Problem **2**:

Let's start with the formal analysis of the fermionic functional integral in a fixed background of gauge fields,

$$Z[A_\mu(x)] \;=\; \iint\!\mathcal{D}[\overline{\Psi}(x)] \iint\!\mathcal{D}[\Psi(x)]\,\exp(-S_E[\overline{\Psi},\Psi,A_\mu]). \tag{S.1}$$

Let's change the integration variables by way of an $x$-dependent axial transform

$$\Psi(x) \;\to\; \exp\bigl(+i\theta(x)\Gamma\bigr)\,\Psi(x), \qquad \overline{\Psi}(x) \;\to\; \overline{\Psi}(x)\exp\bigl(+i\theta(x)\Gamma\bigr). \tag{S.2}$$

Since the Euclidean Lagrangian

$$\mathcal{L}_E \;=\; \overline{\Psi}\gamma_e^\mu D_\mu\Psi \tag{S.3}$$

is invariant under global but not local axial symmetries, we have

$$\mathcal{L}_E \;\to\; \overline{\Psi}e^{i\theta\Gamma}\gamma^\mu D_\mu\bigl(e^{i\theta\Gamma}\Psi\bigr) \;=\; \overline{\Psi}\gamma^\mu D_\mu\Psi + \overline{\Psi}\gamma^\mu\bigl(e^{-i\theta\Gamma}\partial_\mu e^{i\theta\Gamma}\bigr)\Psi \;=\; \mathcal{L}_E + i(\partial_\mu\theta)\times\overline{\Psi}\gamma^\mu\Gamma\Psi. \tag{S.4}$$

In other words,

$$\Delta\mathcal{L}_E \;=\; i(\partial_\mu\theta)\times J_A^\mu \tag{S.5}$$

and therefore

$$\Delta S_E \;=\; i\int\!d^dx_e\,(\partial_\mu\theta(x))\times J_A^\mu(x) \;=\; -i\int\!d^dx_e\,\theta(x)\times\partial_\mu J_A^\mu(x). \tag{S.6}$$

At the same time, changing the integration variables in the functional integral (S.1) involves a Jacobian

$$J \;=\; \mathrm{Det}\bigl(\exp(i\Gamma\Theta)\bigr)\times\mathrm{Det}\bigl(\exp(i\Gamma\Theta)\bigr) \;=\; \mathrm{Det}\Bigl(\exp(2i\Gamma\theta)\Bigr) \tag{S.7}$$

where Det denotes the functional determinant of an operator acting on the fermionic fields with the appropriate Dirac, color, and flavor indices. Altogether, the fermionic functional integral (S.1) becomes

$$\begin{aligned} Z[A_\mu] &= \int\!\!\!\int \mathcal{D}[\overline{\Psi}] \int\!\!\!\int \mathcal{D}[\Psi]\, J \times \exp(-S_E - \Delta S_E) \\ &= \int\!\!\!\int \mathcal{D}[\overline{\Psi}] \int\!\!\!\int \mathcal{D}[\Psi]\, \exp(-S_E[A, \overline{\Psi}, \Psi]) \times \exp\big(-\Delta S_E + \log J\big). \end{aligned} \tag{S.8}$$

From this functional integral's point of view, the axial transform (S.2) is nothing but the change of integration variables, so the integral itself must be invariant for any $\theta(x)$. Consequently, the two term in the red exponent must cancel each other, thus

$$\Delta S_E = \log J = \log \operatorname{Det} \exp(2i\Theta\Gamma) = 2i \operatorname{Tr}(\Theta\Gamma) \tag{S.9}$$

where Tr is the functional trace over positions $x_e$ as well as all the indices of the fermionic fields. In other words

$$\operatorname{Tr}(\Theta\Gamma) = \int d^d x_e \operatorname{tr}(\langle x|\, \Theta\Gamma \,|x\rangle) = \int d^d x_e\, \theta(x) \times \operatorname{tr}(\langle x|\, \Gamma \,|x\rangle) \tag{S.10}$$

where tr is the ordinary matrix trace over Dirac, color, and flavor indices. Plugging this formula into eq. (S.9) and comparing to eq. (S.6), we arrive at

$$-i \int d^d x_e\, \theta(x) \times \partial_\mu J^\mu_A(x) = 2i \int d^d x_e\, \theta(x) \times \operatorname{tr}(\langle x|\, \Gamma \,|x\rangle), \tag{S.11}$$

and this equality must hold for any $\theta(x)$. Consequently, at every point $x$ we must have

$$\partial_\mu J^\mu_A(x) = -2 \operatorname{tr}(\langle x|\, \Gamma \,|x\rangle). \tag{S.12}$$

Eq. (S.12) gives us a formal relation of the axial current non-conservation to the axial anomaly of the functional integral measure for the fermionic fields $\Psi(x)$ and $\overline{\Psi}(x)$. In practice, the functional integral has to be UV-regulated, which means that in eq. (S.12) the matrix element $\langle x|\, \Gamma \,|x\rangle$ must be UV regulated,

$$\langle x|\, \Gamma \,|x\rangle \;\to\; \langle x|\, \Gamma \,|x\rangle_{\text{reg}} = \langle x|\, \Gamma\hat{G} \,|x\rangle \tag{S.13}$$

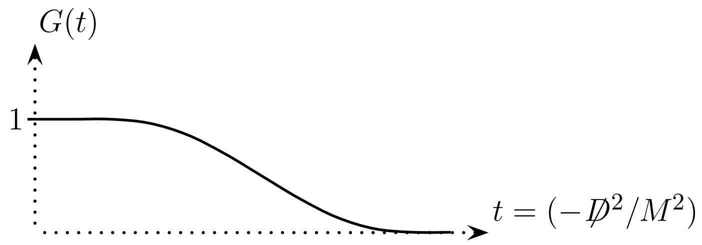
for some UV-cutting operator $G$. Consequently, eq. (S.12) becomes

$$\partial_\mu \big(J^\mu_A(x)\big)_{\text{reg}} = -2 \,\langle x|\, \Gamma\hat{G} \,|x\rangle\,. \tag{S.14}$$

As explained in §22.2–3 of Weinberg's book — *cf.* reading assignment 1.(c), — the regulating operator $\hat{G}$ must commute with the Dirac operator $\not{D}$ in the background gauge field and also with the $\Gamma$ matrix, so $\hat{G}$ should be a smooth function of $\not{D}^2$. Also, $\hat{G}$ should suppress the high-momentum modes $p \gg M$ for some UV cutoff scale $M$ while affecting the low-momentum modes $p \ll M$ as little as possible, hence

$$\hat{G} \;=\; G(-\not{D}^2/M^2). \tag{S.15}$$

for some smooth function $G(t)$ which goes to one for $t \to 0$ and to zero for $t \to \infty$,




(S.16)

Altogether,

$$\partial_\mu J^\mu_A(x) \;=\; 2\,\mathrm{tr}\Big(\langle x|\,\Gamma\hat{G}(-\not{D}^2/M^2)\,|x\rangle\Big). \tag{S.17}$$

Since the covariant derivatives do not commute with each other, $[D_\mu, D_\nu] = i\mathcal{F}_{\mu\nu}$, we have

$$\not{D}^2 \;=\; D^2 \;-\; \tfrac{1}{2}\mathcal{F}_{\mu\nu}\sigma^{\mu\nu} \tag{S.18}$$

and consequently

$$\begin{aligned} G(-\not{D}^2/M^2) \;=\; & G(-D^2/M^2) \;+\; \frac{1}{2M^2}\,G'(-D^2/M^2)\times\mathcal{F}_{\mu\nu}\sigma^{\mu\nu} \\ & +\; \frac{1}{8M^4}\,G''(-D^2/M^2)\times\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}\sigma^{\mu\nu}\sigma^{\alpha\beta} \;+\; \cdots. \end{aligned} \tag{S.19}$$

In $d=4$, the Dirac trace $\mathrm{tr}(\gamma^5\hat{G})$ comes from the second-derivative term in this expansion because we need four $\gamma^\mu$ matrices — or equivalently two $\sigma^{\mu\nu}$ matrices — to accompany the $\gamma^5$. In other even dimensions $d=2n$, we need $d$ $\gamma^\mu$ matrices or $n$ $\sigma^{\mu\nu}$ matrices to accompany the $\Gamma$ inside the

Dirac trace. Specifically, in $2n$ Euclidean dimensions eq. (7) gives us

$$\begin{aligned} \mathrm{Tr}\Big(\Gamma\big(\mathcal{F}_{\alpha\beta}\sigma^{\alpha\beta}\big)^k\Big) &= 0 \quad \text{for } k < n, \\ \mathrm{Tr}\Big(\Gamma\big(\mathcal{F}_{\alpha\beta}\sigma^{\alpha\beta}\big)^n\Big) &= 2^n \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\mathcal{F}_{\alpha_1\beta_1}\cdots\mathcal{F}_{\alpha_n\beta_n}\,. \end{aligned} \tag{S.20}$$

Consequently, the leading term in the Dirac trace $\mathrm{tr}(\Gamma G)$ comes from the $n$–th derivative term in the expansion (S.19),

$$\begin{aligned} \mathrm{tr}_{\mathrm{Dirac}}\Big(\Gamma G(-\not{D}^2/M^2)\Big) &= \frac{1}{n!}\left(\frac{1}{2M^2}\right)^n G^{(n)}(-D^2/M^2) \times \mathrm{tr}_{\mathrm{Dirac}}\Big(\Gamma(F_{\alpha\beta}\sigma^{\alpha\beta})^n\Big) \\ &\quad + \text{ subleading terms} \\ &= \frac{1}{n!}\left(\frac{1}{M^2}\right)^n G^{(n)}(-D^2/M^2) \times \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}F_{\alpha_1\beta_1}\cdots F_{\alpha_n\beta_n} \\ &\quad + \text{ subleading terms.} \end{aligned} \tag{S.21}$$

Fortunately, the subleading terms here carry higher powers of $1/M^2$, so they may be neglected in the $M \to \infty$ limit.

Now let's calculate the matrix element $\langle x|\, G^{(n)}(-D^2/M^2)\,|x\rangle$ of the $n^{\text{th}}$ derivative of the regulator. Going from the coordinate basis to the momentum basis, we have

$$\begin{aligned} \langle x|\, G^{(n)}(-D^2/M^2)\,|x\rangle &= \int\frac{d^dp}{(2\pi)^d}\, e^{-ipx}\, G^{(n)}(-D^2/M^2)\, e^{+ipx} \\ &= \int\frac{d^dp}{(2\pi)^d}\, G^{(n)}\left(\frac{(p^\mu - iD^\mu)^2_E}{M^2}\right) \end{aligned} \tag{S.22}$$

where on the last line the derivative $D^\mu$ acts on the $F(x)\cdots F(x)$ in eq. (S.21) (and also on $A^\nu(x)$ in other $D^\nu$ in the expansion of $G^{(n)}$). In the integral (S.22), the overall scale of the momentum $p$ follows from the regulator $G(p^2/M^2)$, hence $p^\mu = O(M)$ while the derivatives $D^\mu$ are effectively $O$(external momenta) of the background photon or gluon fields. For the regulation purposes, we assume that $M$ is much larger than all such external momenta, hence

$$\frac{1}{M^d}\int\frac{d^dp}{(2\pi)^d}\, G^{(n)}\left(\frac{(p^\mu - iD^\mu)^2}{M^2}\right) = \frac{1}{M^d}\int\frac{d^dp}{(2\pi)^d}\, G^{(n)}(p^2/M^2) \;+\; O\left(\frac{k^2_{\text{ext}}}{M^2}\right) \tag{S.23}$$

while the leading term on the right hand side is a $O(1)$ constant. To calculate this constant, we use spherical coordinates in $2n$ Euclidean dimensions and integrate over the angular variables, thus

$$d^{2n}p \;\to\; \frac{2\pi^n}{(n-1)!}\,p^{2n-1}dp \;=\; \frac{\pi^n}{(n-1)!}\,(p^2)^{n-1}dp^2, \tag{S.24}$$

hence,

$$\begin{aligned}\frac{1}{M^{2n}}\int\frac{d^{2n}p}{(2\pi)^{2n}}\,G^{(n)}(p^2/M^2) \;&=\; \frac{1}{(4\pi)^n(n-1)!}\,\frac{1}{M^{2n}}\int_0^\infty dp^2\,p^{2n-2}G^{(n)}(p^2/M^2)\\ \langle\!\langle\,\text{changing variable from } p_e^2 \text{ to } t=p_e^2/M^2\,\rangle\!\rangle \;&=\; \frac{1}{(4\pi)^n(n-1)!}\int_0^\infty dt\,t^{n-1}G^{(n)}(t),\end{aligned}$$

where the last integral has the same value for any analytic function $G(t)$ such that $G(0)=1$ and $G(\infty)=0$, *cf.* fig. (S.16). Indeed,

$$\frac{1}{(n-1)!}\,t^{n-1}\frac{d^nG}{dt^n} \;=\; \frac{d}{dt}\left(\sum_{k=0}^{n-1}\frac{(-1)^{n-1-k}}{k!}t^k\,\frac{d^kG}{dt^k}\right), \tag{S.25}$$

hence

$$\frac{1}{(n-1)!}\int_0^\infty dt\,t^{n-1}\,\frac{d^nG}{dt^n} = \sum_{k=0}^{n-1}\frac{(-1)^{n-1-k}}{k!}\left(t^k\,\frac{d^kG}{dt^k}\right)\Bigg]_{t=0}^{t=\infty}, \tag{S.26}$$

and for any $k>0$ term in this sum

$$t^k\,\frac{d^kG}{dt^k} \;\longrightarrow\; 0 \quad \text{for both } t\to 0 \text{ and } t\to\infty, \tag{S.27}$$

where the behavior at $t\to\infty$ follows from $G(t)$ being analytic and $G\to 0$ at $t\to\infty$. As to the $k=0$ term, we have

$$\left(t^0G^{\text{itself}}(t)\right)\Big|_{t=0}^{t=\infty} \;=\; G(\infty)\;-\;G(0)\;=\;0\;-\;1\;=\;-1, \tag{S.28}$$

hence altogether

$$\frac{1}{(n-1)!}\int_0^\infty dt\,t^{n-1}\,\frac{d^nG}{dt^n} \;=\; (-1)^{n-1}\times(-1) \;=\; (-1)^n. \tag{S.29}$$

Altogether, we have

$$\langle x|\, G^{(n)}(-D^2/M^2)\,|x\rangle \;=\; M^{2n}\times\frac{(-1)^n}{(4\pi)^n}\;+\;\text{subleading terms,} \tag{S.30}$$

and therefore in the $M\to\infty$ limit

$$\langle x|\,\mathrm{tr}_{\mathrm{Dirac}}\Big(\Gamma G(-\not{D}^2/M^2)\Big)\,|x\rangle \;=\; \frac{1}{n!}\left(\frac{-1}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\mathcal{F}_{\alpha_1\beta_1}(x)\cdots\mathcal{F}_{\alpha_n\beta_n}(x). \tag{S.31}$$

Finally, the axial anomaly follows from this formula via eq. (S.12): All we need is to take the trace of the gauge field product over the fermion species — *e.g.*, the colors and the flavors — and multiply by $-2$, thus

$$\partial_\mu J^\mu_A(x) \;=\; -\frac{2}{n!}\left(\frac{-1}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\,\mathrm{tr}_{\mathrm{species}}\big(\mathcal{F}_{\alpha_1\beta_1}(x)\cdots\mathcal{F}_{\alpha_n\beta_n}(x)\big). \tag{3}$$

*Quod erat demonstrandum.*

Problem **3A**(a):

Let's start with eq. (10.a) for $n=1$. For an abelian vector field $\mathcal{F}=d\mathcal{A}$, hence

$$d\Omega_{(1)} \;=\; d\,\mathrm{tr}(\mathcal{A}) \;=\; \mathrm{tr}(\mathcal{F}) \;=\; Q_{(2)}\,, \tag{S.32}$$

*quod erat demonstrandum.*

For the non-abelian gauge fields for $n>1$, we have $\mathcal{F}=d\mathcal{A}+i\mathcal{A}\wedge\mathcal{A}$. Also, there is a useful Leibniz rule for the covariant derivatives: For any 2 matrix-valued differential forms $\mathcal{B}$ and $\mathcal{C}$ in the adjoint multiplet of the gauge group,

$$\begin{aligned} D\mathcal{B} \;&=\; d\mathcal{B} \;+\; i\mathcal{A}\wedge\mathcal{B} \;-\; (-1)^{\text{degree of }\mathcal{B}} i\mathcal{B}\wedge\mathcal{A},\\ D\mathcal{C} \;&=\; d\mathcal{C} \;+\; i\mathcal{A}\wedge\mathcal{C} \;-\; (-1)^{\text{degree of }\mathcal{C}} i\mathcal{C}\wedge\mathcal{A}, \end{aligned} \tag{S.33}$$

while

$$d\,\mathrm{tr}(\mathcal{B}\wedge\mathcal{C}) \;=\; \mathrm{tr}\big(D(\mathcal{B}\wedge\mathcal{C})\big) \;=\; \mathrm{tr}\big((D\mathcal{B})\wedge\mathcal{C}\big) \;+\; (-1)^{\text{degree of }\mathcal{B}}\,\mathrm{tr}\big(\mathcal{B}(D\mathcal{C})\big). \tag{S.34}$$

Now let's apply this rule to the $\Omega_{(3)}$ form in eq. (10.b):

$$\begin{aligned} d\Omega_{(3)} &= d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}) - \tfrac{i}{3}d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \\ &= \mathrm{tr}(D\mathcal{A}\wedge\mathcal{F}) - \mathrm{tr}(\mathcal{A}\wedge D\mathcal{F}) \\ &\quad - \tfrac{i}{3}\,\mathrm{tr}((D\mathcal{A})\wedge\mathcal{A}\wedge A) + \tfrac{i}{3}\,\mathrm{tr}(\mathcal{A}\wedge(D\mathcal{A})\wedge\mathcal{A}) - \tfrac{i}{3}\,\mathrm{tr}\big(\mathcal{A}\wedge\mathcal{A}\wedge(D\mathcal{A})\big). \end{aligned} \tag{S.35}$$

But by the cyclic symmetry of the trace

$$\mathrm{tr}(\mathcal{B}\wedge\mathcal{C}) = \pm\,\mathrm{tr}(\mathcal{C}\wedge\mathcal{B}) \tag{S.36}$$

where the $\pm$ sign is $-$ if both $\mathcal{B}$ and $\mathcal{C}$ are odd and $+$ otherwise. In particular, we have

$$\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge D\mathcal{A}) = -\,\mathrm{tr}(\wedge\mathcal{A}\wedge D\mathcal{A}\wedge\mathcal{A}) = +\,\mathrm{tr}(D\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}), \tag{S.37}$$

so the 3 terms on the bottom line of eq. (S.35) are equal to each other, thus

$$d\Omega_{(3)} = \mathrm{tr}(D\mathcal{A}\wedge\mathcal{F}) - \mathrm{tr}(\mathcal{A}\wedge D\mathcal{F}) - i\,\mathrm{tr}\,\mathrm{tr}((D\mathcal{A})\wedge\mathcal{A}\wedge A). \tag{S.38}$$

Next, by the homogeneous Yang–Mills equation, $D\mathcal{F}=0$, while

$$D\mathcal{A} = d\mathcal{A} + i\mathcal{A}\wedge\mathcal{A} + i\mathcal{A}\wedge\mathcal{A} = \mathcal{F} + i\mathcal{A}\wedge\mathcal{A}. \tag{S.39}$$

Consequently, eq. (S.38) becomes

$$\begin{aligned} d\Omega_{(3)} &= \mathrm{tr}\big((D\mathcal{A})\wedge(\mathcal{F}-i\mathcal{A}\wedge\mathcal{A})\big) + 0 \\ &= \mathrm{tr}\big((\mathcal{F}+i\mathcal{A}\wedge\mathcal{A})\wedge(\mathcal{F}-i\mathcal{A}\wedge\mathcal{A})\big) \\ &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}) + \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}). \end{aligned} \tag{S.40}$$

Finally, the second term on the bottom line here vanishes by the cyclic symmetry of the trace. Indeed, identifying the product of first 3 $\mathcal{A}$ factors as a 3-form $\mathcal{B}$ while the last $\mathcal{A}$ factor as $\mathcal{C}$, we have

$$\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) = -\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) = -\text{itself} = 0. \tag{S.41}$$

Consequently,

$$d\Omega_{(3)} = \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}), \tag{S.42}$$

*quod erat demonstrandum.*

Finally, consider the Chern–Simons 5-form $\Omega_{(5)}$ from eq. (10.c). Using the Leibniz rule (S.34), the Yang–Mills equation $D\mathcal{F}=0$, and eq.(S.39)— for the (S.39), we have

$$\begin{aligned} d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) &= \mathrm{tr}\big((D\mathcal{A})\wedge\mathcal{F}\wedge\mathcal{F}\big) \\ &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}) \;+\; i\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}), \qquad \text{(S.43)} \\ d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) &= \mathrm{tr}\big((D\mathcal{A})\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F} - \mathcal{A}\wedge(D\mathcal{A})\wedge\mathcal{A}\wedge\mathcal{F} + \mathcal{A}\wedge\mathcal{A}\wedge(D\mathcal{A})\wedge\mathcal{F}\big) \\ &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) - \mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{F}) + \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) \\ &\quad + (i-i+i)\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}), \qquad \text{(S.44)} \\ d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) &= \mathrm{tr}\big((D\mathcal{A})\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big) - \cdots + \mathrm{tr}\big(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge(D\mathcal{A})\big) \\ &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) - \cdots + \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \\ &\quad + (i-i+i-i+i)\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}). \qquad \text{(S.45)} \end{aligned}$$

Furthermore, by the cyclic symmetry of the trace

$$\begin{aligned} \mathrm{tr}(\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) &= -\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) = \cdots \\ &= +\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}), \qquad \text{(S.46)} \\ \mathrm{tr}(\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) &= +\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}), \qquad \text{(S.47)} \\ \text{while}\quad \mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{F}) &= -\text{itself} = 0, \qquad \text{(S.48)} \\ \text{and}\quad \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) &= -\text{itself} = 0. \qquad \text{(S.49)} \end{aligned}$$

Consequently,

$$\begin{aligned} d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}) + i\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}), \qquad \text{(S.50)} \\ d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) &= 2\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) + i\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}), \qquad \text{(S.51)} \\ d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) &= 5\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) + 0, \qquad \text{(S.52)} \end{aligned}$$

and therefore

$$\begin{aligned} d\Omega_{(5)} &= d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) - \tfrac{i}{2}d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) - \tfrac{1}{10}d\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \\ &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}) + i\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) \\ &\quad - i\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) + \tfrac{1}{2}\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \qquad \text{(S.53)} \\ &\quad - \tfrac{1}{2}\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) - 0 \\ &= \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}) + \text{cancellation}. \end{aligned}$$

Altogether,

$$d\Omega_{(5)} \;=\; \mathrm{tr}(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}) \;=\; Q_{(6)} \tag{S.54}$$

*quod erat demonstrandum.*

Problem **3A**(b):

Again, let's start with the $n = 2$ case for the abelian gauge symmetries. An abelian gauge transform acts as $\delta\mathcal{A} = -d\Lambda$, hence

$$\delta\Omega_{(1)} \;=\; \delta\,\mathrm{tr}(\mathcal{A}) \;=\; -d\,\mathrm{tr}(\Lambda) \;=\; -dH_{(0)} \tag{S.55}$$

for the 0-form $H_{(0)} = \mathrm{tr}(\Lambda)$ as in eq. (14).

For the non-abelian gauge symmetries, the infinitesimal gauge transform acts as in eq. (12), or in a more compact form

$$\delta\mathcal{A} \;=\; -D\Lambda \;=\; -d\Lambda \;-\; i[\mathcal{A},\Lambda], \qquad \delta\mathcal{F} \;=\; -i[\mathcal{F},\Lambda]. \tag{S.56}$$

Consequently, for $n = 2$

$$-\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}) \;=\; +\,\mathrm{tr}((d\Lambda)\wedge\mathcal{F}) \;+\; i\,\mathrm{tr}([\mathcal{A},\Lambda]\wedge\mathcal{F}) \;+\; i\,\mathrm{tr}(\mathcal{A}\wedge[\mathcal{F},\Lambda), \tag{S.57}$$

where the last two terms add up to zero:

$$\mathrm{tr}([\mathcal{A},\Lambda]\wedge\mathcal{F}) \;+\; \mathrm{tr}(\mathcal{A}\wedge[\mathcal{F},\Lambda) \;=\; \mathrm{tr}\big([\mathcal{A}\wedge\mathcal{F},\Lambda]\big) \;=\; 0 \tag{S.58}$$

as a trace of a commutator. Thus,

$$-\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}) \;=\; +\,\mathrm{tr}((d\Lambda)\wedge\mathcal{F}). \tag{S.59}$$

Likewise,

$$\begin{aligned} -\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \;=\; &+\,\mathrm{tr}((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}) \;+\; \mathrm{tr}(\mathcal{A}\wedge(d\Lambda)\wedge\mathcal{A}) \;+\; \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge(d\Lambda)) \\ &+\; i\,\mathrm{tr}([\mathcal{A},\Lambda]\wedge\mathcal{A}\wedge\mathcal{A}) \;+\; i\,\mathrm{tr}(\mathcal{A}\wedge[\mathcal{A},\Lambda]\wedge\mathcal{A}) \;+\; i\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge[\mathcal{A},\Lambda]), \end{aligned} \tag{S.60}$$

where the last three terms add up to a trace of a commutator and therefore zero,

$$\begin{aligned} \mathrm{tr}([\mathcal{A},\Lambda]\wedge\mathcal{A}\wedge\mathcal{A}) \;+\; \mathrm{tr}(\mathcal{A}\wedge[\mathcal{A},\Lambda]\wedge\mathcal{A}) \;+\; \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge[\mathcal{A},\Lambda]) \;=\; \\ =\; \mathrm{tr}\big([\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A},\Lambda]\big) \;=\; 0. \end{aligned} \tag{S.61}$$

Also, the three terms on the top line of eq. (S.60) are related to each other by the cyclic symmetry of the trace,

$$\mathrm{tr}((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}) \;=\; \mathrm{tr}(\mathcal{A}\wedge(d\Lambda)\wedge\mathcal{A}) \;=\; \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge(d\Lambda)), \tag{S.62}$$

hence

$$-\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \;=\; +3\,\mathrm{tr}((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}). \tag{S.63}$$

Altogether, the gauge variation of the Chern–Simons 3-form (10.b) amounts to

$$\begin{aligned}
-\delta\Omega_{(3)} \;&=\; -\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}) \;+\; \tfrac{i}{3}\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\\
&=\; +\,\mathrm{tr}((d\Lambda)\wedge\mathcal{F}) \;-\; i\,\mathrm{tr}((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A})\\
&=\; \mathrm{tr}\Big((d\Lambda)\wedge\big(\mathcal{F}\;-\;i\mathcal{A}\wedge\mathcal{A}\big)\Big)\\
&\qquad \text{where } \mathcal{F}\;-\;i\mathcal{A}\wedge\mathcal{A} \;=\; d\mathcal{A}\\
&=\; \mathrm{tr}\big((d\Lambda)\wedge(d\mathcal{A})\big)\\
&=\; d\,\mathrm{tr}\big(\Lambda\wedge d\mathcal{A}\big)\\
&=\; dH_{(2)}
\end{aligned} \tag{S.64}$$

for the 2-form $H_{(2)}$ exactly as in eq. (15).

Finally, consider the gauge variance for the 3 terms in eq. (10.c) comprising the Chern–Simons 5-form $\Omega_{(5)}$. First,

$$-\delta(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) \;=\; +(d\Lambda)\wedge\mathcal{F}\wedge\mathcal{F} \;+\; i\big[(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}),\Lambda],\tag{S.65}$$

hence

$$\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) \;=\; +\,\mathrm{tr}((d\Lambda)\wedge\mathcal{F}\wedge\mathcal{F}) \;+\; 0. \tag{S.66}$$

Second,

$$-\delta(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \;=\; +(d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F} + \mathcal{A}\wedge(d\Lambda)\wedge\mathcal{A}\wedge\mathcal{F} + \mathcal{A}\wedge\mathcal{A}\wedge(d\Lambda)\wedge\mathcal{F} + i\big[(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}),\Lambda\big], \tag{S.67}$$

hence

$$\begin{aligned}
-\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \;&=\; \mathrm{tr}((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \;+\; \mathrm{tr}(\mathcal{A}\wedge(d\Lambda)\wedge\mathcal{A}\wedge\mathcal{F})\\
&\qquad +\; \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge(d\Lambda)\wedge\mathcal{F}) \;+\; 0\\
&\quad\text{by the cyclic symmetries of the traces}\\
&=\; \mathrm{tr}\Big((d\Lambda)\wedge\big(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}+\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A}+\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\big)\Big).
\end{aligned}\tag{S.68}$$

Third,

$$\begin{aligned}
-\delta(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \;&=\; +(d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \;+\; \mathcal{A}\wedge(d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \;+\; \cdots\\
&\qquad +\; \mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge(d\Lambda) \;+\; i\big[(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}),\Lambda],
\end{aligned}\tag{S.69}$$

hence

$$\begin{aligned}
-\delta\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \;&=\; +\,\mathrm{tr}((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \;+\; \mathrm{tr}(\mathcal{A}\wedge(d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\\
&\qquad +\;\cdots\;+\; \mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge(d\Lambda)) \;+\; 0\\
&\quad\text{by the cyclic symmetries of the traces}\\
&=\; 5\times\mathrm{tr}\big((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big).
\end{aligned}\tag{S.70}$$

Combining eq. (S.66), (S.38), and (S.68), we get the net variation of the $\Omega_{(5)}$ form as

$$\begin{aligned}
-\delta\Omega_{(5)} \;&=\; -\delta\Big(\mathrm{tr}(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}) \;-\; \tfrac{i}{2}\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \;-\; \tfrac{1}{10}\,\mathrm{tr}(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\Big)\\
&=\; +\,\mathrm{tr}((d\Lambda)\wedge\mathcal{F}\wedge\mathcal{F})\\
&\qquad -\; \tfrac{i}{2}\,\mathrm{tr}\Big((d\Lambda)\wedge\big(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}+\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A}+\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\big)\Big)\\
&\qquad -\; \tfrac{1}{2}\,\mathrm{tr}\big((d\Lambda)\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\\
&=\; \mathrm{tr}\big((d\Lambda)\wedge K_{(4)}\big)
\end{aligned}\tag{S.71}$$

where

$$
\begin{aligned}
K_{(4)} \;&=\; \mathcal{F}\wedge\mathcal{F} \;-\; \tfrac{i}{2}\big(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F} + \mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A} + \mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A}\big) \;-\; \tfrac{1}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \\
&\qquad \text{expanding}\mathcal{F} \;=\; d\mathcal{A} \;+\; i\mathcal{A}\wedge\mathcal{A} \\
&=\; d\mathcal{A}\wedge d\mathcal{A} \;+\; i\mathcal{A}\wedge\mathcal{A}\wedge d\mathcal{A} \;+\; id\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \;-\; \mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \\
&\qquad -\; \tfrac{i}{2}\big(\mathcal{A}\wedge\mathcal{A}\wedge d\mathcal{A} + \mathcal{A}\wedge d\mathcal{A}\wedge\mathcal{A} + d\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} + 3i\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big) \\
&\qquad -\; \tfrac{1}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \\
&=\; d\mathcal{A}\wedge d\mathcal{A} \;+\; \tfrac{i}{2}\big(\mathcal{A}\wedge\mathcal{A}\wedge d\mathcal{A} - \mathcal{A}\wedge d\mathcal{A}\wedge\mathcal{A} + d\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big) \\
&\qquad +\; \left(-1 + \tfrac{3}{2} - \tfrac{1}{2} \;=\; 0\right) \times \mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A} \\
&=\; d(\mathcal{A}\wedge d\mathcal{A}) \;+\; \tfrac{i}{2}d(\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \;+\; 0 \\
&=\; d\Big(\mathcal{A}\wedge d\mathcal{A} + \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big).
\end{aligned}
\tag{S.72}
$$

Altogether, we have

$$
\begin{aligned}
-\delta\Omega_{(5)} \;&=\; \mathrm{tr}\Big(d\Lambda\wedge d\big(\mathcal{A}\wedge d\mathcal{A} + \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\Big) \\
&=\; d\,\mathrm{tr}\Big(\Lambda\times d\big(\mathcal{A}\wedge d\mathcal{A} + \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\Big) \\
&=\; dH_{(4)}
\end{aligned}
\tag{S.73}
$$

where the $H_{(4)}$ form is exactly as in eq. (16). *Quod erat demonstrandum.*

Problem **3B**(a):

The Chern–Simons forms $\Omega_{(1)}$, $\Omega_{(3)}$, and $\Omega_{(5)}$, — or rather the vectors dual to them in $d = 2, 4, 6$ — are spelled out in eqs. (19), all we need to do is to calculate their divergences and verify eqs. (18) for the appropriate dimensions. Let's start with the $\Omega_{(1)}$ for $d = 2$:

$$
\partial_\mu\Omega^\mu_{(1)} \;\equiv\; 2\epsilon^{\mu\nu}\partial_\mu\,\mathrm{tr}(\mathcal{A}_\nu) \;=\; \epsilon^{\mu\nu}\,\mathrm{tr}(\partial_{[\mu}\mathcal{A}_{\nu]}) \;=\; \epsilon^{\mu\nu}\,\mathrm{tr}(\mathcal{F}_{\mu\nu}) \quad \text{[for the abelian } \mathcal{A}_\nu\text{]}, \tag{S.74}
$$

which is precisely the anomaly (18) in $d = 2$ dimensions.

Next, consider the $\Omega_{(3)}$ for $d = 4$. Using the Leibniz rule for the covariant derivative $D_\mu$ of products of adjoint fields,

$$\partial_\mu \operatorname{tr}(\Phi_1\Phi_2) \;=\; \operatorname{tr}\Big(D_\mu(\Phi_1\Phi_2)\Big) \;=\; \operatorname{tr}\Big((D_\mu\Phi_1)\Phi_2 \,+\, \Phi_1(D_\mu\Phi_2)\Big) \tag{S.75}$$

we obtain

$$\begin{aligned}
\partial_\mu \Omega^\mu_{(3)} \;&\equiv\; 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu \operatorname{tr}\left(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma} \,-\, \frac{2i}{3}\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\right)\\
&=\; 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{F}_{\rho\sigma} \,+\, \mathcal{A}_\nu(D_\mu\mathcal{F}_{\rho\sigma})\Big)\\
&\quad -\,\frac{4i}{3}\,\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma \,+\, \mathcal{A}_\nu(D_\mu\mathcal{A}_\rho)\mathcal{A}_\sigma \,+\, \mathcal{A}_\nu\mathcal{A}_\rho(D_\mu\mathcal{A}_\sigma)\Big).
\end{aligned} \tag{S.76}$$

On the second line here, the second term vanishes by the non-abelian Jacobi identity,

$$\epsilon^{\mu\nu\rho\sigma} D_\mu\mathcal{F}_{\rho\sigma} \;=\; 0, \tag{S.77}$$

while the three terms on the last line of (S.76) are equal to each other because of the cyclic symmetry of the trace and the antisymmetry of the $\epsilon$ tensor,

$$\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma\Big) \;=\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big(\mathcal{A}_\nu(D_\mu\mathcal{A}_\rho)\mathcal{A}_\sigma\Big) \;=\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big(\mathcal{A}_\nu\mathcal{A}_\rho(D_\mu\mathcal{A}_\sigma)\Big). \tag{S.78}$$

Plugging these formulae into eq. (S.76), we get

$$\begin{aligned}
\partial_\mu \Omega^\mu_{(3)} \;&=\; 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{F}_{\rho\sigma}\Big) \;-\; \frac{4i}{3}\,\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma\Big)\times 3\\
&=\; 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu\mathcal{A}_\nu)\times(\mathcal{F}_{\rho\sigma} \,-\, i[\mathcal{A}_\rho,\mathcal{A}_\sigma])\Big).
\end{aligned} \tag{S.79}$$

In this formula, $D_\mu\mathcal{A}_\nu = \partial_\mu\mathcal{A}_\nu + i[\mathcal{A}_\mu,\mathcal{A}_\nu]$, which after antisymmetrization in $\mu\leftrightarrow\nu$ becomes

$$D_{[\mu}\mathcal{A}_{\nu]} \;=\; \partial_{[\mu}\mathcal{A}_{\nu]} \;+\; 2i[\mathcal{A}_\mu,\mathcal{A}_\nu] \;=\; \mathcal{F}_{\mu\nu} \;+\; i[\mathcal{A}_\mu,\mathcal{A}_\nu]. \tag{S.80}$$

Consequently,

$$\begin{aligned}
\partial_\mu \Omega^\mu_{(3)} \;&=\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((\mathcal{F}_{\mu\nu} \,+\, i[\mathcal{A}_\mu,\mathcal{A}_\nu])\times(\mathcal{F}_{\rho\sigma} \,-\, i[\mathcal{A}_\rho,\mathcal{A}_\sigma])\Big)\\
&=\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\Big) \;+\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big([\mathcal{A}_\mu,\mathcal{A}_\nu]\,[\mathcal{A}_\rho,\mathcal{A}_\sigma]\Big)
\end{aligned} \tag{S.81}$$

where on the bottom line

$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big([\mathcal{A}_\mu,\mathcal{A}_\nu]\,[\mathcal{A}_\rho,\mathcal{A}_\sigma]\Big)\ =\ 0, \tag{S.82}$$

as you should have seen in the previous homework#23, problem 1(a). This leaves us with

$$\partial_\mu\Omega^\mu_{(3)}\ =\ \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\Big), \tag{S.83}$$

exactly as in eq. (18) for $d=4$.

Finally, consider the $\Omega_{(5)}$ for $d=6$. Again, using the Leibniz rule (S.75) we obtain

$$\begin{aligned}
\partial_\mu\Omega^\mu_{(5)}\ &=\ 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\partial_\mu\,\mathrm{tr}\left(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\ -\ i\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ -\ \frac{2}{5}\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\right)\\
&=\ 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu(D_\mu\mathcal{F}_{\rho\sigma})\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{F}_{\rho\sigma}(D_\mu\mathcal{F}_{\alpha\beta})\Big)\\
&\quad-\ 2i\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\left(\begin{matrix}(D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu(D_\mu\mathcal{A}_\rho)\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{A}_\rho(D_\mu\mathcal{A}_\sigma)\mathcal{F}_{\alpha\beta}\\ +\ \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma(D_\mu\mathcal{F}_{\alpha\beta})\end{matrix}\right)\\
&\quad-\ \frac{4}{5}\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\ +\ 4\text{ similar terms.}\Big)
\end{aligned} \tag{S.84}$$

In fact, the 5 terms on the last line here are not just similar but exactly equal to each other thanks to the cyclic symmetry of the trace and the antisymmetry of the $\epsilon$ tensor. Moreover, thanks to the non-abelian Jacobi identity (S.77) we may disregard all the terms containing $(D_\mu\mathcal{F}_{\rho\sigma})$ or $(D_\mu\mathcal{F}_{\alpha\beta})$, thus

$$\begin{aligned}
\partial_\mu\Omega^\mu_{(5)}\ &=\ 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\Big)\\
&\quad-\ 2i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu(D_\mu\mathcal{A}_\rho)\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{A}_\rho(D_\mu\mathcal{A}_\sigma)\mathcal{F}_{\alpha\beta}\Big)\\
&\quad-\ 4\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu\mathcal{A}_\nu)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)
\end{aligned} \tag{S.85}$$

Next, let's apply eq. (S.80) — or rather $D_{[\mu}\mathcal{A}_{\nu]}=\mathcal{F}_{\mu\nu}+2i\mathcal{A}_{[\mu}\mathcal{A}_{\nu]}$ — to every $(D_\mu\mathcal{A}_{\ldots})$ derivative in eq. (S.85):

$$\begin{aligned}
\partial_\mu\Omega^\mu_{(5)}\ &=\ \epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\Big)\ +\ 2i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\Big)\\
&\quad-\ i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{F}_{\mu\rho}\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{F}_{\mu\sigma}\mathcal{F}_{\alpha\beta}\Big)\\
&\quad+\ 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{A}_\mu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\mu\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\Big)\\
&\quad-\ 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)\ -\ 4i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)
\end{aligned} \tag{S.86}$$

We may drastically simplify this expression using the cyclic symmetry of the trace and the anti-symmetry of the $\epsilon$ tensor. In particular, the last term here vanishes identically; indeed

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big) &= +\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\beta\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\Big)\\
&\qquad \langle\!\langle\,\text{relabeling indices } \beta\to\mu\to\nu\to\rho\to\sigma\to\alpha\to\beta\,\rangle\!\rangle\\
&= +\epsilon^{\nu\rho\sigma\alpha\beta\mu}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)\\
&= -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)\\
&\equiv -\text{itself}\\
&= 0.
\end{aligned} \tag{S.87}$$

Likewise,

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{F}_{\nu\rho}\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\Big) &= +\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\mathcal{A}_\mu\mathcal{F}_{\nu\rho}\Big)\\
&= +\epsilon^{\sigma\alpha\beta\mu\nu\rho}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{F}_{\nu\rho}\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\Big)\\
&= -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{F}_{\nu\rho}\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\Big)\\
&\equiv -\text{itself}\\
&= 0.
\end{aligned} \tag{S.88}$$

In a similar manner most other terms on the RHS of eq. (S.86) cancel each other:

$$\begin{aligned}
2i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\Big) \;-\; i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} \,+\, \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{F}_{\mu\sigma}\mathcal{F}_{\alpha\beta}\Big)&\\
= i\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(2\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta} - \mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{F}_{\mu\sigma}\mathcal{F}_{\alpha\beta}\Big)&\\
= i\,\mathrm{tr}\Big(2\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\Big)\times\Big(2\epsilon^{\mu\nu\rho\sigma\alpha\beta} - \epsilon^{\alpha\beta\mu\nu\rho\sigma} - \epsilon^{\rho\mu\nu\sigma\alpha\beta}\Big)&\\
= i\,\mathrm{tr}(\cdots)\times\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times(2-1-1=0)&
\end{aligned}$$

and likewise

$$\begin{aligned}
2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu\mathcal{A}_\mu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\mu\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\Big)&\\
- 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)&\\
= 2\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big)\times&\\
\times\Big(\epsilon^{\rho\sigma\alpha\beta\mu\nu} + \epsilon^{\sigma\rho\alpha\beta\mu\nu} + \epsilon^{\alpha\rho\sigma\beta\mu\nu} - \epsilon^{\mu\nu\rho\sigma\alpha\beta}\Big)&\\
= 2\,\mathrm{tr}(\cdots)\times\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times(1-1+1-1=0).&
\end{aligned} \tag{S.89}$$

Altogether, the only surviving term on the RHS of eq. (S.86) is the 6D anomaly:

$$\partial_\mu \Omega^\mu_{(5)} \;=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\Big). \tag{S.90}$$

*Quod erat demonstrandum.*

Problem **3B**(b):

The $d=2$ case is indeed trivial: The infinitesimal gauge variance of an abelian field is $\delta\mathcal{A}_\mu = -\partial_\mu\Lambda$, hence

$$\delta\Omega^\mu_{(1)} \;=\; 2\epsilon^{\mu\nu}\,\mathrm{tr}(-\partial_\nu\Lambda) \;=\; -2\partial_\nu\Big(\epsilon^{\mu\nu}\,\mathrm{tr}(\Lambda)\Big) \;=\; -2\partial_\nu H^{\mu\nu}_{(0)} \quad\text{for}\quad H^{\mu\nu}_{(0)} \;=\; \epsilon^{\mu\nu}\,\mathrm{tr}(\Lambda). \tag{S.91}$$

For $d=4$, the gauge fields may be non-abelian with a more complicated infinitesimal variation

$$\delta\mathcal{A}_\mu \;=\; -D_\mu\Lambda \;=\; -\partial_\mu\Lambda \;-\; i[A_\mu,\Lambda], \qquad \delta\mathcal{F}_{\mu\nu} \;=\; -i[\mathcal{F}_{\mu\nu},\Lambda]. \tag{S.92}$$

Consequently,

$$\begin{aligned}
\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\big) \;&=\; \mathrm{tr}\big(-(\partial_\nu\Lambda)\mathcal{F}_{\mu\nu} \;-\; i[\mathcal{A}_\nu,\Lambda]\mathcal{F}_{\rho\sigma} \;-\; i\mathcal{A}_\nu[\mathcal{F}_{\rho\sigma},\Lambda]\big)\\
&=\; -\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{F}_{\mu\nu}\big) \;-\; i\,\mathrm{tr}\big([\mathcal{A}_\nu\mathcal{F}_{\rho\sigma},\Lambda]\big)\\
&=\; -\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\big)
\end{aligned} \tag{S.93}$$

because the trace of a net commutator like $[\mathcal{A}_\nu\mathcal{F}_{\rho\sigma},\Lambda]$ always vanishes. Similarly,

$$\begin{aligned}
\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\big) \;&=\; \mathrm{tr}\big((-D_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma \;+\; A_\nu(-D_\rho\Lambda)\mathcal{A}_\sigma \;+\; A_\nu\mathcal{A}_\rho(-D_\sigma\Lambda)\big)\\
&=\; -\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma \;+\; A_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma \;+\; A_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\big)\\
&\qquad -\; i\,\mathrm{tr}\big([\mathcal{A}_\nu,\Lambda]\mathcal{A}_\rho\mathcal{A}_\sigma \;+\; \mathcal{A}_\nu[\mathcal{A}_\rho,\Lambda]\mathcal{A}_\sigma \;+\; \mathcal{A}_\nu\mathcal{A}_\rho[\mathcal{A}_\sigma,\Lambda]\big)
\end{aligned}$$

where the trace on the bottom line vanishes as a total commutator,

$$\mathrm{tr}\big([\mathcal{A}_\nu,\Lambda]\mathcal{A}_\rho\mathcal{A}_\sigma \;+\; \mathcal{A}_\nu[\mathcal{A}_\rho,\Lambda]\mathcal{A}_\sigma \;+\; \mathcal{A}_\nu\mathcal{A}_\rho[\mathcal{A}_\sigma,\Lambda]\big) \;=\; \mathrm{tr}\big([\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma,\Lambda]\big) \;=\; 0. \tag{S.94}$$

Furthermore,

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\times\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\big) \;&=\; -\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma \,+\, A_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma \,+\, A_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\big)\\
&\qquad\langle\!\langle\text{ by cyclic symmetry } \nu\to\rho\to\sigma\to\nu \text{ of the } \epsilon \text{ tensor's last 3 indices}\rangle\!\rangle\\
&=\; -\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma \,+\, \mathcal{A}_\sigma(\partial_\nu\Lambda)\mathcal{A}_\rho \,+\, \mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\nu\Lambda)\big)\\
&\qquad\langle\!\langle\text{ by cyclic symmetry of the trace}\rangle\!\rangle\\
&=\; -\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big(3\times(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\big).
\end{aligned} \tag{S.95}$$

The net variation of the $\Omega^\mu_{(3)}$ obtains from adding up eqs. (S.93) and (S.95) with appropriate coefficients, thus

$$\begin{aligned}
\delta\Omega^\mu_{(3)} \;&=\; \epsilon^{\mu\nu\rho\sigma}\times\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma} \,-\, \tfrac{2i}{3}\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\big)\\
&=\; -\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\big) \,+\, \tfrac{2i}{3}\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big(3\times(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\big)\\
&=\; -\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big((\partial_\nu\Lambda)\times(\mathcal{F}_{\rho\sigma} \,-\, 2i\mathcal{A}_\rho\mathcal{A}_\sigma)\big).
\end{aligned} \tag{S.96}$$

Moreover, on the bottom line here

$$\begin{aligned}
\mathcal{F}_{\rho\sigma} \,-\, 2i\mathcal{A}_\rho\mathcal{A}_\sigma \,-\, (\rho\leftrightarrow\sigma) \;&=\; 2\mathcal{F}_{\rho\sigma} \,-\, 2i\mathcal{A}_\rho\mathcal{A}_\sigma \,+\, 2i\mathcal{A}_\sigma\mathcal{A}_\rho\\
&=\; 2\big(\partial_\rho\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\rho + i[\mathcal{A}_\rho,\mathcal{A}_\sigma]\big) \,-\, 2i[\mathcal{A}_\rho,\mathcal{A}_\sigma]\\
&=\; 2\partial_\rho\mathcal{A}_\sigma \,-\, 2\partial_\sigma\mathcal{A}_\rho \,+\, \text{nothing else}\\
&=\; 2\partial_\rho\mathcal{A}_\sigma \,-\, (\rho\leftrightarrow\sigma),
\end{aligned} \tag{S.97}$$

hence in the antisymmetrized context of eq. (S.96) we may replace

$$\mathcal{F}_{\rho\sigma} \,-\, 2i\mathcal{A}_\rho\mathcal{A}_\sigma \;\longrightarrow\; 2\partial_\rho\mathcal{A}_\sigma, \tag{S.98}$$

thus

$$\delta\Omega^\mu_{(3)} \;=\; -2\epsilon^{\mu\nu\rho\sigma}\times\mathrm{tr}\big((\partial_\nu\Lambda)\times(\partial_\rho\mathcal{A}_\sigma)\big) \;=\; -2\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu\,\mathrm{tr}\big(\Lambda\times(\partial_\rho\mathcal{A}_\sigma)\big). \tag{S.99}$$

Or in other words,

$$\delta\Omega^\mu_{(3)} \;=\; -2\partial_\nu H^{\mu\nu}_{(2)} \tag{S.100}$$

$$\text{for}\quad H^{\mu\nu}_{(2)} \;=\; \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\big(\Lambda\times(\partial_\rho\mathcal{A}_\sigma)\big) \;=\; -H^{\nu\mu}_{(2)}, \tag{S.101}$$

*quod erat demonstrandum.*

Now consider the more complicated $d = 6$ case of the $\Omega^\mu_{(5)}$. Similar to eq. (S.93) for the $d = 4$ case, we have

$$\begin{aligned}\delta\big(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\big) &= -(\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta} - i[A_\nu,\Lambda]\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta} - i\mathcal{A}_\nu[\mathcal{F}_{\rho\sigma},\Lambda\mathcal{F}_{\alpha\beta} - i\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}[\mathcal{F}_{\alpha\beta},\Lambda] \\ &= -(\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta} - i[\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta},\Lambda]\end{aligned} \tag{S.102}$$

where the net commutator on the bottom line has zero trace, thus

$$\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\big) = -\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\big). \tag{S.103}$$

Likewise,

$$\begin{aligned}\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\big) &= -\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu\mathcal{A}\rho(\partial_\sigma\Lambda)\mathcal{F}_{\alpha\beta}\big) \\ &\quad - i\,\mathrm{tr}\big([\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta},\Lambda]\big) \\ &= -\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\mathcal{F}_{\alpha\beta}\big),\end{aligned} \tag{S.104}$$

and similarly

$$\begin{aligned}\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big) &= -\,\mathrm{tr}\begin{pmatrix}(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta + \mathcal{A}_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta + \mathcal{A}_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\mathcal{A}_\alpha\mathcal{A}_\beta \\ + \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\Lambda)\mathcal{A}_\beta + \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha(\partial_\beta\Lambda)\end{pmatrix} \\ &\quad - i\,\mathrm{tr}\big([\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta,\Lambda]\big) \\ &= -\,\mathrm{tr}\begin{pmatrix}(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta + \mathcal{A}_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta + \mathcal{A}_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\mathcal{A}_\alpha\mathcal{A}_\beta \\ + \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\Lambda)\mathcal{A}_\beta + \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha(\partial_\beta\Lambda)\end{pmatrix}.\end{aligned} \tag{S.105}$$

Moreover, when we multiply these variations by the $\epsilon$ tensor and use its symmetry WRT to cyclic permutations of its last 5 indices as well as the cyclic symmetry of the trace, we get

$$\begin{aligned}&\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\,\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\big) \\ &\qquad= -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\mathcal{F}_{\alpha\beta}\big) \\ &\qquad= -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\beta(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{F}_{\sigma\alpha} + \mathcal{A}_\alpha\mathcal{A}_\beta(\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\big) \\ &\qquad= -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\big((\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + (\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{F}_{\sigma\alpha}\mathcal{A}_\beta + (\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\mathcal{A}_\alpha\mathcal{A}_\beta\big) \\ &\qquad= -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((\partial_\nu\Lambda)\times\big(\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} + \mathcal{A}_\rho\mathcal{F}_{\sigma\alpha}\mathcal{A}_\beta + \mathcal{F}_{\rho\sigma}\mathcal{A}_\alpha\mathcal{A}_\beta\big)\Big),\end{aligned} \tag{S.106}$$

and likewise

$$
\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\,\delta\,\mathrm{tr}\big(\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big)\\
&\qquad=\ -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\begin{pmatrix}(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\ +\ \mathcal{A}_\nu(\partial_\rho\Lambda)\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\ +\ \mathcal{A}_\nu\mathcal{A}_\rho(\partial_\sigma\Lambda)\mathcal{A}_\alpha\mathcal{A}_\beta\\ +\ \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\Lambda)\mathcal{A}_\beta\ +\ \mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha(\partial_\beta\Lambda)\end{pmatrix}\\
&\qquad=\ -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\begin{pmatrix}(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\ +\ \mathcal{A}_\beta(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\ +\ \mathcal{A}_\alpha\mathcal{A}_\beta(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\\ +\ \mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta(\partial_\nu\Lambda)\mathcal{A}_\rho\ +\ \mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta(\partial_\nu\Lambda)\end{pmatrix}\\
&\qquad=\ -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(5\times(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\Big).
\end{aligned}
\tag{S.107}
$$

Eqs. (S.103), (S.106), and (S.107) give us variances of the three terms comprising the $\Omega^\mu_{(5)}$ Chern–Simons vector. Adding up these equations with appropriate coefficients, we get

$$
\begin{aligned}
\delta\Omega^\mu_{(5)}\ &=\ -2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\begin{pmatrix}(\partial_\nu\Lambda)\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\ -\ i(\partial_\nu\Lambda)\times\big(\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\rho\mathcal{F}_{\sigma\alpha}\mathcal{A}_\beta\ +\ \mathcal{F}_{\rho\sigma}\mathcal{A}_\alpha\mathcal{A}_\beta\big)\\ -\ 2\times(\partial_\nu\Lambda)\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\end{pmatrix}\\
&=\ -2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\partial_\nu\Lambda)\times K_{\rho\sigma\alpha\beta}\Big)
\end{aligned}
\tag{S.108}
$$

for

$$
K_{\rho\sigma\alpha\beta}\ \stackrel{\mathrm{def}}{=}\ \mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta}\ -\ i\big(\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta}\ +\ \mathcal{A}_\rho\mathcal{F}_{\sigma\alpha}\mathcal{A}_\beta\ +\ \mathcal{F}_{\rho\sigma}\mathcal{A}_\alpha\mathcal{A}_\beta\big)\ -\ 2\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\,. \tag{S.109}
$$

Each of the non-abelian $\mathcal{F}$'s here has form

$$
\begin{aligned}
\mathcal{F}_{\rho\sigma}\ &=\ \partial_\rho\mathcal{A}_\sigma\ -\ \partial_\sigma\mathcal{A}_\rho\ +\ i\mathcal{A}_\rho\mathcal{A}_\sigma\ -\ i\mathcal{A}_\sigma\mathcal{A}_\rho\\
&=\ \big(\partial_\rho\mathcal{A}_\sigma\ +\ i\mathcal{A}_\rho\mathcal{A}_\sigma\big)\ -\ (\rho\leftrightarrow\sigma),
\end{aligned}
\tag{S.110}
$$

so in the context of $K_{\rho\sigma\alpha\beta}$ multiplied by the $\epsilon$ tensor which automatically antisymmetrizes all the indices, we may just as well replace

$$
\mathcal{F}_{\rho\sigma}\ \longrightarrow\ 2\big(\partial_\rho\mathcal{A}_\sigma\ +\ i\mathcal{A}_\rho\mathcal{A}_\sigma\big) \tag{S.111}
$$

and likewise for the $\mathcal{F}_{\alpha\beta}$ and $\mathcal{F}_{\sigma\alpha}$. Thus, in this context

$$\begin{aligned}
K_{\rho\sigma\alpha\beta} \;&\cong\; 4\big(\partial_\rho\mathcal{A}_\sigma + i\mathcal{A}_\rho\mathcal{A}_\sigma\big)\big(\partial_\alpha\mathcal{A}_\beta + i\mathcal{A}_\alpha\mathcal{A}_\beta\big)\\
&\quad -\; 2i\mathcal{A}_\rho\mathcal{A}_\sigma\big(\partial_\alpha\mathcal{A}_\beta + i\mathcal{A}_\alpha\mathcal{A}_\beta\big) \;-\; 2i\mathcal{A}_\rho\big(\partial_\sigma\mathcal{A}_\alpha + i\mathcal{A}_\sigma\mathcal{A}_\alpha\big)\mathcal{A}_\beta \;-\; 2i\big(\partial_\rho\mathcal{A}_\sigma + i\mathcal{A}_\rho\mathcal{A}_\sigma\big)\mathcal{A}_\alpha\mathcal{A}_\beta\\
&\quad -\; 2\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\\
&=\; 4(\partial_\rho\mathcal{A}_\sigma)(\partial_\alpha\mathcal{A}_\beta) \;+\; (4-2)i\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\mathcal{A}_\beta) \;+\; (4-2)i(\partial_\rho\mathcal{A}_\sigma)\mathcal{A}_\alpha\mathcal{A}_\beta\\
&\quad -\; 2i\mathcal{A}_\rho(\partial_\sigma\mathcal{A}_\alpha)\mathcal{A}_\beta \;+\; (-4+6-2 \;=\; 0)\times\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\\
&=\; 4(\partial_\rho\mathcal{A}_\sigma)(\partial_\alpha\mathcal{A}_\beta) \;+\; 2i\big(\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\mathcal{A}_\beta) \;-\; \mathcal{A}_\rho(\partial_\sigma\mathcal{A}_\alpha)\mathcal{A}_\beta \;+\; (\partial_\rho\mathcal{A}_\sigma)\mathcal{A}_\alpha\mathcal{A}_\beta\big).
\end{aligned} \tag{S.112}$$

Furthermore, in the context of the $\epsilon$ tensor, permuting the indices $(\rho,\sigma,\alpha,\beta)$ and changing the overall sign as needed, we get

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\mathcal{A}_\rho(\partial_\sigma\mathcal{A}_\alpha)\mathcal{A}_\beta \;&=\; -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\mathcal{A}_\sigma(\partial_\rho\mathcal{A}_\alpha)\mathcal{A}_\beta\,,\\
\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\mathcal{A}_\beta) \;&=\; +\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\mathcal{A}_\sigma\mathcal{A}_\alpha(\partial_\rho\mathcal{A}_\beta),
\end{aligned} \tag{S.113}$$

and hence

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\big(\mathcal{A}_\rho\mathcal{A}_\sigma(\partial_\alpha\mathcal{A}_\beta) \;-\; \mathcal{A}_\rho(\partial_\sigma\mathcal{A}_\alpha)\mathcal{A}_\beta \;+\; (\partial_\rho\mathcal{A}_\sigma)\mathcal{A}_\alpha\mathcal{A}_\beta\big)\\
&\quad=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\big(\mathcal{A}_\sigma\mathcal{A}_\alpha(\partial_\rho\mathcal{A}_\beta) \;+\; \mathcal{A}_\sigma\mathcal{A}_\alpha(\partial_\rho\mathcal{A}_\beta) + (\partial_\rho\mathcal{A}_\sigma)\mathcal{A}_\alpha\mathcal{A}_\beta\big)\\
&\quad=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\times\partial_\rho\big(\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big).
\end{aligned} \tag{S.114}$$

Also,

$$\epsilon^{\mu\nu\rho\sigma\alpha\beta}(\partial_\rho\mathcal{A}_\sigma)(\partial_\alpha\mathcal{A}_\beta) \;=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\partial_\rho\big(A_\sigma\partial_\alpha\mathcal{A}_\beta\big), \tag{S.115}$$

so altogether

$$\epsilon^{\mu\nu\rho\sigma\alpha\beta}K_{\rho\sigma\alpha\beta} \;=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\partial_\rho\big(4A_\sigma\partial_\alpha\mathcal{A}_\beta \;+\; 2i\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big). \tag{S.116}$$

Finally, plugging this formula into eq. (S.108), we get

$$\begin{aligned}
\delta\Omega^\mu_{(5)} \;&=\; -\operatorname{tr}\Big((\partial_\nu\Lambda)\times 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}K_{\rho\sigma\alpha\beta}\Big)\\
&=\; -2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\operatorname{tr}\Big((\partial_\nu\Lambda)\times\partial_\rho\big(4A_\sigma\partial_\alpha\mathcal{A}_\beta \;+\; 2i\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big)\Big)\\
&=\; -2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\partial_\nu\operatorname{tr}\Big(\Lambda\times\partial_\rho\big(4A_\sigma\partial_\alpha\mathcal{A}_\beta \;+\; 2i\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big)\Big).
\end{aligned} \tag{S.117}$$

Or in other words,

$$\delta\Omega^\mu_{(5)} \;=\; -2\partial_\nu H^{\mu\nu}_{(4)} \tag{S.118}$$

$$\text{for} \quad H^{\mu\nu}_{(4)} \;=\; -H^{\nu\mu}_{(4)} \;=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(\Lambda\times\partial_\rho\big(4A_\sigma\partial_\alpha\mathcal{A}_\beta\,+\,2i\mathcal{A}_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\big)\Big), \tag{S.119}$$

*quod erat demonstrandum.*