Problem **1**(a):

In the matrix notation for the non-abelian gauge fields,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \tag{S.1}$$

hence

$$\begin{aligned}\frac{g^2}{16\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(F_{\alpha\beta}F_{\gamma\delta}\Big) = {}& \frac{g^2}{4\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(\partial_\alpha A_\beta\,\partial_\gamma A_\delta\Big) && \text{[two gluons]}\\ &+ \frac{ig^3}{4\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big([A_\alpha, A_\beta]\partial_\gamma A_\delta\Big) && \text{[three gluons]}\\ &- \frac{g^4}{16\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big([A_\alpha, A_\beta]\,[A_\gamma, A_\delta]\Big) && \text{[four gluons].}\end{aligned} \tag{S.2}$$

Thanks to the cyclic symmetry of the trace, for any matrices X, Y, and Z,

$$\mathrm{tr}([X,Y]Z) = \mathrm{tr}(X[Y,Z]). \tag{S.3}$$

Applying this rule to the 4–gluon term in the decomposition (S.2), we have

$$[\text{4–gluon anomaly}] \propto \epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big([A_\alpha, A_\beta]\,[A_\gamma, A_\delta]\Big) = \epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(A_\alpha[A_\beta,[A_\gamma,A_\delta]]\Big). \tag{S.4}$$

In this double-commutator formula, we may use the Jacobi identity

$$[A_\beta,[A_\gamma,A_\delta]] + [A_\gamma,[A_\delta,A_\beta]] + [A_\delta,[A_\beta,A_\gamma]] = 0. \tag{S.5}$$

Since the $\epsilon^{\alpha\beta\gamma\delta}$ is symmetric with respect to *cyclic* permutations of the last three indices $\beta \to \gamma \to \delta \to \beta$, it follows that

$$\begin{aligned}3\epsilon^{\alpha\beta\gamma\delta}[A_\beta,[A_\gamma,A_\delta]] &= \epsilon^{\alpha\beta\gamma\delta}[A_\beta,[A_\gamma,A_\delta]] + \epsilon^{\alpha\gamma\delta\beta}[A_\gamma,[A_\delta,A_\beta]] + \epsilon^{\alpha\delta\beta\gamma}[A_\delta,[A_\beta,A_\gamma]]\\ &= \epsilon^{\alpha\beta\gamma\delta}\times\Big([A_\beta,[A_\gamma,A_\delta]] + [A_\gamma,[A_\delta,A_\beta]] + [A_\delta,[A_\beta,A_\gamma]]\Big)\\ &= 0\end{aligned} \tag{S.6}$$

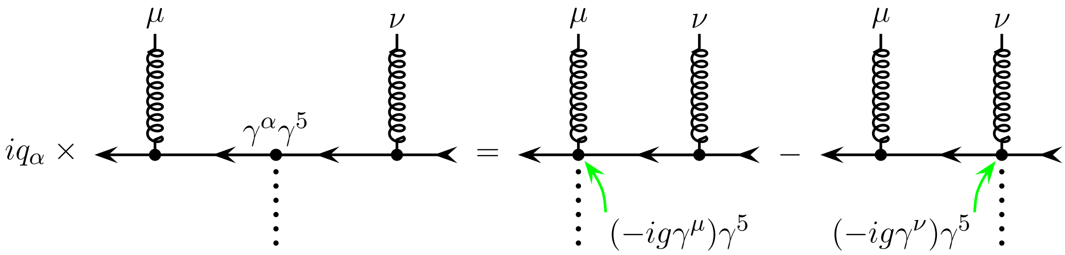
and hence

$$[\text{4–gluon anomaly}] = 0. \tag{S.7}$$

Quod erat demonstrandum.

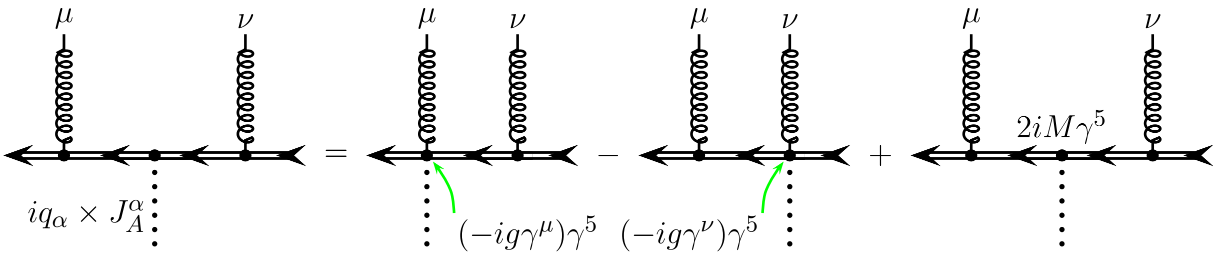
Problem **1**(c):

As explained in class, for a massless fermion



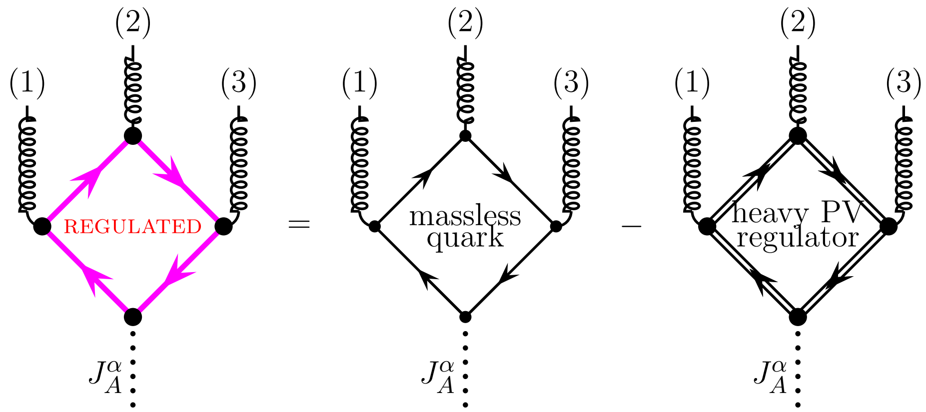
(S.8)

but for a massive fermion such as the Pauli–Villars compensator χ_{PV},



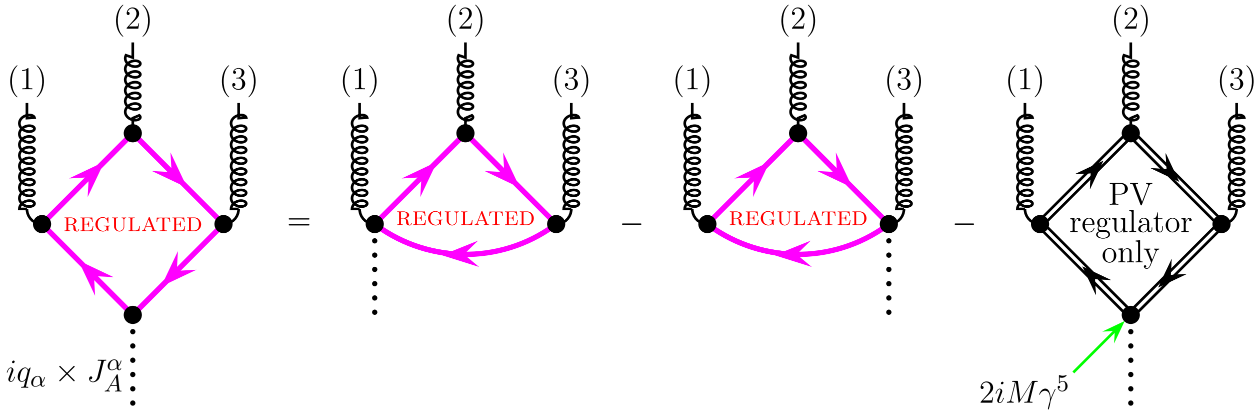
(S.9)

Now, let's apply these rules to the quadrangle anomaly diagram (3) regulated à la Pauli–Villars,



(S.10)

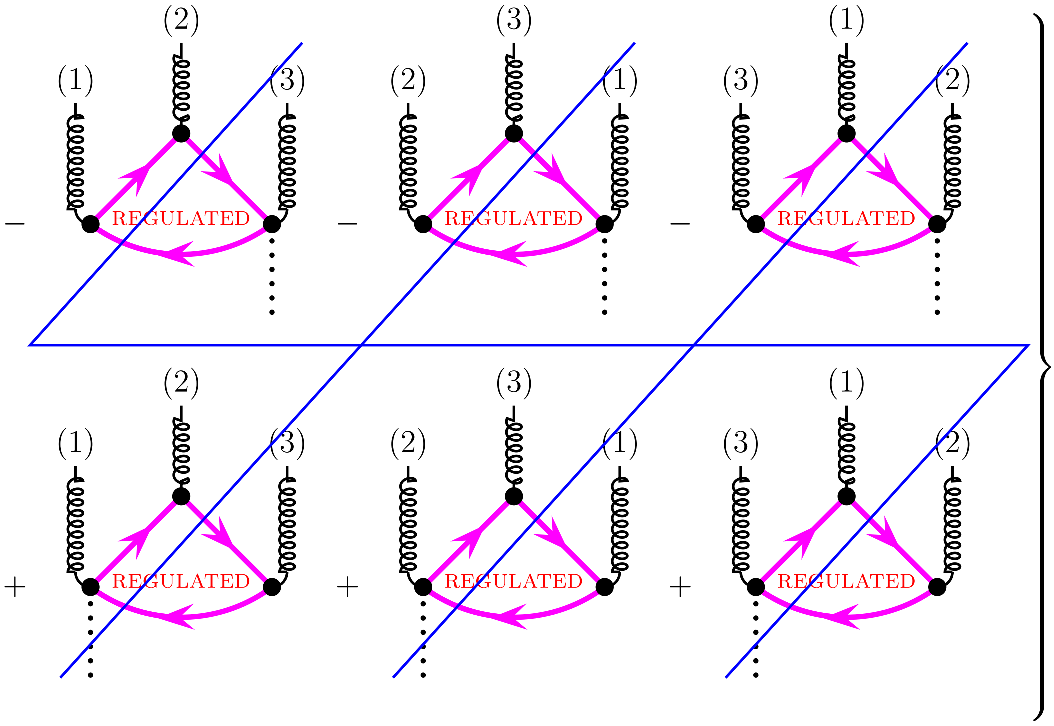
Multiplying the axial current $J^{5\alpha}(q)$ by the iq_α, we get



(S.11)

Without the regulation, the two triangular diagrams on the right hand side would diverge as (UV cutoff scale)$^{+1}$, but subtracting similar loops of the heavy Pauli–Villars fermions makes them converge. Consequently, we may shift the momentum integration variables $p^\mu \to p^\mu + \text{const}$ separately for each diagram, and this would lead to cancellation of the triangle diagrams once we sum over gluon permutations.

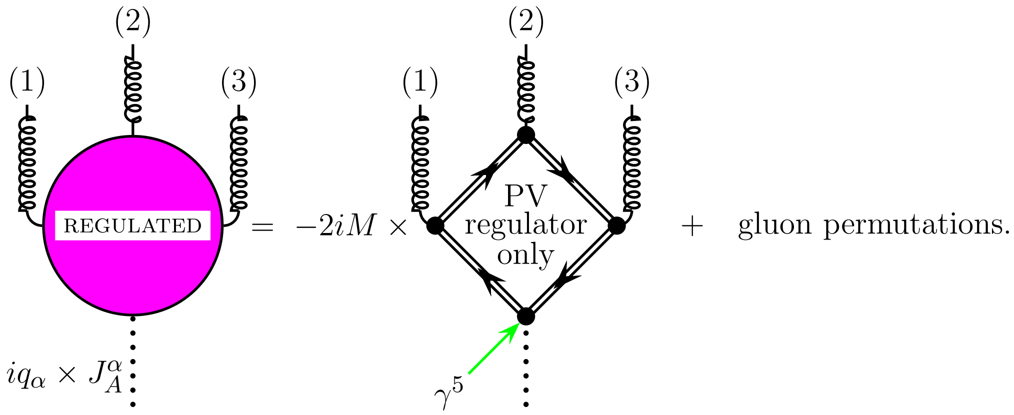
Actually, it is enough to sum over just the cyclic permutations $(1) \to (2) \to (3) \to (1)$ of the three gluons which have the same group factor $\text{tr}(T_{(3)}T_{(2)}T_{(1)})$ thanks to the cyclic symmetry of the trace. Summing the triangular diagrams on the right hand side of eq. (S.11), we have diagram-by-diagram cancellation:



$= 0$ (S.12)

Strictly speaking, this cancellation involves separate shifting of the integration momentum for each diagram, but that's OK since the regulated diagrams are finite.

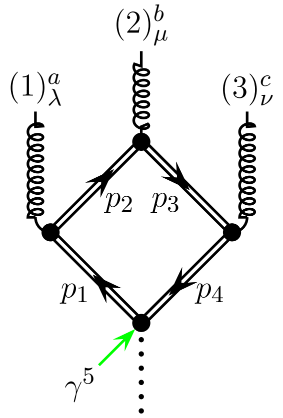
But the quadrangle diagrams involving only the massive PV regulator loops — *cf.* the last term in eq. (S.11) — do not cancel after we sum over gluon permutations, and that's what leads to the quadrangle anomaly:



(S.13)

Problem **1**(d):

For any particular order of the 3 gluons — for example, for the order explicitly shown on the RHS of eq. (S.13), we have

$A(1,2,3) = -2iM \times$  $= \mathrm{tr}_{\mathrm{color}}(T^cT^bT^a) \times -2iM \times \int \frac{d^4p_1}{(2\pi)^4}\,\mathrm{tr}_{\mathrm{Dirac}}(\cdots)$

(S.14)

where

$$\mathrm{tr}_{\mathrm{Dirac}}(\cdots) = \mathrm{tr}\left(\begin{array}{c}\gamma^5\times\frac{i}{\not{p}_4-M+i0}\times(-ig\gamma^\nu)\times\frac{i}{\not{p}_3-M+i0}\times\\ \times(-ig\gamma^\mu)\times\frac{i}{\not{p}_2-M+i0}\times(-ig\gamma^\lambda)\times\frac{i}{\not{p}_3-M+i0}\end{array}\right) \tag{S.15}$$

$$= ig^3\times\frac{\mathcal{N}}{\mathcal{D}}$$

for

$$\mathcal{D} = \prod_{i=1}^{4}(p_i^2-M^2+i0), \tag{S.16}$$

$$\mathcal{N} = \mathrm{tr}\Big(\gamma^5(\not{p}_4+M)\gamma^\nu(\not{p}_3+M)\gamma^\mu(\not{p}_2+M)\gamma^\lambda(\not{p}_1+M)\Big). \tag{S.17}$$

As usual, we may express the denominator in terms of the Feynman parameters x, y, z, w, thus

$$\frac{1}{\mathcal{D}} = 24\int d^4(x,y,z,w)\,\delta(x+y+z+w-1)\times\frac{1}{[\ell^2-O(k^2)-M^2+i0]^4} \tag{S.18}$$

where the details of the $O(k^2)$ expression are not important because the Pauli–Villars mass M is much bigger than all the external momenta. On the other hand, the relations between the shifted loop momentum ℓ^μ and the propagator momenta $p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu$ are rather important for the numerator (S.17), so here they are:

$$\begin{aligned}
p_1 &= \ell + q_1 \quad\text{for}\quad q_1 = -(y+z+w)k_1 - (z+w)k_2 - wk_3\,,\\
p_2 &= \ell + q_2 \quad\text{for}\quad q_2 = +xk_1 - (z+w)k_2 - wk_3\,,\\
p_3 &= \ell + q_3 \quad\text{for}\quad q_3 = +xk_1 + (x+y)k_2 - wk_3\,,\\
p_4 &= \ell + q_4 \quad\text{for}\quad q_4 = +xk_1 + (x+y)k_2 + (x+y+z)k_3\,,
\end{aligned} \tag{S.19}$$

thus

$$\mathcal{N} = \mathrm{tr}\Big(\gamma^5(\not{\ell}+M+\not{q}_4)\gamma^\nu(\not{\ell}+M+\not{q}_3)\gamma^\mu(\not{\ell}+M+\not{q}_2)\gamma^\lambda(\not{\ell}+M+\not{q}_1)\Big). \tag{S.20}$$

Before we evaluate this trace, let's put it in the context of the momentum integral

$$I = M\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}}{[\ell^2-M^2-O(q^2)+i0]^4} \tag{S.21}$$

in the $M\to\infty$ limit. This integral is UV-convergent, so it's dominated by the loop momenta $\ell\sim M$, hence by dimensional analysis we expect $\mathcal{N}=O(M^4)$ and $I=O(M)$. The $q_1,\ldots,q_4$

momenta in the numerator (S.20) and in the $O(q^2)$ term in the denominator make for small corrections, so let's expand the integral (S.21) in powers of q_i/M:

$$I(q_i) \;=\; M\times C^{(0)} \;+\; \sum_i q_i^\alpha C^{(1)}_{i,\alpha} \;+\; O\left(\frac{q^2}{M}\right). \tag{S.22}$$

Note that all the terms involving second or higher powers of q_i carry negative powers of the Pauli–Villars mass M, so in the eventual $M\to\infty$ limit they may be neglected. Thus, all we need to calculate are the q-independent term and the linear-in-q_i terms.

Consequently, we may expand both the numerator $\mathcal{N}$ and the denominator $\mathcal{D}=[\ell^2-m^2=O(q^2)+i0]^4$ in powers of q_i^μ and stop the expansion after the linear terms. For the denominator, this means simply

$$\frac{1}{\mathcal{D}} \;\approx\; \frac{1}{[\ell^2-M^2+i0]^4}\,, \tag{S.23}$$

while for the numerator (S.20) we have

$$\mathcal{N} \;\approx\; \mathcal{N}_0 \;+\; \sum_{i=1}^{4} q_{i,\alpha}\mathcal{N}_i^\alpha \tag{S.24}$$

where

$$\begin{aligned}
\mathcal{N}_0 \;&=\; \mathrm{tr}\Big(\gamma^5(\not\ell+M)\gamma^\nu(\not\ell+M)\gamma^\mu(\not\ell+M)\gamma^\lambda(\not\ell+M)\Big),\\
\mathcal{N}_1 \;&=\; \mathrm{tr}\Big(\gamma^5(\not\ell+M)\gamma^\nu(\not\ell+M)\gamma^\mu(\not\ell+M)\gamma^\lambda\gamma^\alpha\Big),\\
\mathcal{N}_2 \;&=\; \mathrm{tr}\Big(\gamma^5(\not\ell+M)\gamma^\nu(\not\ell+M)\gamma^\mu\gamma^\alpha\gamma^\lambda(\not\ell+M)\Big),\\
\mathcal{N}_3 \;&=\; \mathrm{tr}\Big(\gamma^5(\not\ell+M)\gamma^\nu\gamma^\alpha\gamma^\mu(\not\ell+M)\gamma^\lambda(\not\ell+M)\Big),\\
\mathcal{N}_4 \;&=\; \mathrm{tr}\Big(\gamma^5\gamma^\alpha\gamma^\nu(\not\ell+M)\gamma^\mu(\not\ell+M)\gamma^\lambda(\not\ell+M)\Big).
\end{aligned} \tag{S.25}$$

Now let's evaluate these traces, starting with

$$\begin{aligned}
\mathcal{N}_1 \;&=\; M^3\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\alpha\Big)\\
&\qquad +\; M\times\mathrm{tr}\Big(\gamma^5\big(\not\ell\gamma^\nu\not\ell\gamma^\mu+\not\ell\gamma^\nu\gamma^\mu\not\ell\;+\;\gamma^\nu\not\ell\gamma^\mu\not\ell\big)\gamma^\lambda\gamma^\alpha\Big)\\
&\quad\langle\!\langle\,\text{using }\not\ell\gamma^\nu\not\ell\gamma^\mu+\not\ell\gamma^\nu\gamma^\mu\not\ell\;+\;\gamma^\nu\not\ell\gamma^\mu\not\ell\;=\;4\ell^\nu\ell^\mu\;-\;\ell^2\gamma^\nu\gamma^\mu\,\rangle\!\rangle\\
&=\; M(M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\alpha\Big)\;+\;4M\ell^\nu\ell^\mu\times\mathrm{tr}\Big(\gamma^5\gamma^\lambda\gamma^\alpha\Big)\\
&=\; M(M^2-\ell^2)\times 4i\epsilon^{\nu\mu\lambda\alpha}\;+\;4M\ell^\nu\ell^\mu\times 0\\
&=\; 4iM(M^2-\ell^2)\epsilon^{\nu\mu\lambda\alpha}.
\end{aligned}\tag{S.26}$$

Similarly,

$$\mathcal{N}_4\;=\;4iM(M^2-\ell^2)\epsilon^{\alpha\nu\mu\lambda}.\tag{S.27}$$

In the remaining three traces we move the last $(\not\ell+M)$ factor forward and use

$$(\not\ell+M)\gamma^5(\not\ell+M)\;=\;(\not\ell+M)(M-\not\ell)\gamma^5\;=\;(M^2-\ell^2)\times\gamma^5.\tag{S.28}$$

Consequently,

$$\begin{aligned}
\mathcal{N}_2\;&=\;(M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu(\not\ell+M)\gamma^\mu\gamma^\alpha\gamma^\lambda\Big)\\
&=\;4iM(M^2-\ell^2)\epsilon^{\nu\mu\alpha\lambda},\\
\mathcal{N}_3\;&=\;(M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\alpha\gamma^\mu(\not\ell+M)\gamma^\lambda\Big)\\
&=\;4iM(M^2-\ell^2)\epsilon^{\nu\alpha\mu\lambda},\\
\mathcal{N}_0\;&=\;(M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu(\not\ell+M)\gamma^\mu(\not\ell+M)\gamma^\lambda\Big)\\
&=\;(M^2-\ell^2)\times M\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\{\not\ell,\gamma^\mu\}\gamma^\lambda\Big)\\
&=\;M(M^2-\ell^2)\times 2\ell^\mu\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\lambda\Big)\\
&=\;0.
\end{aligned}$$

Thus, the q-independent term in the numerator vanishes while the linear-in-q terms all have the same form apart from the order of indices in the ϵ tensor. Reordering the indices and changing the sign of a_i as necessary, we arrive at

$$\mathcal{N} \;=\; 4iM(M^2-\ell^2)\epsilon^{\nu\mu\lambda\alpha}(q_1-q_2+q_3-q_4)_\alpha + \; O(M^2q^2). \tag{S.29}$$

Moreover, according to eqs. (S.19)

$$q_1 \,-\, q_2 \,+\, q_3 \,-\, q_4 \;=\; -k_1 \,-\, k_3\,, \tag{S.30}$$

hence

$$\mathcal{N} \;=\; -4iM\epsilon^{\nu\mu\lambda\alpha}(k_1+k_3)_\alpha \times (M^2-\ell^2) \tag{S.31}$$

which does not depend on any Feynman parameters, and only the last factor depends on the loop momentum ℓ. Consequently, plugging this formula for $\mathcal{N}$ into eqs. (S.14) and (S.15) gives us

$$\begin{aligned} A(1,2,3) \;&=\; \mathrm{tr}(T^aT^bT^c)\times 8ig^3M^2\,\epsilon^{\nu\mu\lambda\alpha}(k_1+k_3)_\alpha \times 24\int d^4(x,y,z,w)\,\delta(x+y+z+w-1) \\ &\qquad\times\int\frac{d^4\ell}{(2\pi^4)}\,\frac{(M^2-\ell^2)}{(\ell^2-M^2+i0)^4}\,, \end{aligned} \tag{S.32}$$

where the momentum integral is independent on any of the Feynman parameters (as long as $M \gg$ all q_i). Consequently, the Feynman parameter integral evaluates to

$$24\int_0^1 d^4(x,y,z,w)\,\delta(x+y+z+w-1) \;=\; 1, \tag{S.33}$$

while the momentum integral yields

$$\begin{aligned} \int\frac{d^4\ell}{(2\pi^4)}\,\frac{(M^2-\ell^2)}{(\ell^2-M^2+i0)^4} \;&=\; \int\frac{d^4\ell}{(2\pi^4)}\,\frac{-1}{(\ell^2-M^2+i0)^3} \\ &=\; \int\frac{id^4\ell_e}{(2\pi^4)}\,\frac{+1}{(\ell_e^2+M^2)^3} \\ &=\; \frac{i}{16\pi^2}\int_0^\infty\frac{\ell_e^2\,d(\ell_e^2)}{(\ell_e^2+M^2)^3} \\ &=\; \frac{i}{16\pi^2}\times\frac{1}{2M^2}\,. \end{aligned} \tag{S.34}$$

Plugging these integrals back into eq. (S.32) finally yields

$$A(1,2,3) \;=\; \mathrm{tr}(T^aT^bT^c)\times\frac{-g^3}{4\pi^2}\,\epsilon^{\nu\mu\lambda\alpha}(k_1+k_3)_\alpha\,. \tag{S.35}$$

<u>Problem **1**(e)</u>:

The amplitude (S.35) stems from one particular quadrangle diagram (S.14) for a particular ordering of the 3 gluons. The net 3-gluon amplitude should be summed over all 3! gluon permutations, thus

$$\text{anomaly} \;=\; \Bigl(A(1,2,3)+A(2,3,1)+A(3,1,2)\Bigr)+\Bigl(A(2,1,3)+A(3,2,1)+A(1,3,2)\Bigr). \tag{S.36}$$

For convenience, we have grouped the 6 terms here according to the cyclic orders of the three gluons, because for each cyclic order we have the same color trace

$$\mathrm{tr}(T^aT^bT^c) \;=\; \mathrm{tr}(T^bT^cT^a) \;=\; \mathrm{tr}(T^cT^aT^b) \neq \mathrm{tr}(T^bT^aT^c) \;=\; \mathrm{tr}(T^cT^bT^a) \;=\; \mathrm{tr}(T^aT^cT^b) \tag{S.37}$$

and the same sign of the Levi–Civita tensor,

$$\Bigl(\epsilon^{\nu\mu\lambda\alpha} \;=\; \epsilon^{\mu\lambda\nu\alpha} \;=\; \epsilon^{\lambda\nu\mu\alpha}\Bigr) \;=\; -\Bigl(\epsilon^{\mu\nu\lambda\alpha} \;-\; \epsilon^{\lambda\mu\nu\alpha} \;=\; \epsilon^{\nu\lambda\mu\alpha}\Bigr). \tag{S.38}$$

Consequently,

$$\begin{aligned} A(1,2,3)\;+\;A(2,3,1)\;+\;A(3,1,2) \;&=\; -\frac{g^3}{4\pi^2}\,\mathrm{tr}(T^aT^bT^c)\times\epsilon^{\nu\mu\lambda\alpha}\times \\ &\quad\times\Bigl((k_1+k_3)_\alpha\;+\;(k_2+k_1)_\alpha\;+\;(k_3+k_2)_\alpha\Bigr) \\ &=\; -\frac{g^3}{4\pi^2}\,\mathrm{tr}(T^aT^bT^c)\times\epsilon^{\nu\mu\lambda\alpha}\times 2(k_1+k_2+k_3)_\alpha \\ &=\; -\frac{g^2}{2\pi^2}\,\mathrm{tr}(T^aT^bT^c)\times\epsilon^{\nu\mu\lambda\alpha}q_\alpha\,, \end{aligned} \tag{S.39}$$

and likewise

$$A(2,1,3)\;+\;A(3,2,1)\;+\;A(1,3,2) \;=\; -\frac{g^2}{2\pi^2}\,\mathrm{tr}(T^aT^cT^b)\times -\epsilon^{\nu\mu\lambda\alpha}q_\alpha\,. \tag{S.40}$$

Altogether, we have

$$\begin{aligned}
[\text{quadrangle anomaly}] &= -\frac{g^3}{2\pi^2}\,\epsilon_{\nu\mu\lambda\alpha}q_\alpha \times \operatorname{tr}(T^aT^bT^c \,-\, T^aT^cT^b) \\
&= +\frac{g^3}{2\pi^2}\,q_\alpha\epsilon^{\alpha\lambda\mu\nu}\,\operatorname{tr}(T^a[T^b,T^c]).
\end{aligned} \tag{S.41}$$

Or in terms of the local gluons fields A^a_λ, A^b_μ and A^c_ν,

$$\begin{aligned}
\partial_\alpha J^\alpha_A\big|_{3g} &= \frac{1}{3!}\times\frac{g^3}{2\pi^2}\,i\partial_\alpha\Big(\epsilon^{\alpha\lambda\mu\nu}\operatorname{tr}\big(A_\lambda[A_\mu,A_\nu]\big)\Big) \\
&\quad\langle\!\langle\,\text{using } \operatorname{tr}(A_\lambda[A_\mu,A_\nu]) = \operatorname{tr}(A_\mu[A_\nu,A_\lambda]) = \operatorname{tr}(A_\nu[A_\lambda,A_\mu])\,\rangle\!\rangle \\
&= \frac{ig^3}{12\pi^2}\epsilon^{\alpha\lambda\mu\nu}\Big(\operatorname{tr}\big((\partial_\alpha A_\lambda)[A_\mu,A_\nu]\big) \,+\, \operatorname{tr}\big((\partial_\alpha A_\mu)[A_\nu,A_\lambda]\big) \,+\, \operatorname{tr}\big((\partial_\alpha A_\nu)[A_\lambda,A_\mu]\big)\Big) \\
&\quad\langle\!\langle\,\text{renaming } \lambda\to\mu\to\nu\to\lambda\text{, which also preserves the } \epsilon \text{ tensor}\,\rangle\!\rangle \\
&= \frac{ig^3}{12\pi^2}\times 3\epsilon^{\alpha\lambda\mu\nu}\,\operatorname{tr}\big((\partial_\alpha A_\lambda)[A_\mu,A_\nu]\big).
\end{aligned} \tag{S.42}$$

Comparing this formula to the second line of eq. (S.2) (part (a)), we see that the quadrangle diagrams (3) generate precisely the three–gluon part of the non-abelian anomaly (1).

Problem **3**(a):

For the sake of definiteness, let's focus on the decay of the positive pion, $\pi^+ \to \mu^+\nu_\mu$. In the Fermi's current-current theory (6) of this weak decay, the J^- current annihilates the positive pion while the J^+ current creates the positive muon and the neutrino. In light of eqs. (7) for the currents,

$$\big\langle\mu^+\nu_\mu\big|\,\hat J^{+\alpha}_L\,|0\rangle \;=\; \big\langle\mu^+\nu_\mu\big|\,\tfrac12\overline\Psi_{\nu_\mu}(1-\gamma^5)\gamma^\alpha\Psi_\mu\,|0\rangle \;=\; \tfrac12\bar u(\nu_\mu)(1-\gamma^5)\gamma^\alpha v(\mu^+), \tag{S.43}$$

while

$$\langle 0|\,\hat J^-_{\alpha,L}\,\big|\pi^+\big\rangle \;=\; \frac{\cos\theta_c}{2}\times\Big(\langle 0|\,\overline\Psi_d\gamma_\alpha\Psi_u\,\big|\pi^+\big\rangle \,-\, \langle 0|\,\overline\Psi_d\gamma^5\gamma_\alpha\Psi_u\,\big|\pi^+\big\rangle\Big). \tag{S.44}$$

Inside the () on the RHS, the vector isospin currents have a zero matrix element between the pion and the vacuum states (since the vector isospin symmetry is unbroken), but since the pion is the (pseudo) Goldstone bosons of the spontaneously broken axial symmetry, the axial currents do annihilate the pions with matrix elements

$$\langle \mathrm{vac}| \hat{J}^{a\alpha}_{A} \left|\pi^b\right\rangle \;=\; -if_\pi p^\alpha(\pi)\delta^{ab}. \tag{S.45}$$

However, because of the different normalization of the charged pion states vs. charged currents in terms of the real members of an isotriplet,

$$\left|\pi^+\right\rangle \;=\; \sqrt{\tfrac{1}{2}}\big(\left|\pi^1\right\rangle \,+\, i\left|\pi^2\right\rangle\big) \quad \text{while} \quad J^-_\alpha \;=\; J^1_\alpha \;-\; iJ^2_\alpha, \tag{S.46}$$

the δ^{ab} factor in eq. (S.45) becomes

$$\frac{1+1}{\sqrt{2}} \;=\; \sqrt{2}, \tag{S.47}$$

thus

$$\langle \mathrm{vac}| \hat{J}^{\alpha-}_{A}(\text{isospin}) \left|\pi^+\right\rangle \;=\; \langle \mathrm{vac}| \overline{\Psi}(d)\gamma^\alpha\gamma^5\Psi(u) \left|\pi^+\right\rangle = \; -i\sqrt{2}f_\pi p^\alpha(\pi^+). \tag{S.48}$$

In terms of the charged weak current, this means

$$\langle \mathrm{vac}| \hat{J}^{\alpha-}_{L}[\text{weak} \left|\pi^+\right\rangle \;=\; +i\frac{\sqrt{2}}{2}\cos\theta_c \times f_\pi p^\alpha(\pi^+), \tag{S.49}$$

exactly as in eq. (8). Finally, combining this matrix element for the pion with the leptonic matrix element (S.43) and plugging them into the Fermi's effective Lagrangian (6), we end up with the pion decay amplitude

$$\mathcal{M} \;=\; \left\langle \mu^+\nu_\mu\right| \mathcal{L}_{\text{Fermi}} \left|\pi^+\right\rangle \;=\; iG_F\, f_\pi\, \cos\theta_c \times p_\alpha(\pi) \times \bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha v(\mu^+), \tag{S.50}$$

exactly as in eq. (9).

Problem **3**(b):

Let's sum the amplitude (9) — or rather its mod-square — over the spin states of the outgoing muon and neutrino. As usual, we get a Dirac trace

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}|^2 &= G_F^2 f_\pi^2 \cos^2\theta_c \times \sum_{\text{spins}} \left|\bar{u}_\nu(1-\gamma^5)\,\not{p}_\pi v_\mu\right|^2 \\ &= G_F^2 f_\pi^2 \cos^2\theta_c \times \operatorname{tr}\Big((\not{p}_\nu + m_\nu)\times(1-\gamma^5)\,\not{p}_\pi \times (\not{p}_\mu - M_\mu)\times \overline{(1-\gamma^5)\,\not{p}_\pi}\Big)\end{aligned} \tag{S.51}$$

where

$$\overline{(1-\gamma^5)\,\not{p}_\pi} = \overline{\not{p}_\pi}\times(1-\overline{\gamma^5}) = \not{p}_\pi\times(1+\gamma^5) = (1-\gamma^5)\,\not{p}_\pi\,. \tag{S.52}$$

Since the neutrino's mass m_ν is so much smaller than any other mass and momentum in the process, we may safely neglect it, hence

$$\begin{aligned}\operatorname{tr}(\cdots) &= \operatorname{tr}\Big(\not{p}_\nu(1-\gamma^5)\,\not{p}_\pi(\not{p}_\mu - M_\mu)(1-\gamma^5)\,\not{p}_\pi\Big) \\ &= \operatorname{tr}\Big((1-\gamma^5)\,\not{p}_\pi\,\not{p}_\nu(1-\gamma^5)\,\not{p}_\pi(\not{p}_\mu - M_\mu)\Big) \\ &= \operatorname{tr}\Big((1-\gamma^5)^2\,\not{p}_\pi\,\not{p}_\nu\,\not{p}_\pi(\not{p}_\mu - M_\mu)\Big) \qquad \langle\!\langle \text{ because } \not{p}_\pi\,\not{p}_\nu(1-\gamma^5) = (1-\gamma^5)\,\not{p}_\pi\,\not{p}_\nu \,\rangle\!\rangle \\ &= 2\operatorname{tr}\Big((1-\gamma^5)\,\not{p}_\pi\,\not{p}_\nu\,\not{p}_\pi(\not{p}_\mu - M_\mu)\Big) \qquad \langle\!\langle \text{ because } (1-\gamma^5)^2 = 2(1-\gamma^5) \,\rangle\!\rangle \\ &= 2\operatorname{tr}\Big((1-\gamma^5)\,\not{p}_\pi\,\not{p}_\nu\,\not{p}_\pi\,\not{p}_\mu\Big) \qquad \langle\!\langle \text{ because } \operatorname{tr}(\not{a}\,\not{b}\,\not{c}) = \operatorname{tr}(\gamma^5\not{a}\,\not{b}\,\not{c}) = 0 \,\rangle\!\rangle \\ &= 16(p_\pi p_\nu)(p_\pi p_\mu) - 8(p_\pi^2)(p_\nu p_\mu) - 8i\epsilon_{\alpha\beta\gamma\delta}p_\pi^\alpha p_\nu^\beta p_\pi^\gamma p_\mu^\delta\,.\end{aligned} \tag{S.53}$$

Moreover, the last term on the bottom line vanishes by the antisymmetry of the ϵ tensor $\Longrightarrow$ $\epsilon_{\alpha\beta\gamma\delta}p_\pi^\alpha p_\pi^\gamma = 0$, so we are left with

$$\sum_{\text{spins}} |\mathcal{M}|^2 = G_F^2 f_\pi^2 \cos^2\theta_c \times \Big(16(p_\pi p_\nu)(p_\pi p_\mu) - 8(p_\pi^2)(p_\nu p_\mu)\Big). \tag{S.54}$$

Now consider the kinematics of the pion decay. Since $p_\pi = p_\mu + p_\nu$ and all 3 momenta are on-shell, we have

$$\begin{aligned}2(p_\pi p_\mu) &= (p_\pi)^2 + (\pi_\mu)^2 - (p_\nu = p_\pi - p_\mu)^2 = M_\pi^2 + M_\mu^2 - 0, \\ 2(p_\pi p_\nu) &= (p_\pi)^2 + (\pi_\nu)^2 - (p_\mu = p_\pi - p_\nu)^2 = M_\pi^2 + 0 - M_\mu^2, \\ 2(p_\mu p_\nu) &= (\pi_\pi = \pi_\mu + \pi_\nu)^2 - (\pi_\mu)^2 - (\pi_\nu)^2 = M_\pi^2 - M_\mu^2 - 0.\end{aligned} \tag{S.55}$$

Consequently,

$$\begin{aligned} 16(p_\pi p_\nu)(p_\pi p_\mu) \;-\; 8(p_\pi^2)(p_\nu p_\mu) \;&=\; 4(M_\pi^2 - M_\mu^2)(M_\pi^2 + M_\mu^2) \;-\; 4M_\pi^2(M_\pi^2 - M_\mu^2) \\ &=\; 4(M_\pi^2 - M_\mu^2) \times M_\mu^2, \end{aligned} \tag{S.56}$$

and therefore

$$\sum_{\text{spins}} |\mathcal{M}(\pi^+ \to \mu^+\nu_\mu|^2 \;=\; 4G_F^2 f_\pi^2 \cos^2\theta_c \times (M_\pi^2 - M_\mu^2) \times M_\mu^2. \tag{S.57}$$

Finally, the phase-space factor for the two-body decay of a spinless particle is

$$d(\text{phase space}) \;=\; \frac{|\mathbf{p}'_{\text{c.m.}}|}{32\pi^2 M_\pi^2} \times d^2\Omega_{\text{c.m.}} \quad\Longrightarrow\quad \text{net phase space} \;=\; \frac{|\mathbf{p}'_{\text{c.m.}}|}{8\pi M_\pi^2} \tag{S.58}$$

where

$$|\mathbf{p}'_{\text{c.m.}}| \;=\; E_\nu(\text{pion frame}) \;=\; \frac{2(p_\pi p_\nu)}{2M_\pi} \;=\; \frac{M_\pi^2 - M_\mu^2}{2M_\pi}. \tag{S.59}$$

Altogether, tree-level pion decay rate comes out to be

$$\begin{aligned} \Gamma(\pi^+ \to \mu^+\nu_\mu) \;&=\; \sum_{\text{spins}} |\mathcal{M}(\pi^+ \to \mu^+\nu_\mu|^2 \times \text{net phase space} \\ &=\; 4G_F^2 f_\pi^2 \cos^2\theta_c \times (M_\pi^2 - M_\mu^2) M_\mu^2 \times \frac{M_\pi^2 - M_\mu^2}{16\pi M_\pi^3} \\ &=\; \frac{G_F^2 f_\pi^2 \cos^2\theta_c M_\pi}{4\pi} \times M_\mu^2 \left(1 - \frac{M_\mu^2}{M_\pi^2}\right)^2. \end{aligned} \tag{S.60}$$

Problem **3**(c):

Numerically, for $M_\pi \approx 14\underline{0}$ MeV, $M_\mu \approx 106$ MeV, $f_\pi \approx$ 93 MeV, $G_F = 1.17 \cdot 10^{-5}\,\text{GeV}^{-1}$, and $\theta_c \approx 13^\circ$, eq. (S.60) yields

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) \;\approx\; 25.0 \cdot 10^{-9}\ \text{eV} \quad\Longrightarrow\quad \frac{1}{\Gamma} \;\approx\; 2.63 \cdot 10^{-8}\ \text{seconds}. \tag{S.61}$$

This is quite close to the experimental mean lifetime of a charged pion $\tau = 2.603 \cdot 10^{-8}$ s.

Problem $\mathbf{3}$(d):

The charged weak current $J_L^{+\alpha}$ can create a positron and an electron-type neutrino just as well as it can create a positive muon and a muon-type neutrino. The amplitude for both $\pi^+ \to e^+\nu_e$ and $\pi^+ \to \mu^+\nu_\mu$ processes is given by exactly the same equation (13), hence proceeding exactly as in part (b) we find

$$\Gamma(\pi^+ \to e^+\nu_e) \;=\; \frac{G_F^2 f_\pi^2 \cos^2\theta_c M_\pi}{4\pi} \times M_e^2 \left(1 - \frac{M_e^2}{M_\pi^2}\right)^2 . \tag{S.62}$$

But comparing this formula to eq. (S.60), we find that $M_e \ll M_\mu$ leads to a much smaller decay rate into electrons rather than muons,

$$\begin{aligned}\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} &= \frac{G_F^2 f_\pi^2 \cos^2\theta_c M_\pi}{8\pi} \times M_e^2 \left(1 - \frac{M_e^2}{M_\pi^2}\right)^2 \Bigg/ \frac{G_F^2 f_\pi^2 \cos^2\theta_c M_\pi}{8\pi} \times M_\mu^2 \left(1 - \frac{M_\mu^2}{M_\pi^2}\right)^2 \\ &= \frac{M_e^2}{M_\mu^2} \times \frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \;\approx\; 1.24\cdot 10^{-4}.\end{aligned} \tag{10}$$

To explain this very small ratio of decay rates, consider the chirality structure of the amplitude (9). The $1-\gamma^5$ factor projects the Dirac factors $v(\mu^+)$ and $u(\nu_\mu)$ onto their left-handed components. Indeed, in the frame of the initial pion

$$\begin{aligned}\mathcal{M} &\propto u^\dagger(\nu_\mu)\gamma^0(1-\gamma^5)\gamma^0 v(\mu^+) \\ &= u^\dagger(\nu_\mu)\begin{pmatrix} 1_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2}\end{pmatrix} v(\mu^+) \\ &= u_L^\dagger(\nu_\mu) v_L(\mu^+),\end{aligned} \tag{S.63}$$

so both fermions must have left-handed chirality.

At the same time, the angular momentum conservation requires the two fermions to have similar helicities. Indeed, since the pion has no spin, in its rest frame $\mathbf{J}_{\rm net} = 0$ and hence $\mathbf{J}_\nu = -\mathbf{J}_\mu$. But the muon and the neutrino travel in opposite directions, $\mathbf{p}_\nu = -\mathbf{p}_\mu$, so opposite angular momenta mean similar helicities:

$$\lambda = \frac{\mathbf{p}\cdot\mathbf{J}}{|\mathbf{p}|} \implies \lambda_\nu = +\lambda_\mu\,. \tag{S.64}$$

Finally, for *massless* particles, the helicity follows from the chirality and vice verse, but the relation is different for the particles and the antiparticles:

- $\oplus$ For a particle — such as e^-, or μ^-, or ν — helicity and chirality have the same sign — they are both left or both right.
- $\ominus$ But for an antiparticle — such as e^+ or μ^+ — helicity and chirality have opposite signs.

So if both the neutrino and the charged muon were massless, similar helicities according to eq. (S.64) would require them to have opposite chiralities, one left and the other right. At the same time, the weak interactions (S.63) require left chiralities of both fermions. Since the two requirements contradict each other, the weak decay of a pion into two massless leptons is impossible.

To make the decay happen, one of the two leptons must have mismatched helicity and chirality, and this requires mass. Since the neutrino is much lighter than the charged lepton, the mismatch happens for the μ^+: it has left chirality but also left helicity (because the neutrino has $\lambda = -\frac{1}{2}$), which is a mismatch for an anti-lepton. The cost of this mismatch is a suppression factor $\sqrt{1-\beta} \propto M/E$ in the decay amplitude, which comes from the muon Dirac wave

$$v(\mu^+_L) = \begin{pmatrix} +\sqrt{E-p}\,\eta_L \\ -\sqrt{E+p}\,\eta_L \end{pmatrix} \implies v_L(\mu^+_L) = \sqrt{1-\beta}\times\sqrt{2E}\,\eta_L\,. \tag{S.65}$$

This factor is not too small for the mildly-relativistic muon, but it would be much smaller for an ultra-relativistic positron in the $\pi^+ \to e^+\nu_e$ decay. And that's why spinless mesons do not like to decay into light leptons: the smaller the lepton mass, the smaller the Dirac wave v or u becomes for the mismatched chirality and helicity.

Problem **4**(a):

At the tree level of the effective theory, the $\pi^0 \to \gamma\gamma$ decay amplitude obtains directly from the interaction term in eq. (14):

$$\mathcal{M} \;=\; 2 \times \frac{e^2}{32\pi^2 f_\pi} \times \epsilon^{\alpha\beta\mu\nu} f^*_{\alpha\beta}(1^{\rm st}\ \text{photon}) f^*_{\mu\nu}(2^{\rm nd}\ \text{photon}), \tag{S.66}$$

where the overall factor of 2 comes from 2 identical photons involved in the interactions. Also, $f_{\alpha\beta}$ and $f_{\mu\nu}$ are the polarization tensors of the two photons expressed in terms of the tension field instead of the vector potential. In terms of the polarization vectors,

$$f_{\alpha\beta} \;=\; -ik_\alpha e_\beta \;+\; ik_\beta e_\alpha \tag{S.67}$$

and likewise for the second photon, hence

$$\epsilon^{\alpha\beta\mu\nu} f^*_{\alpha\beta}(1) f^*_{\mu\nu}(2) \;=\; \epsilon^{\alpha\beta\mu\nu} \left(-ik_\alpha e_\beta + ik_\beta e_\alpha\right)^*_1 \left(-ik_\mu e_\nu + ik_\nu e_\mu\right)^*_2 \;=\; -4\epsilon^{\alpha\beta\mu\nu} (k_\alpha e_\beta)^*_1 (k_\mu e_\nu)^*, \tag{S.68}$$

and therefore

$$\mathcal{M} \;=\; -\frac{e^2}{4\pi^2 f_\pi}\, \epsilon^{\alpha\beta\mu\nu} (k_\alpha e_\beta)^*_1 (k_\mu e_\nu)^* \tag{15}$$

Problem **4**(b):

Let's re-express the amplitude (15) as

$$\mathcal{M} \;=\; \mathcal{M}^{\beta\nu} \times e^*_{1\beta} e^*_{2\nu} \quad \text{for} \quad \mathcal{M}^{\beta\nu} \;=\; -\frac{\alpha}{\pi f_\pi}\, \epsilon^{\alpha\beta\mu\nu}\, k_{1\alpha} k_{2\mu}\,. \tag{S.69}$$

As we have learned back in November — *cf.* my notes on Ward identities and polarization sums — summing mod-square of this amplitude over the two photon's polarizations amounts to

$$\overline{|\mathcal{M}|^2} \;=\; \sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; \mathcal{M}^{\mu\nu} \mathcal{M}^*_{\mu\nu}. \tag{S.70}$$

For the problem at hand, this means

$$\overline{|\mathcal{M}|^2} \;=\; \mathcal{M}^{\beta\nu} \mathcal{M}^*_{\beta\nu} \;=\; \left(\frac{\alpha}{\pi f_\pi}\right)^2 \times \left(\epsilon^{\alpha\beta\mu\nu}\, k_{1\alpha} k_{2\mu}\right) \left(\epsilon_{\gamma\beta\lambda\nu}\, k_{1\gamma} k_{2\lambda}\right), \tag{S.71}$$

where

$$\epsilon^{\alpha\beta\mu\nu}\epsilon_{\gamma\beta\lambda\nu} \;=\; -2\delta^\alpha_\gamma\delta^\mu_\lambda \;+\; \delta^\alpha_\lambda\delta^\mu_\gamma \tag{S.72}$$

and therefore

$$\left(\epsilon^{\alpha\beta\mu\nu}\,k_{1\alpha}k_{2\mu}\right)\left(\epsilon_{\gamma\beta\lambda\nu}\,k_{1\gamma}k_{2\lambda}\right) \;=\; -2(k_1^2)(k_2^2) \;+\; 2(k_1k_2)^2. \tag{S.73}$$

Moreover, on shell $k_1^2 = k_2^2 = 0$ while

$$2(k_1k_2) \;=\; (k_1 + k_2 = p_\pi)^2 \;-\; k_1^2 \;-\; k_2^2 \;=\; M_\pi^2, \tag{S.74}$$

which leads to

$$\left(\epsilon^{\alpha\beta\mu\nu}\,k_{1\alpha}k_{2\mu}\right)\left(\epsilon_{\gamma\beta\lambda\nu}\,k_{1\gamma}k_{2\lambda}\right) \;=\; 2(k_1k_2)^2 \;=\; \frac{M_\pi^4}{2} \tag{S.75}$$

and therefore

$$\overline{|\mathcal{M}|^2} \;=\; \frac{\alpha^2 M_\pi^4}{2\pi^2 f_\pi^2}\,. \tag{S.76}$$

Finally, the phase-space factor for a two-body decay (in the rest frame of the original pion) is

$$\frac{d\Gamma}{d\Omega} \;=\; \frac{|\mathbf{k}_\gamma|}{32\pi^2 M_\pi^2}\times\overline{|\mathcal{M}|^2} \tag{S.77}$$

where $|\mathbf{k}_\gamma| = \frac{1}{2}M_\pi$, and the photon direction's span solid angle $4\pi/2$ rather than 4π because of two identical back-to-back photons in the final state. Consequently, the net decay rate is

$$\Gamma(\pi^0\to\gamma\gamma) \;=\; 2\pi\times\frac{M_\pi/2}{32\pi^2 M_\pi^2}\times\overline{|\mathcal{M}|^2} \;=\; \frac{\alpha^2 M_\pi^3}{64\pi^3 f_\pi^2}\,. \tag{S.78}$$

Problem **4**(c):

Plugging the experimental data $M_\pi = 135$ MeV, $f_\pi = 93$ MeV, and $\alpha = 1/137$ into eq. (S.78), we obtain

$$\Gamma(\pi^0 \to \gamma\gamma) \;\approx\; 7.64 \text{ eV}. \tag{S.79}$$

The experimental lifetime of the neutral pion is $\tau = 8.52 \times 10^{-17}$ s, which corresponds to the total decay rate

$$\Gamma_{\text{total}}(\pi^0 \to \text{anything}) \;=\; 7.72 \text{ eV}. \tag{S.80}$$

The experimental and the calculated decay width are fairly close to each other. The small discrepancy between them is due to (1) approximate values of the M_π and f_π we have used, and (2) QED corrections to the two-photon final state. Such corrections allow the neutral pion to decay into one real photon plus one virtual photon which then becomes an electron-positron pair. Experimentally, π^0 decays to two real photons 98.8% of the time and to $\gamma e^+ e^-$ 1.2% of the time; other decay modes exist but have very small branching ratios.