Problem **1**(a):

Classically,

$$\mathcal{L} \;=\; \mathcal{L}_{\rm YM} \;+\; D^\mu\Phi^\dagger D_\mu\Phi \;-\; V(\Phi^\dagger,\Phi) \tag{S.1}$$

where

$$D_\mu\Phi^i \;=\; \partial_\mu\Phi_i \;+\; igA^a_\mu(T^a_{(r)})^i{}_j\Phi^j\,, \qquad D^\mu\Phi^*_i \;=\; \partial^\mu\Phi^*_i \;-\; igA^{a\mu}\Phi^*_j(T^a_{(r)})^j{}_i, \tag{S.2}$$

and $V(\Phi^\dagger,\Phi)$ is some kind of a G–invariant potential. For renormalizability's sake, V should be a polynomial of degree 4 (or less), and to keep my notations simple I assume that V has only the quadratic mass term and the quartic interaction term, thus

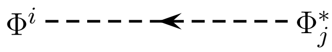
$$V \;=\; m^2\times\Phi^*_i\Phi^i \;+\; \tfrac14\lambda^{ij}_{k\ell}\times\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell \tag{S.3}$$

for a suitable G-invariant coupling array $\lambda^{ij}_{k\ell}$. For example, for scalars Φ^i in the fundamental $\mathbf{N}$ multiplet of the $SU(N)$ gauge group, $\lambda^{ij}_{k\ell}=\lambda\times\bigl(\delta^i_k\delta^j_\ell+\delta^i_\ell\delta^j_k\bigr)$. But the details of the $\lambda^{ij}_{k\ell}$ coupling are not germane for the present problem, so I'll keep them generic as long as they are G-invariant.

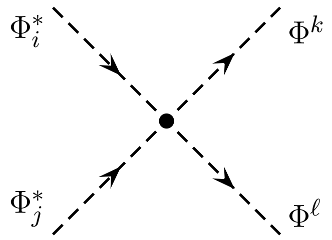
In the quantum field theory, the net bare Lagrangian comprises the classical terms (S.1) plus the ghost Lagrangian, the gauge fixing terms, and the whole slew of counterterms. Altogether, we have

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac14(F^a_{\mu\nu})^2 \;-\; \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \;+\; \partial_\mu\bar c^a D^\mu c^a\\
&\qquad +\; D_\mu\Phi^\dagger D^\mu\Phi \;-\; m^2\Phi^\dagger\Phi \;-\; \tfrac14\lambda^{ij}_{k\ell}\,\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell\\
&-\;\frac{\delta_3}{4}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)^2 \;+\; g\delta_1^{(3g)}f^{abc}A^b_\mu A^c_\nu\partial_\mu A^{a\nu} \;-\; \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A^b_\mu A^c_\nu)^2\\
&+\;\delta_2^{(\rm gh)}\partial_\mu\bar c^a\partial^\mu c^a \;-\; g\delta_1^{(\rm gh)}f^{abc}\partial_\mu\bar c^a A^{b\mu}c^c\\
&+\;\delta_2^{(\phi)}\partial_\mu\Phi^\dagger\partial^\mu\Phi \;+\; ig\delta_1^{(\phi1g)}\,A^a_\mu\times\bigl(\partial^\mu\Phi^\dagger T^a_{(r)}\Phi \;-\; \Phi^\dagger T^a_{(r)}\partial^\mu\Phi\bigr)\\
&\qquad +\; g^2\delta_1^{(\phi2g)}A^a_\mu A^{b\mu}\times\Phi^\dagger T^a_{(r)}T^b_{(r)}\Phi\\
&-\;\delta_m^{(\phi)}\Phi^\dagger\Phi \;-\; \tfrac14(\delta_\lambda)^{ij}_{k\ell}\,\Phi^*_i\Phi^*_j\Phi^k\Phi^\ell.
\end{aligned} \tag{S.4}$$

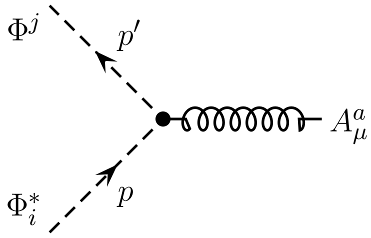
Note that all the terms in this bare Lagrangian which pertain only to the vector and the ghost fields are exactly the same as in the fermionic QCD, *cf.* my notes on QCD Feynman rules. Consequently, in the Feynman rules of the present theory, the gluon propagator, the three-gluon and the four-gluon vertices, the ghost propagator and the ghost-gluon vertex are exactly as in my notes, and I don't need to repeat them here. But let me write down the explicit Feynman rules pertaining to the scalar fields, in particular, the scalar propagator and the scalar vertices:



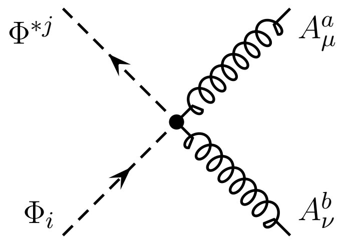
$$= \frac{i\delta^i_j}{p^2 - m^2 + i0}\,, \tag{S.5}$$



$$= -i\lambda^{ij}_{k\ell}\,, \tag{S.6}$$



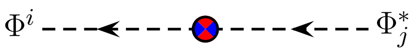
$$= ig(p+p')_\mu (T^a_{(r)})^j{}_i, \tag{S.7}$$



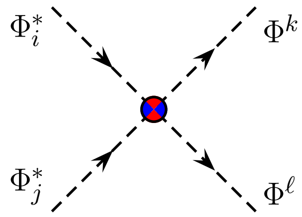
$$= ig^2 g_{\mu\nu}\{T^a_{(r)}, T^b_{(r)}\}^j{}_i. \tag{S.8}$$

Note the anticommutator of the group generators in the two-scalar two-gluon vertex: It follows from permutations of the two gluon lines.

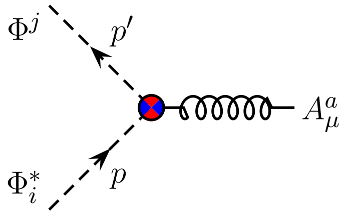
In addition, there are several counterterm vertices involving the scalar fields. Although such vertices are not germane to the present exercise, let me list them here for the completeness sake:



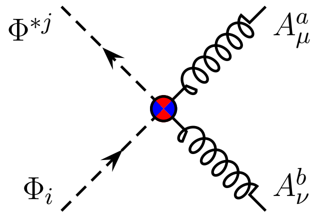
$$= \delta^i_j\bigl(i\delta^{(\phi)}_m - i\delta^{(\phi)}_2 p^2\bigr), \tag{S.9}$$



$$= \; -i\big(\delta_\lambda\big)^{ij}_{k\ell}\,, \tag{S.10}$$



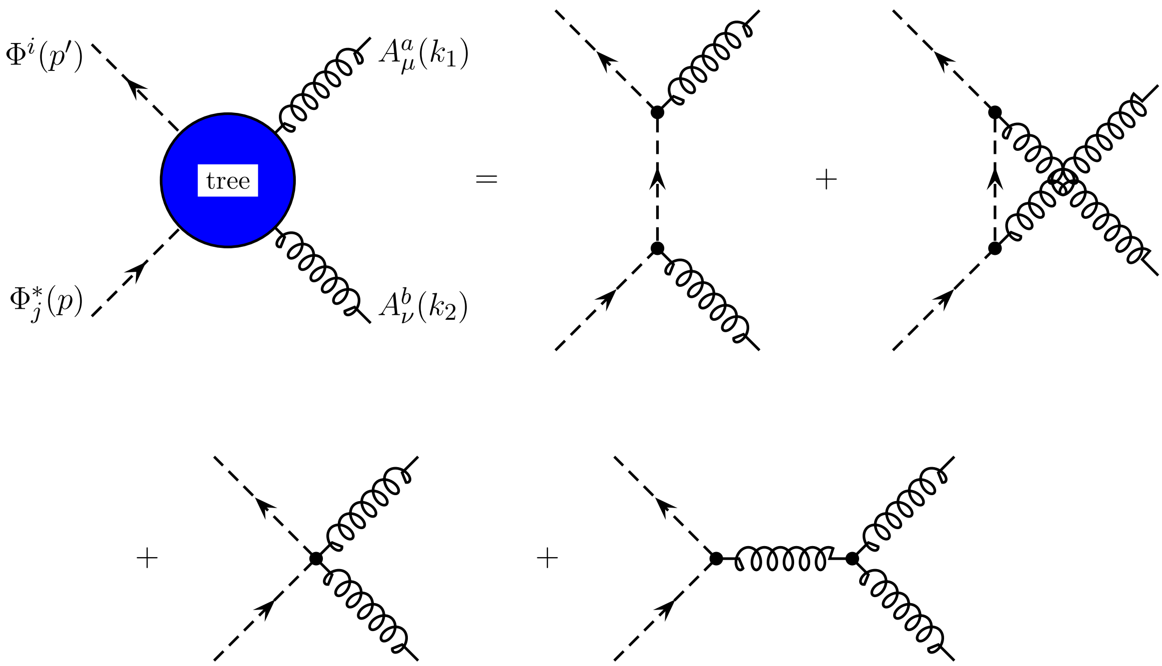
$$= \; ig\delta_1^{(\phi 1g)} \times (p+p')_\mu (T^a_{(r)})^j{}_i, \tag{S.11}$$



$$= \; ig^2\delta_1^{(\phi 2g)} \times g_{\mu\nu}\{T^a_{(r)}, T^b_{(r)}\}^j{}_i. \tag{S.12}$$

Problem **1**(b):

At the tree level there are four diagrams for the $\Phi + \Phi^* \to g + g$ annihilation process, namely



(S.13)

The amplitude stemming from each of these 4 diagrams has form

$$\mathcal{M}[\text{diagram}\#n] \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \mathcal{M}^{\mu\nu}_n \tag{S.14}$$

where

$$\mathcal{M}^{\mu\nu}_1 \;=\; \frac{-g^2}{(p-k_1)^2 - m^2}\,(2p-k_1)^\mu (k_2 - 2p')^\nu \times (T^b T^a)^i{}_j\,, \tag{S.15}$$

$$\mathcal{M}^{\mu\nu}_2 \;=\; \frac{-g^2}{(p'-k_1)^2 - m^2}\,(k_1-2p')^\mu (2p - k_2)^\nu \times (T^a T^b)^i{}_j\,, \tag{S.16}$$

$$\mathcal{M}^{\mu\nu}_3 \;=\; +g^2 g^{\mu\nu} \times \{T^a, T^b\}^i{}_j\,, \tag{S.17}$$

$$\begin{aligned}\mathcal{M}^{\mu\nu}_4 \;=\; &-\frac{ig^2}{(k_1+k_2)^2}\,(p-p')_\lambda\,(T^c)^i{}_j \\ &\times f^{abc}\big(g^{\mu\nu}(k_1-k_2)^\lambda \,+\, g^{\nu\lambda}(2k_2+k_1)^\mu \,+\, g^{\lambda\mu}(-2k_1-k_2)^\nu\big),\end{aligned} \tag{S.18}$$

so the net tree-level amplitude is

$$\mathcal{M}_{\text{net}} \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \mathcal{M}^{\mu\nu}_{\text{net}} \;=\; e^*_{1\mu}\, e^*_{2\nu} \times \big(\mathcal{M}^{\mu\nu}_1 \,+\, \mathcal{M}^{\mu\nu}_2 \,+\, \mathcal{M}^{\mu\nu}_3 \,+\, \mathcal{M}^{\mu\nu}_4\big) \tag{S.19}$$

Problem **1**(c):

Our task is to verify that the net amplitude (S.19) satisfies

$$k_1^\mu e_2^\nu \mathcal{M}_{\mu\nu} \;=\; 0 \tag{S.20}$$

provided $e_2^\nu k_{2\nu} = 0$ and all external momenta are on shell. Let's start by calculating the $k_{1\mu}\mathcal{M}^{\mu\nu}_n$ for each of the 4 diagrams. For the first diagram's amplitude (S.15),

$$k_{1\mu}\mathcal{M}^{\mu\nu}_1 \;=\; -g^2 (k_2 - 2p')^\nu \times \frac{2(pk_1) - k_1^2}{(p-k_1)^2 - m^2} \times (T^b T^a)^j{}_i \tag{S.21}$$

where for the on-shell momenta $p^2 = p'^2 = m^2$, $k_1^2 = k_2^2 = 0$,

$$\frac{2(pk_1) - k_1^2}{(p-k_1)^2 - m^2} \;=\; -1 \tag{S.22}$$

and hence

$$k_{1\mu}\mathcal{M}_1^{\mu\nu} \;=\; +g^2(k_2-2p')^\nu \times (T^bT^a)^j{}_i\,. \tag{S.23}$$

Likewise, for the second diagram's amplitude (S.16),

$$\begin{aligned}
k_{1\mu}\mathcal{M}_2^{\mu\nu} \;&=\; -g^2(2p-k_2)^\nu \times \frac{k_1^2-2(p'k_1)}{(p'-k_1)^2-m^2} \times (T^aT^b)^j{}_i \\
&\quad\langle\!\langle \text{ for the on-shell momenta } \rangle\!\rangle \\
&=\; +g^2(k_2-2p)^\nu \times 1 \times (T^aT^b)^i{}_j\,.
\end{aligned} \tag{S.24}$$

For the third diagram's amplitude (S.17) we have

$$k_{1\mu}\mathcal{M}_3^{\mu\nu} \;=\; +g^2k_1^\nu \times \{T^a,T^b\}^i{}_j \;=\; g^2k_1^\nu \times (T^aT^b)^i{}_j \;+\; g^2k_1^\nu \times (T^bT^a)^i{}_j\,, \tag{S.25}$$

so adding the first three diagrams together, we obtain

$$\begin{aligned}
k_{1\mu}\times\mathcal{M}_{1+2+3}^{\mu\nu} \;&=\; g^2(T^aT^b)^i{}_j \times \bigl((k_2-2p')\,+\,k_1\bigr)^\nu \;+\; g^2(T^bT^a)^i{}_j \times \bigl((k_2-2p)\,+\,k_1\bigr)^\nu \\
&\quad\langle\!\langle \text{ using momentum conservation } k_1+k_2=p+p' \,\rangle\!\rangle \\
&=\; g^2(T^aT^b)^i{}_j \times (p-p')^\nu \;+\; g^2(T^bT^a)^i{}_j \times (p'-p)^\nu \\
&=\; g^2(p-p')^\nu \times (T^aT^b\;-\;T^bT^a)^i{}_j \\
&=\; g^2(p-p')^\nu \times if^{abc}(T^c)^i{}_j\,.
\end{aligned} \tag{S.26}$$

As to the fourth diagram's amplitude (S.18),

$$\begin{aligned}
k_{1\mu}\mathcal{M}_4^{\mu\nu} \;=\; g^2(p'-p)_\lambda \times if^{abc}(T^c)^i{}_j \times \frac{1}{(k_1+k_2)^2} \times& \\
\times\, k_{1\mu}\bigl[g^{\mu\nu}(k_1-k_2)^\lambda \;+\; g^{\nu\lambda}(2k_2+k_1)^\mu \;+\; g^{\lambda\mu}(-2k_1-k_2)^\nu\bigr]&,
\end{aligned} \tag{S.27}$$

where the expression on the second line is exactly similar to its analogue in the fermionic QCD, *cf.* eqs. (37–38) on page 9 of my notes on QCD Ward identities. Just as in my notes, for the on-shell photon momenta

$$k_{1\mu} \times [\cdots] \;=\; (k_1+k_2)^2 g^{\nu\lambda} \;-\; (k_1+k_2)^\nu (k_1+k_2)^\lambda \;+\; k_2^\nu k_2^\lambda \,. \tag{S.28}$$

hence plugging each of the 3 terms here into eq. (S.27), we obtain

$$k_{1\mu}\mathcal{M}_4^{\mu\nu} \;=\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu}\,, \tag{S.29}$$

where

$$k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;=\; g^2(p'-p)^\nu \times if^{abc}(T^c)^i{}_j\,, \tag{S.30}$$

$$k_{1\mu}\mathcal{M}_{4,b}^{\mu\nu} \;=\; -g^2(k_1-k_2)^\nu \times \frac{(p'-p)_\lambda(k_1+k_2)^\lambda}{(k_1+k_2)^2} \times if^{abc}(T^c)^i{}_j\,, \tag{S.31}$$

$$k_{1\mu}\mathcal{M}_{4,c}^{\mu\nu} \;=\; g^2 k_2^\nu \times \frac{(p'-p)_\lambda k_2^\lambda}{(k_1+k_2)^2} \times if^{abc}(T^c)^i{}_j\,. \tag{S.32}$$

By inspection of eqs. (S.30) and (S.26), the first term's contribution precisely cancels the combined contributions of the diagrams 1, 2, and 3,

$$k_{1\mu}\mathcal{M}_{4,a}^{\mu\nu} \;+\; k_{1\mu}\mathcal{M}_{1+2+3}^{\mu\nu} \;=\; 0. \tag{S.33}$$

As to the second term's contribution (S.31), it vanishes for the on-shell scalars' momenta $p^2 = p'^2 = m^2$; indeed,

$$(p'-p)_\lambda(k_1+k_2)^\lambda \;=\; (p'-p)_\lambda(p+p')^\lambda \;=\; p'^2-p^2 \;=\; m^2-m^2 \;=\; 0 \quad\Longrightarrow\quad k_{1\mu}\mathcal{M}_{4,b}^{\mu\nu} \;=\; 0. \tag{S.34}$$

Finally, the third term's contribution (S.32) does not vanish but its ν index belongs to the k_2^ν factor, thus

$$k_{1\mu}\mathcal{M}_{\rm net}^{\mu\nu} \;=\; k_{1\mu}\mathcal{M}_{4,c}^{\mu\nu} \;=\; [\text{stuff}] \times k_2^\nu\,. \tag{S.35}$$

Consequently, when the net amplitude is contracted with the polarization vector $e_{2\nu}^*$ of the second gluon, it vanishes when the second gluon is transversely polarized, $k_2e_2^*=0$, but not if the other gluon's polarization is longitudinal. And this is in accordance to the weak form of Ward Identity: *On-shell amplitudes involving one longitudinal gluon vanish, but only if all the other gluons are transverse.*

Problem **1**(d):

For the first longitudinal gluon $e_1^\mu \propto k_1^\mu$, hence in light of eqs. (S.35) and (S.32),

$$\begin{aligned}\mathcal{M}(\Phi+\Phi^*\to g_L+g_L) &= e^*_{1\mu}\mathcal{M}^{\mu\nu}e^*_{2\nu} = \frac{e_1^*}{k_1}\times k_{1\mu}\mathcal{M}^{\mu\nu}e^*_{2\nu} = \frac{e_1^*}{k_1}\times k_{1\mu}\mathcal{M}^{\mu\nu}_{4,c}e^*_{2\nu} \\ &= \frac{e_1^*}{k_1}\times g^2(k_2e_2^*)\times\frac{(p'-p)_\lambda k_2^\lambda}{(k_1+k_2)^2}\times if^{abc}(T^c)^i{}_j\,.\end{aligned} \tag{S.36}$$

For the specific longitudinal polarizations in question,

$$\frac{e_1^*}{k_1} = \frac{1}{\sqrt{2}\omega_1}\,,\quad (e_2^*k_2) = \sqrt{2}\omega_2\,, \tag{S.37}$$

where in the center-of-mass system

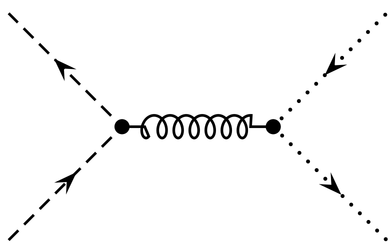
$$\omega_1 = \omega_2 = \tfrac{1}{2}E_{\rm cm}\quad\text{while}\quad (k_1+k_2)^2 = s = E_{\rm cm}^2, \tag{S.38}$$

hence

$$\mathcal{M}(\Phi+\Phi^*\to g_L+g_L) = \frac{g^2}{s}\times((p'-p)k_2)\times if^{abc}(T^c)^i{}_j\,. \tag{S.39}$$

Problem **1**(e):

There is only one tree diagram for the $\mathcal{M}(\Phi+\Phi^*\to\mathrm{gh}+\overline{\mathrm{gh}})$ amplitude, namely



Evaluating this diagram, we get

$$i\mathcal{M} = ig(p-p')^\mu(T^c)^i{}_j\times\frac{-ig_{\mu\nu}}{(k_1+k_2)^2}\times gf^{cab}k_2^\nu \tag{S.40}$$

and hence

$$\mathcal{M}(\Phi+\Phi^*\to\mathrm{gh}+\overline{\mathrm{gh}}) = \frac{g^2}{s}\times((p'-p)k_2)\times if^{abc}(T^c)^i{}_j\,. \tag{S.41}$$

Problem $\mathbf{1}$(f):

By inspection, the tree amplitudes (S.39) and (S.41) for the scalars annihilating into a pair of longitudinal gluons and into a ghost-antighost pair are exactly equal to each other,

$$\mathcal{M}(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}}) \;=\; \mathcal{M}(\Phi+\Phi^*\to g_L+g_L). \tag{S.42}$$

Naively, this means that the partial cross-section of these two processes are also equal to each other. However, In the Hilbert space of both physical and unphysical particles, the longitudinal gluon states have positive norm while the (anti)ghost states have negative norm. Consequently,

$$\frac{d\sigma(\Phi+\Phi^*\to g_L+g_L)}{d\Omega} \;=\; +\frac{|\mathcal{M}|^2}{64\pi^2 s} \quad\text{while}\quad \frac{d\sigma(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} \;=\; -\frac{|\mathcal{M}|^2}{64\pi^2 s}, \tag{S.43}$$

so the two partial cross-sections actually have opposite signs. Therefore, in the net annihilation cross-section into two gluons of any polarizations or into a ghost-antighost pair,

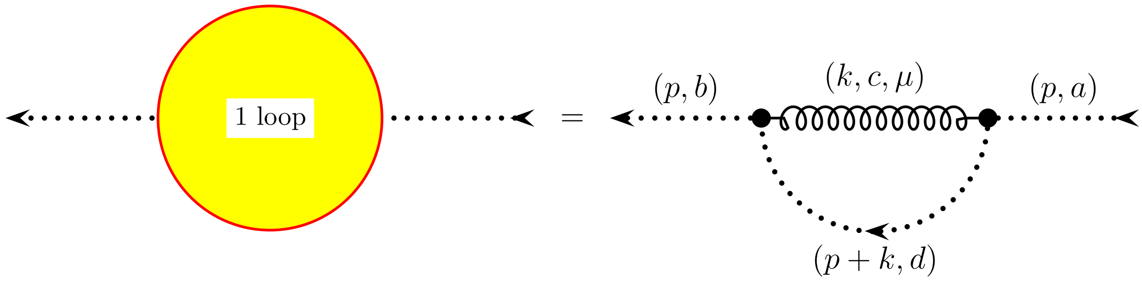
$$\begin{aligned}\frac{d\sigma(\Phi^*+\Phi\to g+g \text{ or } \mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} \;=\;& \frac{d\sigma(\Phi+\Phi^*\to g_T+g_T)}{d\Omega}\\ &+\;\frac{d\sigma(\Phi+\Phi^*\to g_L+g_L)}{d\Omega}\;+\;\frac{d\sigma(\Phi+\Phi^*\to \mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega},\end{aligned} \tag{S.44}$$

the last two terms cancel each other, so the net cross-section is the same as cross-section for annihilation into physical transverse gluons only,

$$\frac{d\sigma(\Phi^*+\Phi\to g+g \text{ or } \mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} = \frac{d\sigma(\Phi+\Phi^*\to g_T+g_T)}{d\Omega}. \tag{S.45}$$

Problem $\mathbf{2}$(a–c):

At the one-loop level, the $\delta_2^{(\mathrm{gh})}$ cancels the divergence of a single diagram



which evaluates (in the Feynman gauge) to

$$-i\Sigma^{ba}_{1\,\mathrm{loop}}(p) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,\frac{i}{(p+k)^2+i0}\times\frac{-i}{k^2+i0}\times -gf^{cad}(p+k)_\mu\times -gf^{cdb}p^\mu. \tag{S.46}$$

In particular, the group factor here is

$$\sum_{c,d} f^{cad}f^{cdb} \;=\; \sum_{c,d}\bigl(-iT^a_{\mathrm{adj}}\bigr)^{dc}\bigl(-iT^b_{\mathrm{adj}}\bigr)^{cd} \;=\; -\,\mathrm{tr}_{\mathrm{adj}}\bigl(T^aT^b\bigr) \;=\; -R(\mathrm{adj})\times\delta^{ab}. \tag{S.47}$$

Using $R(\mathrm{adj}) = C(\mathrm{adj}) \equiv C(G)$ and taking care of all the signs and $\pm i$ factors, we arrive at

$$\Sigma^{ba}_{1\,\mathrm{loop}}(p) \;=\; -ig^2C(G)\delta^{ba}\times p^\mu\times\int\!\frac{d^4k}{(2\pi)^4}\,\frac{(p+k)_\mu}{(k^2+i0)\times((p+k)^2+i0)}\,. \tag{S.48}$$

Note: the p^μ factor from the outgoing ghost vertex can be pulled outside the integral, which reduces its UV divergence form quadratic to linear. Moreover, by Lorentz symmetry the linear divergence cancels out, and the remaining integral becomes $p_\mu \times O(\log\Lambda)$. Consequently, the whole amplitude has form

$$\Sigma^{ba}_{1\,\mathrm{loop}}(p) \;=\; \delta^{ba}\times p^2\times\Pi_{1\,\mathrm{loop}}(p) \tag{S.49}$$

and we do not need the ghost-mass counterterm. Also, the logarithmic divergence of $\Pi(p)$

may be canceled by the $\delta_2^{(\mathrm{gh})}$ counterterm as

$$\Pi(p) \;=\; \Pi_{\mathrm{loop}}(p) \;-\; \delta_2^{(\mathrm{gh})}. \tag{S.50}$$

Indeed, introducing the Feynman parameter x into the momentum integral (S.48), we have

$$\begin{aligned}\int\frac{d^4k}{(2\pi)^4}\,\frac{(p+k)_\mu}{(k^2+i0)\times((p+k)^2+i0)} \;&=\; \int\limits_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu+(1-x)p_\mu}{[\ell^2+x(1-x)p^2+i0]^2}\\ &=\; \int\limits_0^1 dx\,(1-x)p_\mu\times\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+x(1-x)p^2+i0]^2}\end{aligned} \tag{S.51}$$

where the second equality follows from the $\ell\to-\ell$ symmetry of the integral. Using dimensional regularization for the UV divergence of the remaining integral, we have

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+x(1-x)p^2+i0]^2} \;=\; \frac{i}{16\pi^2}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{-x(1-x)p^2}\,+\,\text{finite constant}\right), \tag{S.52}$$

hence

$$\int\limits_0^1 dx\,(1-x)p_\mu\times[\cdots] \;=\; \frac{ip_\mu}{32\pi^2}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{-p^2}\,+\,\text{finite constant}\right) \tag{S.53}$$

and consequently

$$\Sigma^{ba}_{1\,\mathrm{loop}}(p) \;=\; +\frac{g^2C(G)}{32\pi^2}\times\delta^{ba}p^2\times\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{-p^2}\,+\,\text{finite constant}\right). \tag{S.54}$$

To cancel the UV divergence here, we need

$$\delta_2^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; +\frac{g^2C(G)}{32\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.55}$$

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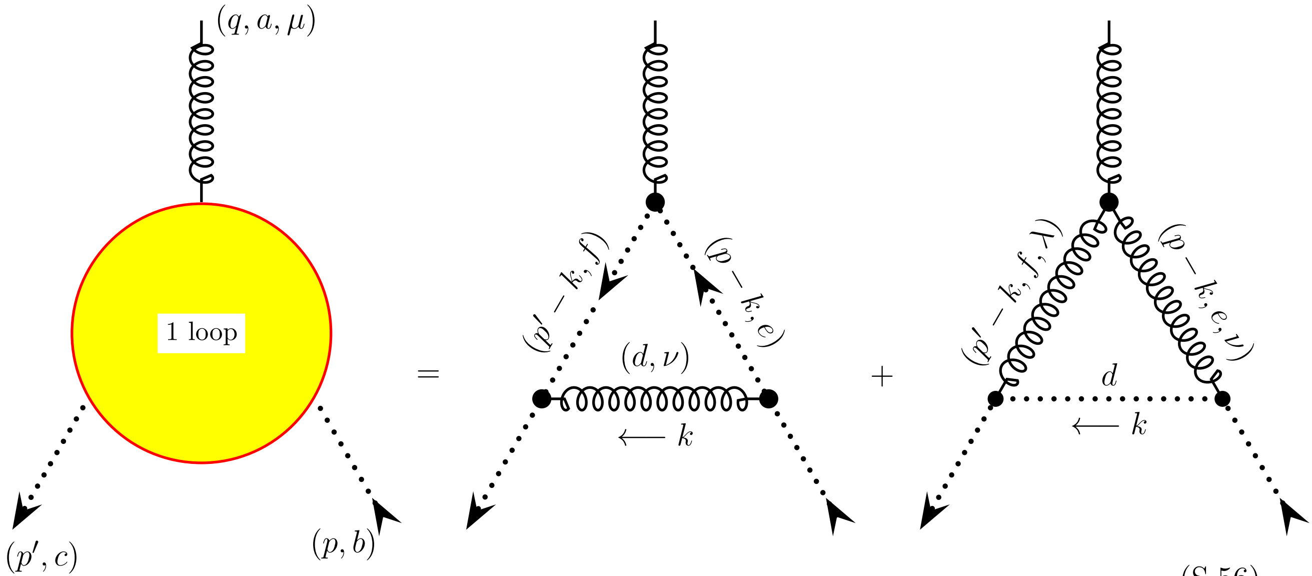
Now consider the $\delta_1^{(\mathrm{gh})}$ counterterm. At the one-loop level, it cancels the UV divergence of two diagrams



(S.56)

The first diagram here evaluates to

$$-igG^{\mu,abc}(1^{\mathrm{st}}) \;=\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{(p-k)^2+i0}\times\frac{i}{(p'-k)^2+i0}\times\frac{-i}{k^2+i0}\times \\ \times -gf^{dbe}(p-k)^\nu\times -gf^{aef}(p'-k)^\mu\times -gf^{dfc}p'_\nu\,. \tag{S.57}$$

In particular, the group factor on the second line here amounts to

$$\begin{aligned} X_1^{abc}\;\equiv\sum_{d,e,f}f^{dbe}f^{aef}f^{dfc} &= -\sum_{d,e,f}f^{dcf}f^{afe}f^{deb} \\ &= -\sum_{d,e,f}\left(-iT^d_{\mathrm{adj}}\right)^{cf}\left(-iT^a_{\mathrm{adj}}\right)^{fe}\left(-iT^d_{\mathrm{adj}}\right)^{eb} \\ &= -i\sum_d\left(T^d_{\mathrm{adj}}T^a_{\mathrm{adj}}T^d_{\mathrm{adj}}\right)^{cb}. \end{aligned}\tag{S.58}$$

The simplest way to take the last sum here is to use the abstract generators $\hat{T}^a$ of the Lie algebra instead of the specific matrices representing them in the adjoint multiplet:

$$\begin{aligned}
\sum_d \hat{T}^d \hat{T}^a \hat{T}^d &= \sum_d \hat{T}^d \hat{T}^d \hat{T}^a \;+\; \sum_d \hat{T}^d \big[\hat{T}^a, \hat{T}^d\big] \\
&= \left(\sum_d \hat{T}^d \hat{T}^d\right) \times \hat{T}^a \;+\; \sum_{d,h} \hat{T}^d \times i f^{adh} \hat{T}^h \\
&= \hat{C}_2 \times \hat{T}^a \;+\; \frac{i}{2} \sum_{d,h} f^{adh} \times \big[\hat{T}^d, \hat{T}^h\big] \\
&= \hat{C}_2 \times \hat{T}^a \;-\; \frac{1}{2} \sum_{d,h,j} f^{adh} f^{dhj} \hat{T}^j \\
&= \hat{C}_2 \times \hat{T}^a \;-\; \frac{C(G)}{2} \times \hat{T}^a
\end{aligned} \tag{S.59}$$

where the last equality follows from eq. (S.47), $\sum_{dh} f^{adh} f^{dhj} = C(G) \times \delta^{aj}$. Consequently, in the adjoint representation of the Lie algebra

$$\sum_d T^d_{\mathrm{adj}} T^a_{\mathrm{adj}} T^d_{\mathrm{adj}} \;=\; C(\mathrm{adj}) \times T^a_{\mathrm{adj}} \;-\; \frac{C(G)}{2} \times T^a_{\mathrm{adj}} \;=\; \frac{C(G)}{2} \times T^a_{\mathrm{adj}} \tag{S.60}$$

and hence

$$X_1^{abc} \;=\; -i\frac{C(G)}{2} \times \left(T^a_{\mathrm{adj}}\right)^{cb} \;=\; +\frac{C(G)}{2} \times f^{acb} \;=\; -\frac{C(G)}{2} \times f^{abc}. \tag{S.61}$$

Plugging this group factor into the loop amplitude (S.57) and pulling all the constant factors outside the integral, we obtain

$$-igG^{\mu,abc}(1^{\mathrm{st}}) \;=\; -\frac{g^3 C(G)}{2} \times f^{abc} p'_\nu \times H_1^{\nu\mu} \tag{S.62}$$

where

$$H_1^{\mu\nu} \;=\; -i\int \frac{d^4k}{(2\pi)^4} \, \frac{(p-k)^\nu (p'-k)^\mu}{[(p-k)^2 + i0] \times [(p'-k)^2 + i0] \times [k^2 + i0]} \,. \tag{S.63}$$

Note that thanks to the k–independent factor p'_ν of the left vertex which we pulled out from the momentum integral, the remaining integral $H^{\mu\nu}$ is only logarithmically divergent.

Consequently, the infinite part of $H^{\mu\nu}$ depends only on the leading terms of the numerator and the denominator (as polynomials in k), thus

$$\begin{aligned} \left[H_1^{\mu\nu}\right]_\infty &= -i\int\frac{d^4k}{(2\pi)^4}\,\frac{k^\nu k^\mu+\cdots}{(k^2+i0)^3+\cdots} = -\int\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^\mu k_E^\nu+\cdots}{(k_E^2)^3+\cdots} \\ &= +\frac{g^{\mu\nu}}{4}\times\int\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2+\cdots}{(k_E^2)^3+\cdots} = +\frac{g^{\mu\nu}}{4}\times\frac{1}{16\pi^2}\times\frac{1}{\epsilon}\,. \end{aligned} \tag{S.64}$$

Altogether, the divergent part of the first diagram amounts to

$$\left[-igG^{\mu,abc}(1^{\rm st})\right]_\infty = -gf^{abc}p'^\mu\times\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.65}$$

Note that its dependence on the colors of the 3 external particle, on the index μ of the gluon, and on the ghosts' momenta have just the right form to be canceled by the $\delta_1^{(\rm gh)}$ counterterm vertex,

$$-igG^{\mu,abc}(\text{counterterm}) = -gf^{abc}p'^\mu\times\delta_1^{(\rm gh)}\,. \tag{S.66}$$

In particular, to cancel just the first diagram we need

$$\delta_1^{(\rm gh)}(1^{\rm st}) = -\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.67}$$

Now consider the second diagram (S.56), which evaluates to

$$\begin{aligned} -igG^{\mu,abc}(2^{\rm nd}) = \int\frac{d^4k}{(2\pi)^4}\,&\frac{-i}{(p-k)^2+i0}\times\frac{-i}{(p'-k)^2+i0}\times\frac{i}{k^2+i0}\times \\ &\times -gf^{ebd}k_\nu\times -gf^{fdc}p'_\lambda\times \\ &\times -gf^{aef}\begin{bmatrix} g^{\mu\lambda}(q-(k-p'))^\nu \\ +g^{\lambda\nu}((k-p')-(p-k))^\mu \\ +g^{\mu\nu}((p-k)-q)^\lambda \end{bmatrix}. \end{aligned} \tag{S.68}$$

Despite other complications, the group factor here is the same as in the first diagram,

$$X_2^{abc} = \sum_{d,e,f}f^{ebd}f^{fdc}f^{aef} = +\sum_{d,e,f}f^{dbe}f^{aef}f^{dfc} = X_1^{abc} = -\frac{C(G)}{2}\times f^{abc} \tag{S.69}$$

Pulling this group factor — as well as other k–independent factors outside of the integral (S.68), we obtain

$$-igG^{\mu,abc}(2^{\text{nd}}) \;=\; -\frac{g^3C(G)}{2}\times f^{abc}p'_\lambda\times H_2^{\lambda\mu} \tag{S.70}$$

where

$$H_2^{\lambda\mu} \;=\; +i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{k_\nu\times\left[g^{\lambda\mu}(q+p'-k)^\nu \;+\; g^{\lambda\nu}(2k-p-p')^\mu \;+\; g^{\mu\nu}(p-q-k)^\lambda\right]}{[(p-k)^2+i0]\times[(p'-k)^2+i0]\times[k^2+i0]}\,. \tag{S.71}$$

Again, thanks to the k–independent factor p'_λ of the left vertex which we pulled out from the momentum integral, the remaining integral $H_2^{\lambda\mu}$ is only logarithmically divergent. Although the numerator in the integral (S.71) is much messier than in the integral (S.63) for the first diagram, its leading term for $k\to\infty$ is fairly simple

$$(\text{numerator}) \;=\; g^{\lambda\mu}\times(-k^2)+k^\lambda\times(2k)^\mu+k^\mu\times(-k)^\lambda+\cdots \;=\; -g^{\lambda\mu}k^2+k^\lambda k^\mu+\cdots \tag{S.72}$$

and that's all we need to get the infinite part of the integral. Specifically,

$$\begin{aligned}
\left[H_2^{\lambda\mu}\right]_\infty \;&=\; +i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{-g^{\mu\lambda}k^2 \;+\; k^\lambda k^\mu \;+\;\cdots}{(k^2+i0)^3 \;+\;\cdots}\\
&=\; +\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{+g^{\lambda\mu}_{\text{Mink}}\times k_E^2 \;+\; k_E^\lambda k_E^\mu \;+\;\cdots}{(k_E^2)^3 \;+\;\cdots}\\
&=\; +\left(g^{\lambda\mu}_{\text{Mink}}-\frac14\,g^{\lambda\mu}_{\text{Mink}}\right)\times\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2+\cdots}{k_E^6+\cdots}\\
&=\; \frac34\,g^{\lambda\mu}\times\frac{1}{16\pi^2}\times\frac{1}{\epsilon}
\end{aligned} \tag{S.73}$$

and hence the infinite part of the second diagram's amplitude

$$\left[-igG^{\mu,abc}(2^{\text{nd}})\right]_\infty \;=\; -gf^{abc}p'^\mu\times\frac{3g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.74}$$

Again, this divergence has exactly the right form to be canceled by the $\delta_1^{(\text{gh})}$ counterterm vertex. This time, to cancel just the divergence of the second diagram we need

$$\delta_1^{(\mathrm{gh})}(2^{\mathrm{nd}}) \;=\; -\frac{3g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.75}$$

Finally, combining the two diagrams' contributions, we get the net one-loop counterterm coefficient

$$\delta_1^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; -\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\;-\;\frac{3g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\;=\;-\frac{g^2C(G)}{32\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.76}$$

Comparing to the $\delta_2^{(\mathrm{gh})}$ for the ghosts' wave function renormalization

$$\delta_2^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; +\frac{g^2C(G)}{32\pi^2}\times\frac{1}{\epsilon}\,, \tag{S.55}$$

we immediately obtain the difference

$$\delta_1^{(\mathrm{gh})}[1\,\mathrm{loop}] \;-\; \delta_2^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; -\frac{g^2C(G)}{16\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.77}$$

As promised, this difference agrees with the $\delta_1-\delta_2$ difference for the quarks we had calculated in class

$$\delta_1^{(\mathrm{q})}[1\,\mathrm{loop}] \;-\; \delta_2^{(\mathrm{q})}[1\,\mathrm{loop}] \;=\; -\frac{g^2C(G)}{16\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.78}$$

Problem **3**:

In my notes on QCD beta function (eq. (122) on page 25), I gave a general formula for the coefficient b of a one-loop beta function $\beta = b\times g^3/(16\pi)^2$ of any gauge theory. For the gauge theories with product gauge groups $G = G_1\otimes G_2\otimes\cdots$, each factor G_i has its own gauge coupling g_i with one-loop beta-function $\beta_i = b_i\times g_i^3/(16\pi)^2$ where

$$b_i \;=\; \sum_{\substack{\text{all physical}\\ \text{multiplets}}} R_i(\text{multiplet})\times\begin{cases} -\frac{11}{3} & \text{for the gauge fields,}\\ +\frac{4}{3} & \text{for Dirac fermions,}\\ +\frac{2}{3} & \text{for Majorana fermions,}\\ +\frac{2}{3} & \text{for chiral Weyl fermions,}\\ +\frac{1}{3} & \text{for complex scalar fields,}\\ +\frac{1}{6} & \text{for real scalar fields,}\end{cases} \tag{S.79}$$

and R_i is the index of the multiplet in question with respect to the group factor G_i.

Problem $\mathbf{3}$(a):

The Standard Model has gauge symmetry $G=SU(3)\times SU(2)\times U(1)$, 3 families of quarks and leptons, and one Higgs doublet of scalar fields, hence

$$b_i \;=\; -\tfrac{11}{3}\times R_i(\text{gauge fields}) \;+\; 3\times\tfrac{2}{3}\times R_i(\text{family}) \;+\; \tfrac{1}{3}\times R_i(\text{Higgs}) \tag{S.80}$$

where each family counts as a reducible 15–plet of Weyl fermions

$$\text{family} \;=\; (\mathbf{3},\mathbf{2},+\tfrac16)_L \;+\; (\mathbf{1},\mathbf{2},-\tfrac12)_L \;+\; (\mathbf{3},\mathbf{1},+\tfrac23)_R \;+\; (\mathbf{3},\mathbf{1},-\tfrac13)_R \;+\; (\mathbf{1},\mathbf{1},-1)_R, \tag{S.81}$$

and the Higgses form an irreducible doublet of complex scalars,

$$\text{Higgs} \;=\; (\mathbf{1},\mathbf{2},+\tfrac12). \tag{S.82}$$

In my notations, $(\mathbf{m},\mathbf{n},y)$ is an $\mathbf{m}$ of $SU(3)_C$, an $\mathbf{n}$ of $SU(2)_W$, and its $U(1)$ charge is y. Consequently, this multiplet acts as n copies of $\mathbf{m}$ under the $SU(3)$, m copies of $\mathbf{n}$ under the $SU(2)$, and $n\times m$ copies of a y–charge field under the $U(1)$, hence

$$R_3(\mathbf{m},\mathbf{n},y) \;=\; n\times R_3(\mathbf{m}),\quad R_2(\mathbf{m},\mathbf{n},y) \;=\; m\times R_2(\mathbf{n}),\quad R_1(\mathbf{m},\mathbf{n},y) \;=\; nm\times y^2. \tag{S.83}$$

In particular

$$\begin{array}{llll}
(\mathbf{3},\mathbf{2},+\tfrac16)\text{ has} & R_3=2\times\dfrac12=1, & R_2=3\times\dfrac12=\dfrac32\,, & R_1=6\times\dfrac{1}{6^2}=\dfrac16\,,\\
(\mathbf{1},\mathbf{2},-\tfrac12)\text{ has} & R_3=2\times0=0, & R_2=1\times\dfrac12=\dfrac12\,, & R_1=2\times\dfrac{1}{2^2}=\dfrac12\,,\\
(\mathbf{3},\mathbf{1},+\tfrac23)\text{ has} & R_3=1\times\dfrac12=\dfrac12\,, & R_2=3\times0=0, & R_1=3\times\dfrac{2^2}{3^2}=\dfrac43\,,\\
(\mathbf{3},\mathbf{1},-\tfrac13)\text{ has} & R_3=1\times\dfrac12=\dfrac12\,, & R_2=3\times0=0, & R_1=3\times\dfrac{1^2}{3^2}=\dfrac13\,,\\
(\mathbf{1},\mathbf{1},-1)\text{ has} & R_3=1\times0=0, & R_2=1\times0=0, & R_1=1\times1^2=1,
\end{array} \tag{S.84}$$

and hence

$$\begin{aligned}
R_3(\text{family}) \;&=\; 1\,+\,0+\,\frac12\,+\,\frac12\,+\,0 \;=\; 2,\\
R_2(\text{family}) \;&=\; \frac32\,+\,\frac12\,+\,0\,+\,0\,+\,0 \;=\; 2,\\
R_1(\text{family}) \;&=\; \frac16\,+\,\frac12\,+\,\frac43\,+\,\frac13\,+\,1 \;=\; \frac{10}{3}\,.
\end{aligned} \tag{S.85}$$

Also,

$$R_3(\text{Higgs}) \;=\; 0, \quad R_2(\text{Higgs}) \;=\; \frac{1}{2}\,, \quad R_1(\text{Higgs}) \;=\; \frac{1}{2}\,. \tag{S.86}$$

Finally, the gauge fields are in the adjoint multiplets of the respective groups,

$$A^a_\mu \;\in\; (\mathbf{8},\mathbf{1},0) \,+\, (\mathbf{1},\mathbf{3},0) \,+\, (\mathbf{1},\mathbf{1},0) \tag{S.87}$$

hence

$$R_3(\text{gauge fields}) \;=\; 3, \quad R_2(\text{gauge fields}) \;=\; 2, \quad R_2(\text{gauge fields}) \;=\; 0, \tag{S.88}$$

since for the non-abelian factors $R\big(\text{adjoint of } SU(N)\big) = N$ while the abelian $U(1)$ factor has its gauge field neutral and hence $R = 0$.

At this point, all we need to do is to plug eqs. (S.85), (S.86), and (S.88) into eqs. (S.80), thus

$$\begin{aligned}
b_3 &= -\frac{11}{3}\times 3 \;+\; 3\times\frac{2}{3}\times 2 \;+\; \frac{1}{3}\times 0 \;=\; -7,\\
b_2 &= -\frac{11}{3}\times 2 \;+\; 3\times\frac{2}{3}\times 2 \;+\; \frac{1}{3}\times\frac{1}{2} \;=\; -\frac{19}{6},\\
b_1 &= -\frac{11}{3}\times 0 \;+\; 3\times\frac{2}{3}\times\frac{10}{3} \;+\; \frac{1}{3}\times\frac{1}{2} \;=\; +\frac{41}{6}\,.
\end{aligned} \tag{S.89}$$

Problem **3**(b):

In the MSSM, we have the same gauge fields as in the Standard Model, but each gauge field A^a_μ is accompanied by the Majorana fermion λ^a (the gaugino). Likewise, we have the same 3 families of quarks and leptons as the SM, but each Weyl quark or lepton is accompanies by a scalar quark or slepton in exactly similar multiplet of the gauge group. Finally, all Higgses belong to the same $(\mathbf{1},\mathbf{2},+\frac{1}{2})$ doublet as in the SM, but now we have two such doublets of scalars and one doublet of Dirac Fermions. Therefore, for each gauge coupling we have

$$\begin{aligned} b_i \;&=\; R_i(\text{adjoint}_i)\times\left(-\frac{11}{3}+\frac{2}{3}\;=\;-3\right)\\ &+\;3\times R_i(\text{family})\times\left(+\frac{2}{3}+\frac{1}{3}\;=\;+1\right)\\ &+\;R_i\begin{pmatrix}\text{Higgs}\\ \text{doublet}\end{pmatrix}\times\left(2\times\frac{+1}{3}+\frac{4}{3}\;=\;+2\right). \end{aligned} \tag{S.90}$$

Since we have already computed the net indices of 1 whole family, of 1 Higgs doublet, and of the gauge fields, all we need to do now is to plug eqs. (S.85), (S.86), and (S.88) into eq. (S.90), thus

$$\begin{aligned} b_3 \;&=\; -3\times 3\;+\;3\times 1\times 2\;+\;2\times 0\;=\;-3,\\ b_2 \;&=\; -3\times 2\;+\;3\times 1\times 2\;+\;2\times\frac{1}{2}\;=\;+1,\\ b_1 \;&=\; -3\times 0\;+\;3\times 1\times\frac{10}{3}\;+\;2\times\frac{1}{2}\;=\;+11. \end{aligned} \tag{S.91}$$

Problem **3**(c):

At the one-loop level, the renormalization group equations for the 3 gauge couplings of the SM (or the MSSM) are independent from each other,

$$\forall i,\quad \frac{dg(E)}{d\log(E)}\;=\;\beta_i(g_i)\;\approx\;b_i\times\frac{g_i^3}{16\pi^2}\quad\text{regardless of the other } g_j\,. \tag{S.92}$$

Consequently, we may separately integrate each of these equations to obtain

$$\log\frac{E_2}{E_2}\;=\int\limits_{g_i(E_1)}^{g_i(E_2)}\frac{dg_i}{\beta_i(g_i)}\;\approx\;\frac{16\pi^2}{b_i}\times\int\limits_{g_i(E_1)}^{g_i(E_2)}\frac{dg}{g^3}\;=\;\frac{8\pi^2}{b_i}\left(\frac{1}{g_i^2(E_1)}\;-\;\frac{1}{g_i^2(E_2)}\right) \tag{S.93}$$

and hence

$$\frac{4\pi}{g_i^2(E_1)}\;=\;\frac{4\pi}{g_i^2(E_2)}\;+\;\frac{b_i}{2\pi}\times\log\frac{E_2}{E_1}\,. \tag{S.94}$$

In particular, for $E_2 = M_{\text{GUT}}$ and the SM (or MSSM) couplings, we have

$$\forall i=3,2,1,\quad \frac{1}{\alpha_i(E)}\;=\;\frac{1}{\alpha_i(M_{\text{GUT}})}\;+\;\frac{b_i}{2\pi}\times\log\frac{M_{\text{GUT}}}{E}\,. \tag{S.95}$$

Combining these formulae with the GUT-scale relations (3) of the Unified theory, we immediately arrive at the Georgi–Quinn–Weinberg equations (4).

Problem **3**(d–e):

The 3 Georgi–Quinn–Weinberg equations (4) have 2 unknown parameters — the GUT scale $M_{\rm GUT}$ and the unified coupling $\alpha_{\rm GUT}$ — so they impose one constraint on the 3 SM gauge couplings at $M_t \approx 173$ GeV. Indeed, taking the differences between the GQW equations, we obtain

$$\begin{aligned} \frac{1}{\alpha_2(M_t)} - \frac{1}{\alpha_3(M_t)} &= (b_2 - b_3) \times \frac{1}{2\pi} \log \frac{M_{\rm GUT}}{M_t}\,, \\ \frac{3/5}{\alpha_1(M_t)} - \frac{1}{\alpha_2(M_t)} &= \left(\tfrac{3}{5} b_1 - b_2\right) \times \frac{1}{2\pi} \log \frac{M_{\rm GUT}}{M_t}\,, \end{aligned} \tag{S.96}$$

and hence the ratio

$$X \equiv \left. \frac{\dfrac{3/5}{\alpha_1} - \dfrac{1}{\alpha_2}}{\dfrac{1}{\alpha_2} - \dfrac{1}{\alpha_3}} \right|_{E=M_t} = \frac{\frac{3}{5} b_1 - b_2}{b_2 - b_3}\,. \tag{S.97}$$

Experimentally,

$$X_{\rm exp} = \left. \frac{\dfrac{3/5}{\alpha_1} - \dfrac{1}{\alpha_2}}{\dfrac{1}{\alpha_2} - \dfrac{1}{\alpha_3}} \right|_{E=M_t} \approx \frac{\frac{3}{5} \times 97.84 - 30.03 \pm 0.01}{30.03 - 9.18 \pm 0.12} = 1.375 \pm 0.008. \tag{S.98}$$

On the other hand, using the b_3, b_2, b_1 coefficients from parts (a) and (b), we have

$$X_{\rm SM} = \frac{\frac{3}{5} \times \frac{41}{6} + \frac{19}{6}}{-\frac{19}{6} + 7} = \frac{218}{115} \approx 1.895 \tag{S.99}$$

for the non–SUSY minimal Standard Model, and

$$X_{\rm MSSM} = \frac{\frac{3}{5} \times 11 - 1}{1 + 3} = \frac{28}{20} = 1.400 \tag{S.100}$$

for the MSSM. By inspection, the predicted X ratio of the minimal non-SUSY Standard Model is rather off the mark. On the other hand, the MSSM prediction is much closer to the experimental value. The agreement is not perfect, but that's OK because the Georgi–Quinn–Weinberg equations (3) are not exact but follow from the one-loop approximation to the gauge couplings' β-functions. A more accurate analysis would take into account the two-loop corrections to these β-functions as well as the one-loop threshold corrections at the GUT scale. But that would take us well beyond the scope of this class, let alone this homework assignment.

Instead of delving into this issue, let's calculate the GUT scale $M_{\rm GUT}$ and the unified coupling $\alpha_{\rm GUT}$ for the MSSM. Using one of eqs. (S.96) — for example the top equation — we obtain

$$\big(b_2 - b_3 \;=\; 4\big) \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{M_t} \;=\; \frac{1}{\alpha_2(M_t)} \;-\; \frac{1}{\alpha_3(M_t)} \;=\; 20.85 \pm 0.12 \qquad \text{(S.101)}$$

hence

$$\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{M_t} \;=\; 5.21 \pm 0.03 \qquad \text{(S.102)}$$

and consequently

$$M_{GUT} \;\approx\; (2.9 \pm 0.5)\cdot 10^{16}\,\mathrm{GeV}. \qquad \text{(S.103)}$$

Given this value of the GUT scale, the unified coupling $\alpha_{\rm GUT}$ follows from any one of the GQW equations, for example

$$\frac{1}{\alpha_{\rm GUT}} \;=\; \frac{1}{\alpha_2(M_t)} \;-\; (b_2 = +1)\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{M_t} \;\approx\; 30.03 \;-\; 1\times 5.21 \;=\; 24.82. \qquad \text{(S.104)}$$

Problem **3**(f):

For the model at hand, the beta-functions of the gauge couplings — and their respective one-loop coefficients $b_{1,2,3}$ — depend on the energy scale: Above $M_{\rm top}$ but below the SUSY threshold $M_S = 2$ TeV the effective theory is the minimal non-SUSY Standard Model, thus

$$b_i \;=\; b_i^a \quad \text{from part (a),} \qquad \text{(S.105)}$$

while the SUSY threshold but below the GUT scale the effective theory is the MSSM and hence

$$b_i \;=\; b_i^b \quad \text{from part (b).} \tag{S.106}$$

Consequently, the gauge couplings at the SUSY threshold scale obey two sets of equations: On one hand, they are related to the experimental couplings (5) at $M_{\rm top}$ as

$$\forall i=1,2,3: \qquad \frac{1}{\alpha_i(M_S)} \;=\; \frac{1}{\alpha_i(M_{\rm top})} \;-\; b_i^a\times\frac{1}{2\pi}\log\frac{M_S}{M_{\rm top}} \tag{S.107}$$

where the coefficients b_i^a are as in the non-SUSY minimal SM, *cf.* part (a); but on the other hand, they are related to the GUT coupling $\alpha_{\rm GUT}$ by the Georgi–Quinn–Weinberg equations

$$\begin{aligned} \frac{1}{\alpha_3(M_S} &= \frac{1}{\alpha_{\rm GUT}} \;+\; b_3^b\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{M_S}\,, \\ \frac{1}{\alpha_2(M_S} &= \frac{1}{\alpha_{\rm GUT}} \;+\; b_2^b\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{M_S}\,, \\ \frac{1}{\alpha_1(M_S} &= \frac{5/3}{\alpha_{\rm GUT}} \;+\; b_1^b\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{M_S}\,, \end{aligned} \tag{S.108}$$

where the coefficients b_i^b are as in the MSSM, *cf.* part (b).

Evaluating eqs. (S.107) numerically, we obtain

$$\frac{1}{2\pi}\log\frac{M_S}{M_{\rm top}} \;=\; 0.390\pm0.002 \tag{S.109}$$

and hence

$$\begin{aligned} \frac{1}{\alpha_3(M_S)} &= (9.18\pm0.12) \;+\; 7\times(0.390\pm0.002) \;=\; 11.91\pm0.12, \\ \frac{1}{\alpha_2(M_S)} &= (30.028\pm0.005) \;+\; \frac{19}{6}\times(0.390\pm0.002) \;=\; 31.263\pm0.008, \\ \frac{1}{\alpha_1(M_S)} &= (97.84\pm0.01) \;-\; \frac{41}{6}\times(0.390\pm0.002) \;=\; 95.17\pm0.02.\pm \end{aligned} \tag{S.110}$$

To compare these couplings to the GQW relations (S.108), let us proceed as in part (e):

calculate the ratio

$$X(M_S) = \frac{\frac{3/5}{\alpha_1(M_S)} - \frac{1}{\alpha_2(M_S)}}{\frac{1}{\alpha_2(M_S)} - \frac{1}{\alpha_3(M_S)}} \tag{S.111}$$

for the couplings at the SUSY threshold, and then compare to the GQW prediction for the MSSM beta functions

$$X_{\text{MSSM}} = \frac{\frac{3}{5} \times b_1^b - b_2^b}{b_2^b - b_3^b} = \frac{28}{20} = 1.400. \tag{S.112}$$

Thus, plugging the numeric values (S.110) into eq. (S.111), we get

$$X(M_S) = \frac{\frac{3}{5} \times 95.170 - 31.263 \pm 0.013}{31.26 - 11.91 \pm 0.12} = 1.335 \pm 0.008. \tag{S.113}$$

This value is further away from the MSSM+GUT prediction (S.112) than the MSSM with the light super-partners from part (e), but not so far as to rule it out. Perhaps the two-loop corrections to the β-functions and the threshold corrections at the GUT scale can bridge the difference, but such analysis is way beyond the scope of this homework.