Problem **1**(a):

For the $(0+1)$ dimensional (zero space, one time) free complex Grassmann field $\psi(t)$, we have quadratic Euclidean action

$$S_E = \int\limits_0^\beta dt_E\,\bar{\psi}(\partial+\omega)\psi \tag{S.1}$$

and hence partition function

$$Z = \mathrm{Det}[\partial+\omega]\,. \tag{S.2}$$

All physical observables of this system must be periodic in Euclidean time, so the odd Grassmann variables such as the fermionic fields themselves should be either periodic or antiperiodic. Consequently, the 'momentum' modes should be quantized as either integers or half integers,

$$k = \frac{2\pi}{\beta}\times n \qquad \text{or} \quad k = \frac{2\pi}{\beta}\times(n+\tfrac{1}{2}), \tag{S.3}$$

which produces two distinct expressions for the partition function.

In the periodic case, the partition function evaluates to

$$\begin{aligned} Z_+ &= \prod_{n=-\infty}^{+\infty}\left(\frac{2\pi i}{\beta}\times n+\omega\right) = \omega\times\prod_{n=1}^{\infty}\left(\omega^2+\left(\frac{2\pi n}{\beta}\right)^2\right) \\ &= \text{const}\times\beta\omega\times\prod_{n=1}^{\infty}\left(1+\left(\frac{\beta\omega}{2\pi n}\right)^2\right) = \text{const}\times 2\sinh(\beta\omega/2) \end{aligned} \tag{S.4}$$

where the last equality follows from the analytic properties of the infinite product as a function of $\beta\omega/2$: no poles, zeros at imaginary integers $i\times n$, and no exponential growth for $\mathrm{Im}\,\beta\omega\to\pm\infty$.

For the antiperiodic boundary conditions we have a quite different partition function:

$$\begin{aligned}
Z_- \;&=\; \prod_{n=-\infty}^{+\infty}\left(\frac{2\pi i}{\beta}\times(n-\tfrac{1}{2})\,+\,\omega\right)\;=\;\prod_{n=1}^{\infty}\left(\omega^2+\left(\frac{2\pi}{\beta}\right)^2\times(n-\tfrac{1}{2})^2\right)\\
&=\;\text{const}\times\prod_{n=1}^{\infty}\left(1\,+\,\left(\frac{\beta\omega}{2\pi(n-\frac{1}{2})}\right)^2\right)\\
&=\;\text{const}\times\prod_{m=1}^{\infty}\left(1\,+\,\left(\frac{\beta\omega}{2\pi(m/2)}\right)^2\right)\Big/\prod_{m=1}^{\infty}\left(1\,+\,\left(\frac{\beta\omega}{2\pi m}\right)^2\right)\\
&=\;\text{const}\times\sinh(\beta\omega)\Big/\sinh(\beta\omega/2)\;=\;\text{const}\times 2\cosh(\beta\omega/2).
\end{aligned}\tag{S.5}$$

Expanding both partition functions into exponentials $e^{-\beta E}$, we obtain

$$Z_\pm(\beta,\omega)\;=\;\text{const}\times\left(e^{+\beta\omega/2}\,\mp\,e^{-\beta\omega/2}\right).\tag{S.6}$$

The pre-exponential coefficients in a partition function correspond to the multiplicities of states, so they must be positive. Consequently, the periodic Z_+ — which contains a negative term $-E^{-\beta\omega/2}$ — is not a valid partition function of any physical quantum system. On the other hand, the antiperiodic Z_- is a perfectly good partition function for a two-level quantum system with energies $E=\mp\frac{1}{2}\omega$. The two levels have equal multiplicities — the overall constant factor in eq. (S.5), whatever it happens to evaluate to in a more accurate calculation than we did. For const $=1$, the Z_- agrees with the partition function of a two-state quantum system generated by the single-mode fermionic creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.

The bottom line is, *to get the correct partition function at a finite temperature, the fermionic fields should be anti-periodic in the Euclidean time*,

$$\psi(t_e+\beta)\;=\;-\psi(t_e).\tag{S.7}$$

Likewise, in the Euclidean spacetime of $D>1$ dimensions,

$$\Psi(\mathbf{x},t_e+\beta)\;=\;-\Psi(\mathbf{x},t_e).\tag{3}$$

Problem $\mathbf{1}$(b):

In light of eq. (3), the partition function of a free Dirac field in $D = 3 + 1$ dimensions obtains from the Euclidean functional integral

$$Z = \iint \mathcal{D}[\text{antiperiodic } \Psi(x) \text{ and } \overline{\Psi}(x)] \, \exp(-S_E[\Psi, \overline{\Psi}]). \tag{S.8}$$

where

$$S_E = \int d^4x \, \overline{\Psi}(\partial\!\!\!/_e + m)\Psi \tag{S.9}$$

(for $\gamma_E^4 = \gamma_M^0$ and $\vec{\gamma}_E = -i\vec{\gamma}_M$). The fermionic functional integral (S.8) has a Gaussian form, so it formally evaluates to

$$Z = \operatorname{Det}[\partial\!\!\!/_e + m] \tag{S.10}$$

where Det is a functional determinant in the Hilbert space of 4-component wave functions $\psi_\alpha(x_1, x_2, x_3, x_4)$ that are anti-periodic in x_4, $\psi_\alpha(\mathbf{x}, x_4 + \beta) = -\psi_\alpha(\mathbf{x}, x_4)$. Consequently, the free energy of the Dirac field at a finite temperature $\mathcal{T} = 1/\beta$ is

$$F(\mathcal{T}) \stackrel{\text{def}}{=} -\mathcal{T} \log Z(\mathcal{T}) = -\mathcal{T} \log \operatorname{Det}[\partial\!\!\!/_e + m] = -\mathcal{T} \operatorname{Tr}[\log(\partial\!\!\!/_e + m)] \tag{S.11}$$

where Tr is the functional trace in the same Hilbert space as Det. This verifies eq. (4) for the free energy of the Dirac spinor field.

To evaluate the functional trace in eq. (S.11), we go to the Euclidean momentum basis of states $|\mathbf{p}, p_4, \alpha\rangle$, where α is the Dirac index, $\mathbf{p}$ is the 3D momentum, and p_4 is the momentum in the Euclidean time direction. In light of the antiperiodic 'boundary' condition in the Euclidean time direction, the p_4 has a discrete spectrum comprising

$$p_4 = \frac{2\pi}{\beta} \times \text{half-integers} = 2\pi\mathcal{T} \times \left(\pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots\right). \tag{7}$$

As to the 3-momentum $\mathbf{p}$, its spectrum is continuous in infinite space, but when we put the fields in a large but finite box of volume L^3, the spectrum of $\mathbf{p}$ becomes discrete but almostcontinuous, with density

$$d\#\mathbf{p} \;=\; \frac{L^3}{(2\pi)^3}\,d^3\mathbf{p}. \tag{S.12}$$

In the Euclidean momentum basis

$$\big\langle \mathbf{p}', p_4', \beta \big| \log(\partial\!\!\!/_e + m) \left|\mathbf{p}, p_4, \alpha\right\rangle \;=\; \delta_{\mathbf{p}',\mathbf{p}} \times \delta_{p_4',p_4} \times \big(\log(i\,p\!\!\!/_e + m)\big)_{\alpha,\beta}\,, \tag{S.13}$$

so the functional trace in eq. (S.11) becomes

$$\begin{aligned}
\mathrm{Tr}[\log(\partial\!\!\!/_E + m)] \;&=\; \sum_{\mathbf{p}}\sum_{p_4}\sum_{\alpha} \langle \mathbf{p}, p_4, \alpha | \log(\partial\!\!\!/_e + m) \left|\mathbf{p}, p_4, \alpha\right\rangle \\
&=\; \sum_{\mathbf{p}}\sum_{p_4}\sum_{\alpha} \big(\log(i\,p\!\!\!/_e + m)\big)_{\alpha,\alpha} \\
&=\; L^3 \int\!\frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{p_4} \mathrm{tr}_{\mathrm{Dirac}}\big(\log(i\,p\!\!\!/_e + m)\big).
\end{aligned} \tag{S.14}$$

In terms of this trace, the free energy *density* of the Dirac spinor field is

$$\begin{aligned}
\mathcal{F}(T) \;\stackrel{\mathrm{def}}{=}\; \frac{F(T)}{L^3} \;&=\; -\frac{\mathcal{T}}{L^3}\,\mathrm{Tr}[\log(\partial\!\!\!/_e + m)] \qquad \langle\!\langle\, \textit{cf.}\text{ eq. (S.11)}\,\rangle\!\rangle \\
&=\; -\mathcal{T}\int\!\frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{p_4} \mathrm{tr}_{\mathrm{Dirac}}\big(\log(i\,p\!\!\!/_e + m)\big),
\end{aligned} \tag{S.15}$$

exactly as in eq. (5)

Finally, we evaluate the trace over the Dirac indices in eq. (5) using

$$\mathrm{tr}\big(\log(i\,p\!\!\!/_e + m)\big) \;=\; \log\big(\det(i\,p\!\!\!/_e + m)\big) \tag{S.16}$$

where the determinant obtains from the Dirac matrix identity

$$\gamma^5(i\,p\!\!\!/_e + m)\gamma^5(i\,p\!\!\!/_e + m) \;=\; (-i\,p\!\!\!/_e + m)(i\,p\!\!\!/_e + m) \;=\; (p_e^2 + m^2) \times \mathbf{1}_{4\times 4}\,. \tag{S.17}$$

Thanks to this identity,

$$\det\big(\gamma^5(i\,p\!\!\!/_e + m)\gamma^5(i\,p\!\!\!/_e + m)\big) \;=\; (p_e^2 + m^2)^4, \tag{S.18}$$

but on the other hand

$$\begin{aligned}\det\bigl(\gamma^5(i\not{p}_e+m)\gamma^5(i\not{p}_e+m)\bigr)\ &=\ \det{}^2(i\not{p}_e+m)\times\det{}^2(\gamma^5)\\ \langle\!\langle\,\text{since }\det(\gamma^5)=1\,\rangle\!\rangle\ &=\ \det{}^2(i\not{p}_e+m),\end{aligned}\tag{S.19}$$

which together give us

$$\det{}^2(i\not{p}_e+m)\ =\ (p_e^2+m^2)^4\tag{S.20}$$

and consequently,

$$\operatorname{tr}\bigl(\log(i\not{p}_e+m)\bigr)\ =\ \log\bigl(\det(i\not{p}_e+m)\bigr)\ =\ \frac{4}{2}\times\log(p_e^2+m^2).\tag{S.21}$$

Plugging this formula into eq. (5) for the energy of the Dirac spinor field, we arrive at

$$\mathcal{F}(\mathcal{T})\ =\ -\mathcal{T}\int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{p_4}\frac{4}{2}\log(p_e^2+m^2),\tag{S.22}$$

exactly as in eq. (6).

Problem **1**(c):

In light of the spectrum (7) of the discrete p_4 momentum in eq. (6), we may rewrite the latter as

$$\mathcal{F}(\mathcal{T})\ =\ -\!\int\!\frac{d^4p_e}{(2\pi)^4}\,2\log(p_e^2+m^2)\times\sum_n^{\text{half-integer}}\delta\left(\frac{p_4}{2\pi\beta}-n\right),\tag{S.23}$$

and then apply the Poisson resummation to the sum over δ-functions here. In the preamble of the previous homework#20, we saw that

$$\sum_n^{\text{integer}}\delta(x-n)\ =\ \sum_{\ell=-\infty}^{+\infty}e^{2\pi i\ell x}.\tag{S.24}$$

For the sum over half-integer n, we have a similar formula

$$\sum_n^{\text{half-integer}}\delta(x-n)\ =\ \sum_{\ell=-\infty}^{+\infty}(-1)^\ell\times e^{2\pi i\ell x},\tag{S.25}$$

which follows from plugging (a half-integer n) $= \frac{1}{2} +$ (an integer m) into eq. (S.24):

$$\sum_{n}^{\text{half-integer}} \delta(x-n) \;=\; \sum_{m}^{\text{integer}} \delta(x-m-\tfrac{1}{2}) \\ =\; \sum_{\ell=-\infty}^{+\infty} e^{2\pi i\ell(x-\frac{1}{2})} \\ =\; \sum_{\ell=-\infty}^{+\infty} (-1)^\ell \times e^{2\pi i\ell x}. \tag{S.26}$$

Applying the half-integer Poisson resummation formula (S.25) to the Dirac fermion's free energy (S.23), we arrive at

$$\mathcal{F}(\mathcal{T}) \;=\; -\!\int\!\frac{d^4p_e}{(2\pi)^4}\,2\log(p_e^2+m^2)\times\sum_{\ell=-\infty}^{+\infty}(-1)^\ell\times e^{i\ell\beta p_4} \\ =\; -2\sum_{\ell=-\infty}^{+\infty}(-1)^\ell\int\!\frac{d^4p_e}{(2\pi)^4}\,\log(p_e^2+m^2)\times e^{i\ell\beta p_4}. \tag{S.27}$$

Similar to the bosonic case we have dealt with in the previous homework#20, in the zero-temperature limit $\beta\to+\infty$, the sum on the bottom line of eq. (S.27) is dominated by the $\ell=0$ term. Indeed, for any other $\ell\neq 0$, the integral is suppressed by the rapidly oscillating factor $\exp(i\beta\ell p_4)$, hence

$$\mathcal{F}(0) \;=\; -2\int\!\frac{d^4p_e}{(2\pi)^4}\,\log(p_e^2+m^2). \tag{S.28}$$

Note that the integral here is exactly as in the $\ell=0$ term in eq. (S.27) for any temperature, so subtraction of the zero-temperature free energy density (S.28) from the finite-temperature free energy density (S.27) amounts to simply skipping the $\ell=0$ term in the sum, thus

$$\mathcal{F}(\mathcal{T}) \;-\; \mathcal{F}(0) \;=\; -2\sum_{\ell\neq 0}(-1)^\ell\int\!\frac{d^4p_e}{(2\pi)^4}\,\log(p_e^2+m^2)\times e^{i\ell\beta p_4}. \tag{S.29}$$

Moreover, in the remaining sum the terms for $+\ell$ and $-\ell$ have exactly the same integrals — they are related by the variable change $p_4\to -p_4$, — so we may sum over just the positive ℓ

and double-count their contributions. Thus,

$$\begin{aligned}\mathcal{F}(\mathcal{T})\,-\,\mathcal{F}(0) \;&=\; -2\times 2\sum_{\ell=1}^{\infty}(-1)^{\ell}\int\!\frac{d^4p_e}{(2\pi)^4}\,\log(p_e^2+m^2)\times e^{i\ell\beta p_4}\\ &=\; 4\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\times\int\!\frac{d^4p_e}{(2\pi)^4}\,\log(p_e^2+m^2)\times e^{i\ell\beta p_4},\end{aligned}\tag{S.30}$$

exactly as in eq. (8).

To partially evaluate the sum and the integral in eq. (8), let's reorganize it as

$$\mathcal{F}(\mathcal{T})\,-\,\mathcal{F}(0) \;=\; 4\!\int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\times\int\!\frac{dp_4}{2\pi}\,\log(m^2+\mathbf{p}^2+p_4^2)\times e^{i\ell\beta p_4}.\tag{S.31}$$

The p_4 integral here was evaluated in the previous homework#20:

$$\int\!\frac{dp_4}{2\pi}\,\log(E^2+p_4^2)\times e^{i\ell\beta p_4} \;=\; -\frac{\exp(-\ell\beta E)}{\ell\beta}\tag{HW20.17}$$

for $E=\sqrt{m^2+\mathbf{p}^2}$. Consequently, the sum over ℓ becomes a Taylor series expansion of the logarithm,

$$\log(1+x) \;=\; \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\times\frac{x^\ell}{\ell}\,.\tag{S.32}$$

Indeed, plugging in the integral (HW20.17) into the sum in eq. (S.31) gives us

$$\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\times\frac{-\exp(-\ell\beta E)}{\ell\beta} \;=\; -\mathcal{T}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\times\frac{[\exp(-\beta E)]^\ell}{\ell} \;=\; -\mathcal{T}\log\bigl(1+\exp(-\beta E)\bigr),\tag{S.33}$$

and hence

$$\mathcal{F}(\mathcal{T})\,-\,\mathcal{F}(0) \;=\; -4\mathcal{T}\int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\log\bigl(1+\exp(-\beta E_{\mathbf{p}})\bigr),.\tag{9}$$

Now let's compare eq. (9) to the conventional Fermi–Dirac statistics of the quantum fermionic field. From the grand canonical ensemble point of view, the Dirac field comprises an infinite family of independent fermionic modes — 4 modes for each $\mathbf{p}$, from 2 spin states for the particle plus 2 more for the antiparticle. For each mode of frequency $E = E_{\mathbf{p}}$, we have a 2-state system with free energy

$$F_{\text{mode}}(\mathcal{T}) \;=\; -\mathcal{T}\log\bigl(Z = e^{+\beta E/2} + e^{-\beta E/2}\bigr) \;=\; -\tfrac{1}{2}E \;-\; \mathcal{T}\log\bigl(1+e^{-\beta E}\bigr), \tag{S.34}$$

where $-\frac{1}{2}E$ can be identifies as the zero-point energy of the fermionic mode, same as the free energy at zero temperature. Thus, for each mode of energy E

$$F_{\text{mode}}(\mathcal{T}) \;-\; F_{\text{mode}}(0) \;=\; -\mathcal{T}\log\bigl(1+e^{-\beta E}\bigr), \tag{S.35}$$

and hence for the whole Dirac field

$$F(\mathcal{T}) \;-\; F(0) \;=\; 4\int\frac{L^3\,d^3\mathbf{p}}{(2\pi)^3}\,(-\mathcal{T})\log\bigl(1+e^{-\beta E}\bigr), \tag{S.36}$$

or in terms of the free energy *density*

$$\mathcal{F}(\mathcal{T}) \;-\; \mathcal{F}(0) \;=\; \frac{F(\mathcal{T})-F(0)}{L^3} \;=\; -4\mathcal{T}\int\frac{L^3\,d^3\mathbf{p}}{(2\pi)^3}\,\log\bigl(1+e^{-\beta E}\bigr), \tag{S.37}$$

exactly as in eq. (9).

Problem **1**(d–e):

Similarly to other bosonic fields, at finite temperature $\mathcal{T} = 1/\beta$, the EM field $A^\mu(x_e)$ becomes periodic in the Euclidean time direction,

$$A^\mu(\mathbf{x}, x_4 = 0) \;=\; A^\mu(\mathbf{x}, x_4 = \beta), \qquad \mu = 1,2,3,4. \tag{S.38}$$

Its *local* properties however remain exactly the same; in particular, we still have local gauge transformations

$$A'^\mu(x_e) \;=\; A^\mu(x_e) \;-\; \partial^\mu\Lambda(x_e) \tag{S.39}$$

albeit subject to the periodicity condition

$$\partial^\mu\Lambda(\mathbf{x}, x_4 = 0) \;=\; \partial^\mu\Lambda(\mathbf{x}, x_4 = \beta). \tag{S.40}$$

Consequently, the proper construction of the Euclidean Functional integral over the EM field configurations requires the same Fadde'ev–Popov gauge-fixing procedure as for $\mathcal{T} = 0$ with suitable modifications to reflect the fields' periodicities. Thus, the EM Partition Function is

$$Z_{\text{EM}} = C \iint\limits_{\text{periodic}} \mathcal{D}[A^\mu(x_E)]\,\Delta_{\text{FP}} e^{-S_E[A^\mu(x_E)]} \tag{S.41}$$

where the Euclidean action

$$S_E[A^\mu(x_E)] = \int d^3\mathbf{x} \int_0^\beta dx_4 \left\{ \tfrac{1}{4} F_{\mu\nu}^2 + \tfrac{1}{2\xi} (\partial A)^2 \right\} \tag{S.42}$$

includes the gauge-fixing term. The Δ_{FP} in eq. (S.41) is the Fadde'ev–Popov functional determinant

$$\Delta_{\text{FP}} = \text{Det}[-\partial_e^2]_{\text{periodic}}; \tag{S.43}$$

the determinant is over periodic functions of x_4 because the gauge transforms (S.40) are periodic at finite temperature. Finally, the normalization factor

$$C = \left[\iint \mathcal{D}[\omega(x_e)]\, e^{-\frac{1}{2\xi}\int \omega^2 d^4 x_e} \right]^{-1} \tag{S.44}$$

compensating for the averaging over the gauge conditions $\partial_\mu A^\mu = \omega$ should also involve properly periodic $\omega(x_E)$.

For the free EM field, the Euclidean action functional (S.42) is quadratic and the functional integral (S.41) is purely Gaussian, but keeping in mind the Fadde'ev–Popov determinant factor Δ_{FP}, we have

$$Z_{\text{EM}} = C\,\text{Det}(-\partial^2) \times \left[\text{Det}\left(-\partial^2 \delta^{\mu\nu} + (1-\xi^{-1})\partial^\mu \partial^\nu \right) \right]^{-1/2} \tag{S.45}$$

where all the determinants are over periodic fields. Hence, in the momentum basis

$$\log Z_{\text{EM}} = \int \frac{L^3\, d^3\mathbf{k}}{(2\pi)^3} \sum_{k_4} \left\{ \tfrac{1}{2}\log(\xi^{-1}) + \log(k_E^2) - \tfrac{1}{2}\log\det\left(k_E^2 \delta^{\mu\nu} - (1-\xi^{-1}) k_E^\mu k_E^\nu \right) \right\} \tag{S.46}$$

where the last determinant acts on the Euclidean indices μ, ν only.

The 4×4 matrix $(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu)$ has three eigenvalues equal to k_E^2 (transverse eigenvectors) and one eigenvalue equal to k_E^2/ξ (eigenvector parallel to the k_E). Consequently

$$\det\left(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu\right) \;=\; \frac{(k_E^2)^4}{\xi}\,, \tag{S.47}$$

and therefore

$$\tfrac{1}{2}\log(\xi^{-1}) \;+\; \log(k_E^2) \;-\; \tfrac{1}{2}\log\det\left(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu\right) \;=\; -\log(k_E^2). \tag{S.48}$$

Plugging this formula into eq. (S.46) finally gives us the electromagnetic partition function,

$$\log Z_{\mathrm{EM}} \;=\int\!\frac{L^3\,d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{-1\times\log(k_e^2)\right\}. \tag{S.49}$$

By comparison, a (real) scalar field has

$$\log Z_\phi \;=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{-\tfrac{1}{2}\times\log(k_e^2+m^2)\right\}, \tag{S.50}$$

which means the EM field has the partition function of *two species* of a massless scalar — or equivalently, two physical polarizations states of one massless boson — the photon. *Quod erat demonstrandum.*

Problem **2**(a):

Let us evaluate the trace of the Casimir operator C_2 over an irreducible multiplet (r). On one hand,

$$\begin{aligned}\mathrm{tr}_{(r)}\left(C_2 \stackrel{\mathrm{def}}{=} \sum_a T^aT^a\right) &= \sum_a \mathrm{tr}_{(r)}\left(T^aT^a\right) \;=\; \sum_a \mathrm{tr}\left(T^a_{(r)}T^b_{(r)}\right)\\ \langle\!\langle\text{by eq. (12)}\rangle\!\rangle\quad &= \sum_a R(r)\times(\delta^{aa}=1) \;=\; R(r)\times\dim(G)\end{aligned} \tag{S.51}$$

where $\dim(G) \stackrel{\mathrm{def}}{=} \dim(\mathrm{Adj}(G))$ is the number of the generators of the symmetry group G — which is also the dimension of the adjoint representation of G, hence the notation. On the other hand,

$$\operatorname{tr}_{(r)}(C_2) \;=\; \operatorname{tr}_{(r)}\Big(C_2\big|_{(r)}\Big) \;=\; \operatorname{tr}\big(C(r)\times\mathbf{1}_{(r)}\big) \;=\; C(r)\times\dim(r). \tag{S.52}$$

Together, eqs. (S.51) and (S.52) immediately imply eq. (14). *Quod erat demonstrandum.*

For the special case of $G=SU(2)$, an irreducible multiplets of isospin I has $C=\mathbf{I}^2=I(I+1)$ and dimension $2I+1$, hence

$$R(I) \;=\; C(I)\times\frac{\dim(I)}{\dim(G)} \;=\; I(I+1)\times\frac{2I+1}{3}\,, \tag{S.53}$$

exactly as in eq. (15).

<u>Problem **2**(b)</u>:

Unlike the Casimir value $C(r)$, the index $R(r)$ is well defined for any complete multiplet (r), irreducible or otherwise. For a reducible multiplet

$$(r) \;=\; \bigoplus_{i=1}^{n}(r_i) \;\equiv\; (r_1)\oplus(r_2)\oplus\cdots\oplus(r_n)$$

one has

$$\begin{aligned}\operatorname{tr}_{(r)}\Big(T^aT^b\Big) \;&=\; \operatorname{tr}\left(T^aT^b\Big|_{\bigoplus_{i=1}^n(r_i)}\right) \;=\sum_{i=1}^{n}\operatorname{tr}\left(T^aT^b\Big|_{(r_i)}\right)\\ &=\sum_{i=1}^{n}\Big(R(r_i)\times\delta^{ab}\Big) \;=\; \delta^{ab}\times\sum_{i=1}^{n}R(r_i)\end{aligned} \tag{S.54}$$

and thus

$$R(r) \;=\sum_{i=1}^{n}R(r_i). \tag{S.55}$$

In particular, a reducible multiplet

$$(r) \;=\; \bigoplus_{i=1}^{n}(I_i)$$

of the isospin group $SU(2)$ has index

$$R(r) \;=\sum_{i=1}^{n}\tfrac{1}{3}I(I+1)(2I+1). \tag{S.56}$$

Now consider a bigger symmetry group G which contains the 'isospin' $SU(2)$ as a subgroup. Then any complete multiplet (r) of G is automatically a complete multiplet of the $SU(2)\subset G$. However, irreducible multiplets of G usually become reducible from the $SU(2)$ point of view, $(r)=(I_1)\oplus(I_2)\oplus\cdots\oplus(I_n)$; for example, the adjoint multiplet of $SU(3)$ decomposes into $(0)\oplus(\frac{1}{2})\oplus(\frac{1}{2})\oplus(1)$ of the $SU(2)\subset SU(3)$. Let T^1, T^2, and T^3 be generators of the $SU(2)$ subgroup of G. Then according to eq. (S.56),

$$\text{for } a,b=1,2,3,\quad \operatorname{tr}_{(r)}\left(T^aT^b\right) \;=\; \delta^{ab}\times\sum_{i=1}^{n}\tfrac{1}{3}I(I+1)(2I+1). \tag{S.57}$$

Now, let us suppose that the Lie group G is *simple*, that is, all its generators are related to each other by the symmetry G itself. In this case, for any complete multiplet (r) of G

$$\operatorname{tr}_{(r)}\left(T^aT^b\right) \;=\; R(r)\times\delta^{ab},\quad \text{same } R(r)\ \forall a,b=1,\ldots,\dim(G). \tag{S.58}$$

Combining this formula with eq. (S.57) we immediately obtain

$$R(r) \;=\sum_{i=1}^{n}\tfrac{1}{3}I(I+1)(2I+1), \tag{17}$$

Quod erat demonstrandum..

<u>*Caveat:*</u> We have silently assumed that $T^{1,2,3}$ have the same normalization as generators of G as they have as generators of the $SU(2)\subset G$. This assumption is correct for the $SU(2)\subset SU(N)$ discussed in parts (c) and (d) of this problem, but it would fail for a different (*i.e.*, in-equivalent) $SU(2)$ subgroup. In general, properly normalized $SU(2)\subset G$ generators $I^{1,2,3}$ are related to the properly normalized generators of G as

$$I^a \;=\; T^{(a)}\times\sqrt{k} \tag{S.59}$$

where $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ are three generators of G which happen to satisfy $[T^{(a)},T^{(b)}]=i\epsilon^{abc}T^{(c)}/\sqrt{k}$. The k here is always a positive integer; it's called *the level of embedding of* *the* $SU(2)$ *into* G. For example, consider the $SU(2)$ subgroup of $SU(3)$ which acts on the fundamental triplet as real $SO(3)$ rotations. This subgroup is generated by the

$$I^1 = \sqrt{4} \times T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{pmatrix}, \quad I^2 = \sqrt{4} \times T^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}, \quad I^3 = \sqrt{4} \times T^2 = \begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{S.60}$$

(note $T^a = \frac{1}{2}\lambda^a$), so its embedding level is $k = 4$.

When you decompose a multiplet (r) of G into irreducible multiplets of an $SU(2)$ subgroup, you should take into account the level at which this $SU(2)$ is embedded into G. As written, eq. (17) works only for the $k = 1$ subgroups; for other embedding levels,

$$R(r) \;=\; \frac{1}{k}\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.61}$$

Note that the decomposition of the G multiplet (r) into $SU(2)$ multiplets depends on the $SU(2)$ embedding into G. For example, under the $k = 1$ subgroup $SU(2) \subset SU(3)$

$$\text{triplet} \;=\; (\tfrac{1}{2}) \oplus (0), \qquad \text{octet} \;=\; (1) \oplus (\tfrac{1}{2}) \oplus (\tfrac{1}{2}) \oplus (0), \tag{S.62}$$

while under the $k = 4$ subgroup (S.60)

$$\text{triplet} \;=\; (1), \qquad \text{octet} \;=\; (1) \oplus (2). \tag{S.63}$$

In both cases, eq. (S.61) produces the same index R for each $SU(3)$ multiplet, for example $R(\text{triplet}) = \frac{1}{2}$ and $R(\text{octet}) = 3$, but only if you remember the $1/k$ factor in front of the sum.

Problem **2**(c):

From the $SU(2) \subset SU(N)$ point of view, the fundamental representation $\mathbf{N}$ of the $SU(N)$

decomposes into one doublet plus $(N-2)$ singlets,

$$\mathbf{N} \;=\; \mathbf{2} \;+\; (N-2)\times\mathbf{1} \;\equiv\; (I=\tfrac{1}{2}) \;+\; (N-2)\times(I=0), \tag{S.64}$$

hence according to eq. (17),

$$R(\mathbf{N}) \;=\; R(I=\tfrac{1}{2}) \;+\; (N-2)\times R(I=0) \;=\; \tfrac{1}{2} \;+\; (N-2)\times 0 \;=\; \tfrac{1}{2} \tag{18.a}$$

and consequently

$$C(\mathbf{N}) \;=\; R(\mathbf{N})\times\frac{\dim(G)}{\dim(\mathbf{N})} \;=\; \frac{1}{2}\times\frac{N^2-1}{N} \;=\; \frac{N^2-1}{2N} \tag{18.b}$$

Now consider the adjoint representation of the $SU(N)$. Let us form a tensor product of the fundamental representation $\mathbf{N}$ and the conjugate (anti-fundamental) representation $\overline{\mathbf{N}}$. Given the transformation laws

$$\begin{aligned} \Psi &\to U\Psi, \qquad i.e.\quad \Psi'_j \;=\; U_j^{\,k}\Psi_k\,,\\ \overline{\Psi} &\to \overline{\Psi}U^\dagger, \qquad i.e.\quad \overline{\Psi}'^{\ell} \;=\; \overline{\Psi}^m\,U^{*\ell}_{\ m}\,, \end{aligned}$$

it follows that the tensor product is a hermitian $N\times N$ matrix $\Phi_j^{\,k}$ which transforms as

$$\Phi' \;=\; U\Phi U^\dagger \qquad i.e.\quad \Phi'^{\,\ell}_j \;=\; U_j^{\,k}\Phi_k^{\,m}U^{*\ell}_{\ m}\,. \tag{S.65}$$

This matrix is a reducible multiplet $\mathrm{Adj}+\mathbf{1}$ of the $SU(N)$: The trace $\mathrm{tr}(\Phi)$ is an invariant singlet, while the traceless part $\Phi_i^{\,j}-\delta_i^j\times\mathrm{tr}(\Phi)/N$ forms the adjoint multiplet, *cf.* homework#6 from the Fall semester. In other words,

$$\mathbf{N}\otimes\overline{\mathbf{N}} \;=\; \mathrm{Adj}\oplus\mathbf{1} \tag{S.66}$$

In $SU(2)$ $\bar{\mathbf{2}}=\mathbf{2}$, so from the $SU(2)\subset SU(N)$ point of view, both the fundamental and the anti-fundamental multiplets of the $SU(N)$ decompose into similar sets of one doublet and

$N-2$ singlets. Therefore,

$$\begin{aligned}
[\text{Adj} \ + \ \mathbf{1}]_{SU(N)} \ &= \ \left[\mathbf{N}\otimes\overline{\mathbf{N}}\right]_{SU(N)} \\
&= \ [\mathbf{2} \ + \ (N-2)\times\mathbf{1}]_{SU(2)} \ \otimes \ [\mathbf{2} \ + \ (N-2)\times\mathbf{1}]_{SU(2)} \\
&= \ \left[(\mathbf{2}\otimes\mathbf{2}) \ + \ 2(N-2)\times(\mathbf{2}\otimes\mathbf{1}) \ + \ (N-2)^2\times(\mathbf{1}\otimes\mathbf{1})\right]_{SU(2)} \qquad \text{(S.67)} \\
&= \ \left[\mathbf{3} \ + \ \mathbf{1} \ + \ 2(N-2)\times\mathbf{2} \ + \ (N-2)^2\times\mathbf{1}\right]_{SU(2)} \ , \\
\textit{i.e.}, \quad [\text{Adj}]_{SU(N)} \ &= \ \left[\mathbf{3} \ + \ 2(N-2)\times\mathbf{2} \ + \ (N-2)^2\times\mathbf{1}\right]_{SU(2)} \ ,
\end{aligned}$$

and consequently

$$\begin{aligned}
R(\text{Adj}) \ &= \ R_{SU(2)}(\mathbf{3}) \ + \ 2(N-2)\times R_{SU(2)}(\mathbf{2}) \ + \ (N-2)^2\times R_{SU(2)}(\mathbf{1}) \\
&= \ 2 \ + \ 2(N-2)\times\tfrac{1}{2} \ + \ (N-2)^2\times 0 \ = \ N.
\end{aligned} \tag{19.a}$$

Finally,

$$C(G) \ \stackrel{\text{def}}{=} \ C(\text{Adj}(G)) \ = \ R(\text{Adj})\times\frac{\dim(G)}{\dim(G)} \ = \ R(\text{Adj}) \ = N. \tag{19.b}$$

Problem **2**(d):

Consider the two-index symmetric tensor $S_{(ij)}$ representation of the $SU(N)$ symmetry group. Denote the index $i=\alpha$ if $i=1,2$ or $i=\mu$ if $i=3,4,\ldots,N$ and likewise $j=\beta$ if $j=1,2$ and $j=\nu$ if $j=3,4,\ldots,N$. Thus, the complete set of independent $S_{(ij)}$ decomposes into $S_{(\alpha\beta)}$, $S_{\alpha,\mu}\equiv S_{\mu,\alpha}$ and $S_{(\mu\nu)}$. The $SU(2)\subset SU(N)$ acts on indices $\alpha,\beta=1,2$ and ignores indices $\mu,\nu=3,4,\ldots,N$, so from the $SU(2)$ point of view, $S_{(\alpha\beta)}$ is a triplet, $S_{\alpha,\mu}$ are $N-2$ separate doublets, and $S_{(\mu\nu)}$ are $(N-2)(N-1)/2$ singlets. Consequently,

$$\begin{aligned}
R(S) \ &= \ R_{SU(2)}(\mathbf{3}) \ + \ (N-2)\times R_{SU(2)}(\mathbf{2}) \ + \ \tfrac{1}{2}(N-1)(N-2)\times R_{SU(2)}(\mathbf{1}) \\
&= \ 2 \ + \ (N-2)\times\tfrac{1}{2} \ + \ \tfrac{1}{2}(N-1)(N-2)\times 0 \ = \ \tfrac{1}{2}(N+2),
\end{aligned} \tag{S.68}$$

and hence

$$C(S) = \ R(S)\times\frac{\dim(G)}{\dim(S)} \ = \ \frac{N+2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N+1)} \ = \ \frac{N^2+N-2}{N}\,. \tag{S.69}$$

Similarly, the two-index anti-symmetric tensor $A_{[ij]}$ decomposes into $A_{[\alpha\beta]}$, $A_{\alpha,\mu}$, and $A_{[\mu\nu]}$. In $SU(2)$, the $A_{[\alpha\beta]}$ is equivalent to the trivial singlet $A\times\epsilon_{[\alpha\beta]}$, the $A_{\alpha,\mu}$ are $N-2$ doublets, and $A_{[\mu\nu]}$ are $(N-2)(N-3)/2$ singlets. Altogether

$$(A) \;=\; (N-2)\times\mathbf{2} \;+\; \text{singlets},$$

therefore

$$R(A) \;=\; (N-2)\times\tfrac{1}{2} \;+\; 0 \;=\; \tfrac{1}{2}(N-2) \tag{S.70}$$

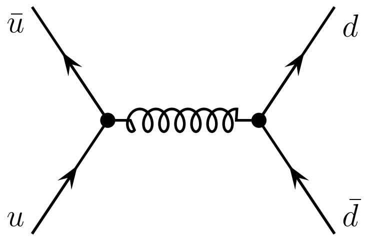
and

$$C(A) = \; R(A)\times\frac{\dim(G)}{\dim(A)} \;=\; \frac{N-2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N-1)} \;=\; \frac{N^2-N-2}{N}\,. \tag{S.71}$$

Problem **3**:

At the tree level of QCD,

$$i\mathcal{M}(u\bar{u}\to d\bar{d}) \;=$$



$$= \frac{ig^2}{s}\times\bar{v}(\bar{u})\gamma^\mu u(u)\,(T^a)^i{}_j\times\bar{u}(d)\gamma_\mu v(d)\,(T^a)^k{}_\ell \tag{S.72}$$

where $s = E^2_{\text{c.m.}}$, the quarks and the antiquarks have color indices i,j,k,ℓ, the virtual gluon has adjoint color index a, and the summation over a is implicit. Except for the color indices, the $u\bar{u}\to d\bar{d}$ process in QCD is completely analogous to the $e^-e^+\to\mu^-\mu^+$ pair production in QED, *cf.* my notes from the Fall semester. In particular, summing / averaging $|\mathcal{M}|^2$ over the fermion's spins yields

$$\begin{aligned}\tfrac{1}{4}\sum_{\text{all spins}}\left|\bar{v}(\bar{u})\gamma^\mu u(u)\,\bar{u}(d)\gamma_\mu v(d)\right|^2 \;&\approx\; \tfrac{1}{4}\,\mathrm{tr}(\not{p}_{\bar{u}}\gamma^\mu\,\not{p}_u\gamma^\nu)\times\mathrm{tr}(\not{p}_{\bar{d}}\gamma_\mu\,\not{p}_d\gamma_\nu)\\ &=\; 2(t^2+u^2) \;=\; s^2(1+\cos^2\theta_{\text{c.m.}})\end{aligned} \tag{S.73}$$

where the approximation is neglecting the quark masses m_u and m_d.

The new part of this exercise is summing / averaging over the color indices. By hermiticity of the Lie Algebra matrices T^a, we have

$$\Bigl((T^a)^i{}_j\,(T^a)^k{}_\ell\Bigr)^* \;=\; (T^a)^j{}_i\,(T^a)^\ell{}_k \;=\; (T^b)^j{}_i\,(T^b)^\ell{}_k \tag{S.74}$$

— note the implicit summation over a or b — and hence

$$\begin{aligned}\sum_{i,j,k,\ell}\Bigl|(T^a)^i{}_j\,(T^a)^k{}_\ell\Bigr|^2 \;&=\; \sum_{i,j,k,\ell}(T^a)^i{}_j\,(T^a)^k{}_\ell\times(T^b)^j{}_i\,(T^b)^\ell{}_k\\ &= \sum_{ij}(T^a)^i{}_j\,(T^b)^j{}_i\times\sum_{k,\ell}(T^a)^k{}_\ell\,(T^b)^\ell{}_k\\ &=\; \mathrm{tr}(T^aT^b)\times\mathrm{tr}(T^aT^b)\end{aligned} \tag{S.75}$$

For the moment, let us consider ‘quarks’ belonging to some generic multiplet (r) of some generic gauge group G. In such a generic case, $\mathrm{tr}(T^aT^b)=R(r)\times\delta^{ab}$ where $R(r)$ is the index of the quark multiplet (*cf.* problem 2) and therefore

$$\sum_{a,b}\mathrm{tr}(T^aT^b)\times\mathrm{tr}(T^aT^b) \;=\; R^2(r)\times\sum_{a,b}\delta^{ab}\delta^{ab} \;=\; R^2(r)\times\dim(G). \tag{S.76}$$

Thus,

$$\sum_{i,j,k,\ell}\Bigl|\sum_a(T^a)^i{}_j\,(T^a)^k{}_\ell\Bigr|^2 \;=\; R^2(r)\times\dim(G),$$

or, for the average over the initial ‘colors’ i and j,

$$\frac{1}{\dim^2(r)}\sum_{i,j}\sum_{k,\ell}\Bigl|\sum_a(T^a)^i{}_j\,(T^a)^k{}_\ell\Bigr|^2 \;=\; \frac{R^2(r)\,\dim(G)}{\dim^2(r)} \;=\; \frac{C^2(r)}{\dim(G)}\,. \tag{S.77}$$

Specializing to the ‘quarks’ in the fundamental representation of an $SU(N)$ gauge group, we have $R(r)=\frac{1}{2}$, $\dim(r)=N$ and $\dim(G)=N^2-1$, hence eq. (S.77) evaluates to $(N^2-1)/(4N^2)$; for the actual QCD $N=3$ and the color sum / average (S.77) gives 2/9.

Altogether, $|\mathcal{M}|^2$ summed / averaged over both spins and colors of all the fermions is

$$\frac{2}{9} \times g^4 (1 + \cos^2 \theta_{\text{c.m.}}) \tag{S.78}$$

and hence the cross section

$$\frac{d\sigma(u\bar{u} \to d\bar{d})}{d\Omega_{\text{cm}}} = \frac{2}{9}\, \alpha_{\text{QCD}}^2 \times \frac{1 + \cos^2 \theta_{\text{cm}}}{4s}\,. \tag{S.79}$$