 - t_1)}\right) = \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu\, \exp\left(\frac{iM(x_2 - x_1 + \nu L)^2}{2\hbar(t_2 - t_1)}\right) \times e^{2\pi i \ell \nu}. \tag{S.12}
$$

To calculate the integral here, we combine the exponentials and re-arrange the net exponent so as to extract a full square for the ν variable:

$$\begin{aligned}
&\frac{iM}{2\hbar(t_2-t_1)}\,(x_2-x_1+\nu L)^2 \;+\; 2\pi i\ell\times\nu \;= \\
&\qquad= \frac{iML^2}{2\hbar(t_2-t_1)}\left(\nu+\frac{x_2-x_1}{L}\right)^2 \;+\; 2\pi i\ell\left(\nu+\frac{x_2-x_1}{L}\right) \;-\; \frac{2\pi i\ell(x_2-x_1)}{L} \\
&\qquad= \frac{iML^2}{2\hbar(t_2-t_1)}\left(\nu+\frac{x_2-x_1}{L}+\frac{2\pi\ell\hbar(t_2-t_1)}{ML^2}\right)^2 \\
&\qquad\quad -\frac{i\hbar(t_2-t_1)(2\pi\ell)^2}{2ML^2} \;-\; \frac{2\pi i\ell(x_2-x_1)}{L} \\
&\qquad= \frac{iML^2}{2\hbar(t_2-t_1)}\big(\nu+\text{const}\big)^2 \;+\; \nu\text{–independent},
\end{aligned} \tag{S.13}$$

hence the integral yields

$$\begin{aligned}
&\int\limits_{-\infty}^{+\infty}\!d\nu\,\exp\left(\frac{iM}{2\hbar(t_2-t_1)}\right)\times e^{2\pi i\ell\nu} \;= \\
&\qquad= \exp\big(\nu\text{–independent}\big)\times\int\limits_{-\infty}^{+\infty}\!d\nu\,\exp\left(\frac{iML^2}{2\hbar(t_2-t_1)}\times(\nu+\text{const})^2\right) \\
&\qquad= \exp\big(\nu\text{–independent}\big)\times\sqrt{\frac{2\pi i\hbar(t_2-t_1)}{ML^2}}\,. \\
&\qquad= \exp\left(-\frac{i\hbar(t_2-t_1)(2\pi\ell)^2}{2ML^2}\;-\;\frac{2\pi i\ell(x_2-x_1)}{L}\right)\times\sqrt{\frac{2\pi i\hbar(t_2-t_1)}{ML^2}}\,.
\end{aligned} \tag{S.14}$$

Finally, plugging this integral into the sum (S.12), we arrive at

$$\begin{aligned}
U_{\text{circle}}(x_2,t_2;x_1,t_1) \;&=\; \sqrt{\frac{M}{2\pi i\hbar(t_2-t_1)}}\times\sqrt{\frac{2\pi i\hbar(t_2-t_1)}{ML^2}}\times \\
&\quad\times\sum_{\ell=-\infty}^{+\infty}\exp\left(-2\pi i\ell\,\frac{x_2-x_1}{L}\;-\;\frac{i(2\pi\ell)^2\hbar(t_2-t_1)}{ML^2}\right) \\
&=\; \frac{1}{L}\times\sum_{\ell=-\infty}^{+\infty}\exp(+ip_\ell x_2/\hbar)\times\exp(-ip_\ell x_1/\hbar)\times\exp(-iE_\ell(t_2-t_1)/\hbar)
\end{aligned} \tag{S.15}$$

where

$$p_\ell \;=\; -\frac{2\pi\hbar\ell}{L} \;=\; -\frac{\hbar\ell}{R} \quad\text{and}\quad E_\ell \;=\; \frac{p_\ell^2}{2M}\,. \tag{S.16}$$

Problem **1**(c): This is obvious from eqs. (S.15) and (S.16).

Problem **11.1**(a):

The easiest way to compute the correlation function of $\exp(+i\Phi(x_1))$ and $\exp(-i\Phi(x_2)$ is in terms of functional integrals:

$$\left\langle \mathbf{T} e^{+i\hat{\Phi}(x_1)} e^{-i\hat{\Phi}(x_2)} \right\rangle = \frac{\iint \mathcal{D}[\Phi(x)]\, e^{iS[\Phi(x)]}\, e^{+i\Phi(x_1)} e^{-i\Phi(x_2)}}{\iint \mathcal{D}[\Phi(x)]\, e^{iS[\Phi(x)]}} = \frac{\iint \mathcal{D}[\Phi(x)]\, \exp\left(i\int (\mathcal{L} + J\Phi) d^d x\right)}{\iint \mathcal{D}[\Phi(x)]\, \exp\left(i\int \mathcal{L}\, d^d x\right)} \equiv \frac{Z[J]}{Z[0]} \tag{S.17}$$

where

$$J(x) = \delta^{(d)}(x - x_1) - \delta^{(d)}(x - x_2). \tag{S.18}$$

Moreover, for a *free* scalar field $\Phi(x)$

$$Z[J] = Z[0] \times \exp\left(-\frac{1}{2}\int d^d x \int d^d y\, J(x) G^F(x-y) J(y)\right). \tag{S.19}$$

For the source as in eq. (S.18), the double integral inside the exponential is simply

$$\int d^d x \int d^d y\, J(x) G^F(x-y) J(y) = 2G^F(0) - 2G^F(x-y), \tag{S.20}$$

hence

$$\left\langle \mathbf{T} e^{+i\hat{\Phi}(x_1)} e^{-i\hat{\Phi}(x_2)} \right\rangle = \frac{Z[J]}{Z[0]} = \exp\Big(G^F(x_1 - x_2) - G^F(0)\Big). \tag{S.21}$$

Problem **11.1**(b):

Under the axionic symmetry $\Phi(x) \mapsto \Phi(x) - \alpha$, the derivatives $\partial_\mu \Phi$, $\partial_\mu \partial_\nu \Phi$, *etc.* are invariant but the field Φ is not. Consequently, the most general effective Lagrangian for the quantum $\Phi(x)$ field must be a function of its derivatives only,

$$\mathcal{L} = \frac{\rho}{2}(\partial\Phi)^2 + \frac{A}{2}(\partial^2\Phi)^2 + \frac{B}{4}(\partial\Phi)^4 + \cdots. \tag{S.22}$$

Now consider the renormalizability. In d spacetime dimensions, the scalar field Φ has dimensionality $\frac{d}{2}-1$, hence a Lagrangian term involving n fields and m derivatives has dimensionality

$$\Delta = n\left(\frac{d}{2}-1\right) + m = \frac{nd}{2} + (m-n). \tag{S.23}$$

Renormalizability requires $\Delta \leq d$ and hence

$$\frac{d}{2}(n-2) + (m-n) \leq 0. \tag{S.24}$$

On the other hand, axionic symmetry requires $m \geq n$ (no field without a derivative) while any term with $n < 2$ can be disregarded as a total derivative. All these conditions leave just one possibility (assuming $d > 0$), namely $m = n = 2$ and hence

$$\mathcal{L}_{\text{renorm.}} = \frac{\rho}{2}(\partial\Phi)^2 + \text{ nothing else.} \tag{S.25}$$

In other words, in the infrared regime where the non-renormalizable interactions become irrelevant, $\Phi(x)$ is a free massless field.

Problem **11.1**(c):

Suppose a theory with a global phase symmetry $U(1)$ appears to have a symmetry-breaking VEV of a complex field

$$S(x) = A(x) \times e^{i\Phi(x)}, \qquad \langle A\rangle > 0. \tag{S.26}$$

The radial field A is massive, so its fluctuations decouple from the low-energy effective theory. All we have at low energies is the VEV $\langle A\rangle > 0$ and the massless Goldstone field $\Phi(x)$ — and we saw in part (b) that $\Phi(x)$ is effectively a free field with Lagrangian (S.25). It's non-canonically normalized, so

$$\langle \mathbf{T}\Phi(x)\Phi(y)\rangle \;=\; \frac{1}{\rho}\times G_0(x-y) \tag{S.27}$$

where $G_0(x-y)$ is the usual Feynman propagator for $m^2=0$. Consequently, according to eq. (S.21),

$$\begin{aligned}\left\langle \mathbf{T}S(x)S^\dagger(y)\right\rangle \;&=\; \langle A\rangle^2\times\left\langle \mathbf{T}e^{+i\hat{\Phi}(x_1)}e^{-i\hat{\Phi}(x_2)}\right\rangle\\ &=\; \langle A\rangle^2\times\exp\left(\frac{G_0(x-y)-G_0(0)}{\rho}\right)\\ &\equiv\; C^2\times\exp(G_0(x-y)/\rho)\end{aligned} \tag{S.28}$$

where $C=\langle A\rangle\times e^{-G_0(0)/2\rho}$ absorbs the UV corrections to the bare A parameter.

Similarly, in the Euclidean d-dimensional space of the statistical mechanics, the correlation function becomes

$$\left\langle S(x)S^\dagger(y)\right\rangle \;=\; C^2\times\exp(G_0^E(x-y)/\rho) \tag{S.29}$$

where

$$G_0^E(x-y) \;=\; \int\frac{d^dp_E}{(2\pi)^E}\,\frac{e^{ip(x-y)}}{p_E^2} \tag{S.30}$$

is the Euclidean propagator of a massless scalar. By the $SO(d)$ symmetry, it depends only on the Euclidean distance $r=|x-y|$, and for $m^2=0$ this dependence is a pure power law:

$$G_0^E(x) \;=\; \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}\,r^{2-d}. \tag{9}$$

Specifically, in $d=3$ Euclidean dimensions $D_0^E=1/4\pi r$ and hence

$$\left\langle S(x)S^\dagger(y)\right\rangle_{d=3} \;=\; C^2\times\exp\left(\frac{1}{4\pi\rho}\times\frac{1}{r}\right). \tag{S.31}$$

Likewise, in $d=4$ dimensions,

$$\left\langle S(x)S^\dagger(y)\right\rangle_{d=4} \;=\; C^2\times\exp\left(\frac{1}{2\pi^2\rho}\times\frac{1}{r^2}\right). \tag{S.32}$$

In both cases, we see extremely strong self-correlations of the $S(x)$ field at very short distances. On the other hand, in the long-distance limit the correlated expectation values (S.31) and (S.32) remain finite. Indeed, in any dimension greater then two

$$\left\langle S(x)S^\dagger(y)\right\rangle_{d>2} \xrightarrow[r\to\infty]{} C^2 > 0. \tag{S.33}$$

Physically, such asymptotic behavior is characteristic of non-trivial vacuum expectation values: By cluster expansion,

$$\langle S(x)S^*(y)\rangle_{d>2} \xrightarrow[r\to\infty]{} \langle S(x)\rangle \times \langle S^*(y)\rangle \equiv |\langle S\rangle|^2, \tag{S.34}$$

thus the physical meaning of the limit (S.33) is

$$|\langle S(x)\rangle| = C > 0. \tag{S.35}$$

In other words, *the $U(1)$ symmetry of the complex $S(x)$ field is spontaneously broken.*

One the other hand, in one Euclidean dimension $G_0^E = -\frac{1}{2}r$, hence

$$\left\langle S(x)S^\dagger(y)\right\rangle_{d=1} = C^2 \times \exp\left(-\frac{r}{2\rho}\right). \tag{S.36}$$

This correlated expectation value remains finite at short distance and decreases exponentially at large distances. Indeed, for any dimension d less than two $G_0^E(x-y) \to -\infty$ for $r = |x-y| \to \infty$ and hence

$$\left\langle S(x)S^\dagger(y)\right\rangle_{d<2} \xrightarrow[r\to\infty]{} 0. \tag{S.37}$$

In terms of the cluster expansion (S.34), this means $\langle S\rangle = 0$ — *despite the classical formula $|S(x)| \equiv A > 0$, the quantum theory has a zero VEV in $d < 2$ and the $U(1)$ phase symmetry remains unbroken.*

In the borderline case of exactly two dimensions $D_0^E = \frac{-1}{2\pi}\log r + \text{const}$, hence

$$\langle S(x)S^*(y)\rangle_{d=2} = \text{const} \times r^{-1/2\pi\rho}. \tag{S.38}$$

Such scaling behavior correspond to the absence of dimensionful parameters in the theory — the spin wave modulus ρ is dimensionless for $d = 2$. Among other things, the scaling behavior (S.38) provides for vanishing of the correlated expectation value in the infinite distance limit. Thus, similarly to the $d < 2$ case, we again have $\langle S\rangle = 0$ and the unbroken $SO(2)$ symmetry.

The bottom line of this exercise is to illustrate the general rule: In $d > 2$ spacetime dimensions, exact symmetries of the action may become spontaneously broken by vacuum expectation values such as $\langle S \rangle$. *But in $d \leq 2$ dimensions, quantum effects destroy any VEV that would break a continuous symmetry.* However, the discrete symmetries may be spontaneously broken in two dimensions or in fractional dimensions $d > 1$.

Problem **3**(a):

A point of notation: $\mathcal{T}$ is the temperature, not the time.

In class, we have learned the path-integral formula for the "partition function" of a quantum particle,

$$Z(t) \equiv \exp\bigl(-it\hat{H}\bigr) = \iint\limits^{\mathbf{x}(t)=\mathbf{x}(0)} \mathcal{D}[\mathbf{x}(t')] \exp\bigl(iS[\mathbf{x}(t')]\bigr). \tag{S.39}$$

In statistical mechanics, the partition function at temperature $\mathcal{T}$ is defined as

$$Z(\mathcal{T}) \equiv \operatorname{Tr}\exp\bigl(-\beta\hat{H}\bigr), \qquad \beta = \frac{1}{\mathcal{T}}. \tag{S.40}$$

The two partition functions are related by analytic continuation of the real time t to the imaginary time $-i\beta = -i/\mathcal{T}$, thus

$$Z_{\rm SM}(\mathcal{T}) = Z_{\rm QM}(t = -i/\mathcal{T}). \tag{S.41}$$

In terms of the path integrals, this relation corresponds to going from the Minkowski path integral to the Euclidean path integral

$$Z(\mathcal{T}) = \iint\limits^{\mathbf{x}(\beta)=\mathbf{x}(0)} \mathcal{D}[\mathbf{x}(t_e)] \exp\bigl(-S_E[\mathbf{x}(t_e)]\bigr) \tag{S.42}$$

where the Euclidean action is

$$S_E[\mathbf{x}(t_e)] = \int_0^\beta dt_e \left(\frac{m}{2}\left(\frac{d\mathbf{x}}{dt_e}\right)^2 + V(\mathbf{x})\right). \tag{S.43}$$

Note the boundary conditions for the Euclidean path integral (S.42): after the Euclidean time interval $\beta = 1/\mathcal{T}$, the particle must come back to its starting point. In other words, *at finite temperatures, the motion in Euclidean time is periodic with period $\beta = 1/\mathcal{T}$.*

Generalizing eq. (S.42) from particle mechanics to field theory is quite straightforward. For a real scalar field field $\phi(x)$ with a Euclidean Lagrangian

$$\mathcal{L}_E \;=\; \tfrac{1}{2}(\partial\phi)_e^2 \;+\; V(\phi) \tag{S.44}$$

we have finite-temperature Partition function

$$Z(\beta) \;=\; \iint^{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)} \mathcal{D}[\phi(\mathbf{x},x_4)]\,\exp\left[-\int d^3\mathbf{x}\int_0^\beta dx_4\left(\tfrac{1}{2}(\partial\phi)_e^2 \;+\; V(\phi)\right)\right]. \tag{S.45}$$

Again, the finite temperature translates into the geometry of the Euclidean 4D spacetime: The Euclidean time $x_4 = it$ is of finite extent $\beta = 1/\mathcal{T}$, and the scalar field is subject to the periodic boundary condition; the other 3 dimensions x_1, x_2, x_3 are infinite as usual.

For the free scalar field, the Euclidean action is a quadratic functional

$$S_E[\phi(x_E)] \;=\; \frac{1}{2}\int d^4x_e\,\phi(m^2-\partial_e^2)\phi\,, \tag{S.46}$$

so the path integral (S.45) is a Gaussian integral that can be formally evaluated as the determinant

$$Z(\beta) \;=\; \text{const}\times\Big(\text{Det}[m^2-\partial_e^2]\Big)^{-1/2} \tag{S.47}$$

Or in terms of the Helmholtz's free energy,

$$F \;\equiv\; -\mathcal{T}\log Z \;=\; \text{const} \;+\; \frac{\mathcal{T}}{2}\log\text{Det}[m^2-\partial_e^2] \;=\; \text{const} + \frac{\mathcal{T}}{2}\,\text{Tr}\log[m^2-\partial_e^2]. \tag{13}$$

Problem **3**(b):

To actually evaluate the functional trace in eq. (13), we diagonalize the $m^2-\partial_E^2$ operator via Fourier transform to the momentum space. However, because of the periodicity of the Euclidean time coordinate, the Euclidean "energies" k_4 have discrete rather than continuous spectrum,

$$k_4 \;=\; \frac{2\pi}{\beta}\times\text{integer}. \tag{S.48}$$

As to the 3-space components $\mathbf{k} = (k_1,k_2,k_3)$ of the momentum, they have a continuous spectrum in the infinite space, while in the large but finite box of volume L^3 the spectrum becomes discrete-but-near-continuous with density

$$d\#\mathbf{k} \;=\; \frac{L^3}{(2\pi)^3}\times d^3\mathbf{k}. \tag{S.49}$$

Consequently, the functional trace in eq. (13) evaluates to

$$\operatorname{Tr}\log[m^2-\partial_E^2] \;=\; \sum_{\mathbf{k}}\sum_{k_4}\log(m^2+\mathbf{k}^2+k_4^2) \;=\; L^3\times\!\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4=2\pi\mathcal{T}n}\log(m^2+\mathbf{k^2}+k_4^2), \tag{S.50}$$

and the free energy of the quantum field becomes

$$\begin{aligned} F \;&=\; \text{const} \;+\; \frac{\mathcal{T}}{2}\times L^3\!\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4=2\pi\mathcal{T}n}\log(m^2+k_e^2) \\ &=\; \text{const} \;+\; \frac{L^3}{2}\!\int\!\frac{d^4k_e}{(2\pi)^4}\sum_n^{\text{integer}}\log(m^2+k_e^2)\times\delta\left(\frac{k_4}{2\pi\mathcal{T}}-n\right). \end{aligned} \tag{S.51}$$

Or in terms of the free energy *density*,

$$\mathcal{F} \;\stackrel{\text{def}}{=}\; \frac{F}{L^3} \;=\; \text{const} \;+\; \frac{1}{2}\!\int\!\frac{d^4k_e}{(2\pi)^4}\sum_n^{\text{integer}}\log(m^2+k_e^2)\times\delta\left(\frac{k_4}{2\pi\mathcal{T}}-n\right). \tag{14}$$

Now let's apply the Poisson's resummation formula

$$\sum_n^{\text{integer}}\delta(x-n) = \sum_{\ell=-\infty}^{+\infty}\exp(2\pi i\ell x) \tag{1}$$

to eq. (14):

$$\begin{aligned} \mathcal{F}(\mathcal{T}) \;&=\; \text{const} \;+\; \frac{1}{2}\!\int\!\frac{d^4k_e}{(2\pi)^4}\,\log(m^2+k_e^2)\times\sum_n^{\text{integer}}\delta\left(\frac{k_4}{2\pi\mathcal{T}}-n\right) \\ &=\; \text{const} \;+\; \frac{1}{2}\!\int\!\frac{d^4k_e}{(2\pi)^4}\,\log(m^2+k_e^2)\times\sum_{\ell=-\infty}^{+\infty}e^{i\ell\beta k_4} \\ &=\; \text{const} \;+\; \frac{1}{2}\sum_{\ell=-\infty}^{+\infty}\int\!\frac{d^4k_e}{(2\pi)^4}\,\log(m^2+k_e^2)\times e^{i\ell\beta k_4}. \end{aligned} \tag{15}$$

In the zero-temperature limit $\beta\to\infty$, the sum $\sum_\ell$ in this formula reduces to just the $\ell=0$

term while all the other terms are suppressed by the rapidly oscillating phase $\exp(i\ell\beta k_4)$. Thus,

$$\mathcal{F}_0 \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{T}=0) \;=\; \text{const} \;+ \int\!\frac{d^4k_e}{(2\pi)^4}\,\log(k_e^2+m^2) \tag{S.52}$$

for the same constant term as in eq. (15). In the general spirit of subtracting the zero-point energy contribution, let's focus on the difference of free energies at finite and zero temperatures. Taking the difference between eqs. (15) and (S.52), we get rid of the unknown constant terms as well as the $\ell=0$ term in the sum over ℓ, thus

$$\widehat{\mathcal{F}}(\mathcal{T}) \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{T}) \;-\; \mathcal{F}_0 \;=\; \frac{1}{2}\sum_{\ell\neq 0}\int\!\frac{d^4k_e}{(2\pi)^4}\,\log(k_e^2+m^2)\times\exp(i\ell\beta k_4). \tag{S.53}$$

Moreover, the integrals for the $+\ell$ and the $-\ell$ terms in the sum are related by changing the integration variable $+k_4\to -k_4$, so they are equal to each other. Thus, adding up such terms in pairs, we arrive at a sum over positive ℓ only,

$$\widehat{\mathcal{F}} \;=\; \sum_{\ell=1}^{\infty}\int\!\frac{d^4k_e}{(2\pi)^4}\,\log(k_e^2+m^2)\times\exp(i\ell\beta k_4). \tag{16}$$

Problem **3**(c):

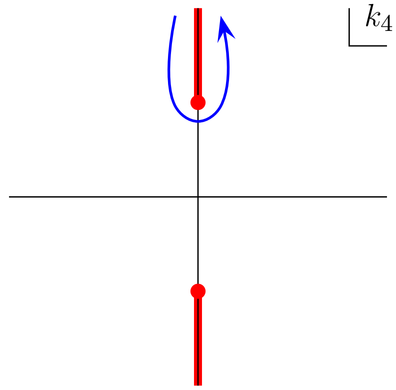
Formula (16) has a nice 4D form, but for the purpose of comparison with the ordinary statistical mechanics, let us integrate over the k_4 before we integrate over the 3–momentum $\mathbf{k}$. For fixed $\mathbf{k}$ and ℓ, we need to calculate

$$I \;=\int\!\frac{dk_4}{2\pi}\,e^{i\beta\ell k_4}\,\log(k_4^2+E^2) \qquad \text{for } E^2=m^2+\mathbf{k}^2, \tag{S.54}$$

as on the LHS of eq. (17).

The best way to evaluate this integral is to deform the integration contour in the complex k_4 plane away from the real axis. Since the exponential factor rapidly decreases for $\operatorname{Im}k_4\to+\infty$, we should move the contour up as far as we can without crossing any singularities of the $\log(k_4^2+E^2)$ factor. Specifically, that log factor has a branch cut at real and negative values of the log's argument $k_4^2+E^2$, which makes for two separate branch cuts in the complex k_4 plane:

one cut runs from $+iE$ to $+i\infty$ along the imaginary axis, and the other cut runs from $-iE$ to $-i\infty$ (also along the imaginary axis). Thus, we move the integration contour up (towards $+i\infty$) until it wraps around the upper branch cut:



(S.55)

Along the descending left half of this contour we may take $k_4 = iE(1+x+i\epsilon)$ for real x running from $+\infty$ to zero, while along the ascending right half of the contour we may take $k_4 = iE(1+x-i\epsilon)$ for real x running back from zero to $+\infty$. Consequently $E^2 + k_4^2 = E^2 \times (-x^2 - 2x \mp i\epsilon)$, and the integral becomes

$$
\begin{aligned}
I \;&=\; \frac{iE}{2\pi}\int\limits_{+\infty}^{0} dx\, e^{-\beta\ell E(1+x)} \times \log\bigl[E^2(-x^2-2x-i\epsilon)\bigr] \\
&\qquad +\; \frac{iE}{2\pi}\int\limits_{0}^{+\infty} dx\, e^{-\beta\ell E(1+x)} \times \log\bigl[E^2(-x^2-2x+i\epsilon)\bigr] \\
&=\; \frac{iE}{2\pi}\int\limits_{0}^{+\infty} dx\, e^{-\beta\ell E(1+x)} \times \Bigl(\log\bigl[E^2(-2x-x^2+i\epsilon)\bigr] \;-\; \log\bigl[E^2(-2x-x^2-i\epsilon)\bigr] \;=\; 2\pi i\Bigr) \\
&=\; -E\int\limits_{0}^{+\infty} dx\, e^{-\beta\ell E(1+x)} \;=\; -\frac{e^{-\beta\ell E}}{\beta\ell}\,,
\end{aligned}
\tag{S.56}
$$

exactly as on the RHS of eq. (17).

Problem **3**(d):

Eq. (16) for the free energy density involves a 4D integral over the Euclidean momentum k_e^μ.

Using eq. (17) to evaluate the $\int dk_4$ integral, we arrive at

$$\widehat{\mathcal{F}} \;=\; \mathcal{F}(\mathcal{T}) \;-\; \mathcal{F}(0) \;=\; \sum_{\ell=1}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{-e^{-\beta\ell E(\mathbf{k})}}{\beta\ell}\,. \tag{S.57}$$

Next, let's sum over ℓ before integrating over the 3D momentum $\mathbf{k}$:

$$\widehat{\mathcal{F}} \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\ell=1}^{\infty} \frac{-e^{-\beta\ell E(\mathbf{k})}}{\beta\ell} \tag{S.58}$$

where the sum over ℓ has a form of the Taylor series for $\log(1-x)$,

$$\sum_{\ell=1}^{\infty} \frac{-(e^{-\beta E})^{\ell}}{\ell} \;=\; \log\Bigl(1 \;-\; e^{-\beta E}\Bigr), \tag{S.59}$$

Therefore,

$$\widehat{\mathcal{F}}(\mathcal{T},m) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \mathcal{T} \times \log\Bigl(1 \;-\; e^{-\beta E_{\mathbf{k}}}\Bigr). \tag{S.60}$$

Now let us compare our result (S.60) for the free energy of the free scalar quantum field with the conventional statistical mechanics of identical spinless relativistic bosons. In the SM of identical bosons, a free quantum field is equivalent to a $\mathbf{k}$–labeled family of harmonic oscillators, thus

$$F(\mathcal{T},m) \;=\; L^3 \times \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, F^{\rm harmonic}_{\rm oscillator}(\mathcal{T},E_{\mathbf{k}}) \tag{S.61}$$

where each oscillator mode contributes

$$F^{\rm harmonic}_{\rm oscillator}(\mathcal{T},E) \;=\; -\mathcal{T}\log Z^{\rm harmonic}_{\rm oscillator} \;=\; \mathcal{T}\log(2\sinh(\beta E/2)) \;=\; \tfrac{1}{2}E \;+\; \mathcal{T}\log\Bigl(1 \;-\; e^{-\beta E}\Bigr). \tag{S.62}$$

Subtracting the zero-point energy $\frac{1}{2}E$ of the oscillator, and integrating over the all the oscillators comprising the free scalar field — *cf.* eq. (S.61), — we get

$$\begin{aligned} \mathcal{F}(T) \;-\; \mathcal{F}(0) \;&=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Bigl(F^{\rm harmonic}_{\rm oscillator}(\mathcal{T},E_{\mathbf{k}}) \;-\; \tfrac{1}{2}E_{\mathbf{k}}\Bigr) \\ &=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \mathcal{T}\log\Bigl(1 \;-\; e^{-\beta E}\Bigr), \end{aligned} \tag{S.63}$$

in perfect agreement with the path-integral based formula (S.60).