Free Fields, Harmonic Oscillators, and Identical Bosons

A free quantum field and its canonical conjugate are equivalent to a family of harmonic oscillators (one oscillator for each plane wave), which is in turn equivalent to a quantum theory of free identical bosons. In these notes, I will show how all of this works for the relativistic scalar field $\hat{\varphi}(x)$ and its conjugate $\hat{\pi}(x)$. And then in a separate set of notes I shall turn around and show that a quantum theory of any kind of identical bosons is equivalent to a family of oscillators. (Harmonic for the free particles, non-harmonic if the particles interact with each other.) Moreover, for the non-relativistic particles, the oscillator family is in turn equivalent to a non-relativistic quantum field theory.

In these notes we shall work in the Schrödinger picture of Quantum Mechanics because it's more convenient for dealing with the eigenstates and the eigenvalues. Consequently, all operators — including the quantum fields such as $\hat{\varphi}(\mathbf{x})$ — are time-independent.

From Relativistic Fields to Harmonic Oscillators

Let us start with the relativistic scalar field $\hat{\varphi}(\mathbf{x})$ and its conjugate $\hat{\pi}(\mathbf{x})$; they obey the canonical commutation relations

$$[\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{x}')] = 0, \quad [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] = 0, \quad [\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \tag{1}$$

and are governed by the Hamiltonian

$$\hat{H} = \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\pi}^2(\mathbf{x}) + \tfrac{1}{2}(\nabla\hat{\varphi}(\mathbf{x}))^2 + \tfrac{1}{2}m^2\hat{\varphi}^2(\mathbf{x}) \right). \tag{2}$$

We want to expand the fields into plane-wave modes $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$, and to avoid technical difficulties with the oscillators and their eigenstates, we want discrete modes. Therefore, we replace the infinite $\mathbf{x}$ space with a finite but very large box of size $L \times L \times L$, and impose periodic boundary conditions — $\hat{\varphi}(x + L, y, z) = \hat{\varphi}(x, y + L, z) = \hat{\varphi}(x, y, z + L) = \hat{\varphi}(x, y, z)$, *etc.*, *etc.* For large L, the specific boundary conditions are unimportant, so I have chosen the periodic conditions since they give us particularly simple plane-wave modes

$$\psi_{\mathbf{k}}(\mathbf{x}) \;=\; L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \qquad \text{where} \quad k_x, k_y, k_z \;=\; \frac{2\pi}{L} \times \text{an integer.} \tag{3}$$

Expanding the quantum fields into such modes, we get

$$\begin{aligned} \hat{\varphi}(\mathbf{x}) \;&=\; \sum_{\mathbf{k}} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \times \hat{\varphi}_{\mathbf{k}}, \qquad & \hat{\varphi}_{\mathbf{k}} \;&=\; \int d^3\mathbf{x}\, L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \times \hat{\varphi}(\mathbf{x}), \\ \hat{\pi}(\mathbf{x}) \;&=\; \sum_{\mathbf{k}} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \times \hat{\pi}_{\mathbf{k}}, \qquad & \hat{\pi}_{\mathbf{k}} \;&=\; \int d^3\mathbf{x}\, L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \times \hat{\pi}(\mathbf{x}). \end{aligned} \tag{4}$$

A note on hermiticity: The classical fields $\varphi(\mathbf{x})$ and $\pi(\mathbf{x})$ are real (*i.e.*, their values are real numbers), so the corresponding quantum fields are hermitian, $\hat{\varphi}^\dagger(\mathbf{x}) = \hat{\varphi}(\mathbf{x})$ and $\hat{\pi}^\dagger(\mathbf{x}) = \hat{\pi}(\mathbf{x})$. However, the mode operators $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ are not hermitian; instead, eqs. (4) give us $\hat{\varphi}^\dagger_{\mathbf{k}} = \hat{\varphi}_{-\mathbf{k}}$ and $\hat{\pi}^\dagger_{\mathbf{k}} = \hat{\pi}_{-\mathbf{k}}$.

The commutation relations between the mode operators follow from eqs. (1), namely

$$[\hat{\varphi}_{\mathbf{k}}, \hat{\varphi}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{\pi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{\varphi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] \;=\; i\,\delta_{\mathbf{k},-\mathbf{k}'}\,. \tag{5}$$

The first two relations here are obvious, but the third needs a bit of algebra:

$$\begin{aligned} [\hat{\varphi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] \;&=\; \int d^3\mathbf{x} \int d^3\mathbf{x}'\, L^{-3} e^{-i\mathbf{k}\mathbf{x}} e^{-i\mathbf{k}'\mathbf{x}'} \times [\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')] \\ &=\; \int d^3\mathbf{x} \int d^3\mathbf{x}'\, L^{-3} e^{-i\mathbf{k}\mathbf{x}} e^{-i\mathbf{k}'\mathbf{x}'} \times i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \\ &=\; i\, L^{-3} \int\limits_{\text{box}} d^3\mathbf{x}\, e^{-i\mathbf{x}(\mathbf{k}+\mathbf{k}')} \\ &=\; i\,\delta_{\mathbf{k},-\mathbf{k}'}\,. \end{aligned} \tag{6}$$

Equivalently,

$$[\hat{\varphi}_{\mathbf{k}}, \hat{\pi}^\dagger_{\mathbf{k}'}] \;=\; [\hat{\varphi}^\dagger_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] \;=\; i\,\delta_{\mathbf{k},\mathbf{k}'}\,, \quad [\hat{\varphi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] \;=\; [\hat{\varphi}^\dagger_{\mathbf{k}}, \hat{\pi}^\dagger_{\mathbf{k}'}] \;=\; i\,\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\,. \tag{7}$$

Now let's express the Hamiltonian (2) in terms of the modes. For the first term, we have

$$\begin{aligned}
\int d^3\mathbf{x}\,\hat{\pi}^2(\mathbf{x}) &= \int d^3\mathbf{x}\,\hat{\pi}^\dagger(\mathbf{x})\hat{\pi}(\mathbf{x}) = \int d^3\mathbf{x}\sum_{\mathbf{k}}\sum_{\mathbf{k}'} L^{-3} e^{-i\mathbf{k}\mathbf{x}} e^{+i\mathbf{k}'\mathbf{x}} \times \hat{\pi}^\dagger_{\mathbf{k}}\hat{\pi}_{\mathbf{k}'} \\
&= \sum_{\mathbf{k},\mathbf{k}'} \hat{\pi}^\dagger_{\mathbf{k}}\hat{\pi}_{\mathbf{k}'} \times \left(L^{-3}\int_{\text{box}} d^3\mathbf{x}\, e^{i\mathbf{x}(\mathbf{k}'-\mathbf{k})} = \delta_{\mathbf{k},\mathbf{k}'} \right) \\
&= \sum_{\mathbf{k}} \hat{\pi}^\dagger_{\mathbf{k}}\hat{\pi}_{\mathbf{k}}\,.
\end{aligned} \tag{8}$$

Similarly, the last term becomes

$$\int d^3\mathbf{x}\,\hat{\varphi}^2(\mathbf{x}) = \sum_{\mathbf{k}} \hat{\varphi}^\dagger_{\mathbf{k}}\hat{\varphi}_{\mathbf{k}}\,, \tag{9}$$

while in the second term

$$\nabla\hat{\varphi}(\mathbf{x}) = \sum_{\mathbf{k}} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \times i\mathbf{k}\,\hat{\varphi}_{\mathbf{k}} = \sum_{\mathbf{k}} L^{-3/2} e^{-i\mathbf{k}\mathbf{x}} \times -i\mathbf{k}\,\hat{\varphi}^\dagger_{\mathbf{k}}\,, \tag{10}$$

hence

$$\int d^3\mathbf{x}\,(\nabla\hat{\varphi}(\mathbf{x}))^2 = \sum_{\mathbf{k}} \mathbf{k}^2\,\hat{\varphi}^\dagger_{\mathbf{k}}\hat{\varphi}_{\mathbf{k}}\,. \tag{11}$$

Altogether, the Hamiltonian (2) becomes

$$\hat{H} = \sum_{\mathbf{k}} \left(\tfrac{1}{2}\,\hat{\pi}^\dagger_{\mathbf{k}}\hat{\pi}_{\mathbf{k}} + \tfrac{1}{2}(\mathbf{k}^2+m^2)\,\hat{\varphi}^\dagger_{\mathbf{k}}\hat{\varphi}_{\mathbf{k}} \right). \tag{12}$$

Clearly, this Hamiltonian describes a bunch of harmonic oscillators with frequencies $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$ (in the $\hbar = c = 1$ units). But since the mode operators are not hermitian, converting them into creation and annihilation operators takes a little more work then usual:

We define

$$\hat{a}_{\mathbf{k}} \;=\; \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(\omega_{\mathbf{k}}\hat{\varphi}_{\mathbf{k}} \,+\, i\,\hat{\pi}_{\mathbf{k}}\right),$$

and consequently

$$\begin{aligned}
\hat{a}^\dagger_{\mathbf{k}} \;&=\; \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(\omega_{\mathbf{k}}\hat{\varphi}^\dagger_{\mathbf{k}} \,-\, i\,\hat{\pi}^\dagger_{\mathbf{k}}\right),\\
\hat{a}_{-\mathbf{k}} \;&=\; \frac{1}{\sqrt{2\omega_{-\mathbf{k}}}}\left(\omega_{-\mathbf{k}}\hat{\varphi}_{-\mathbf{k}} \,+\, i\,\hat{\pi}_{-\mathbf{k}}\right) \;=\; \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(\omega_{\mathbf{k}}\hat{\varphi}^\dagger_{\mathbf{k}} \,+\, i\,\hat{\pi}^\dagger_{\mathbf{k}}\right),\\
\hat{a}^\dagger_{-\mathbf{k}} \;&=\; \frac{1}{\sqrt{2\omega_{-\mathbf{k}}}}\left(\omega_{-\mathbf{k}}\hat{\varphi}^\dagger_{-\mathbf{k}} \,-\, i\,\hat{\pi}^\dagger_{-\mathbf{k}}\right) \;=\; \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(\omega_{\mathbf{k}}\hat{\varphi}_{\mathbf{k}} \,-\, i\,\hat{\pi}_{\mathbf{k}}\right).
\end{aligned} \tag{13}$$

Note that $\hat{a}^\dagger_{\mathbf{k}} \neq \hat{a}_{-\mathbf{k}}$ and $\hat{a}_{-\mathbf{k}} \neq \hat{a}^\dagger_{\mathbf{k}}$; instead, we have independent creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ for every mode $\mathbf{k}$.

The commutations relations between these operators are

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{14}$$

Indeed,

$$\begin{aligned}
[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \;&=\; \frac{1}{\sqrt{4\omega\omega'}}\Big(\omega\omega'\,[\hat{\varphi}_{\mathbf{k}}, \hat{\varphi}_{\mathbf{k}'}] \,+\, i\omega'\,[\hat{\pi}_{\mathbf{k}}, \hat{\varphi}_{\mathbf{k}'}] \,+\, i\omega\,[\hat{\varphi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] \,-\, [\hat{\pi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}]\Big)\\
&=\; \frac{1}{\sqrt{4\omega\omega'}}\left(0 \,+\, i\omega' \times -i\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \,+\, i\omega \times +i\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \,+\, 0\right)\\
&=\; \delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \times \frac{\omega' - \omega}{\sqrt{4\omega\omega'}}\\
&=\; 0 \qquad \text{because } \omega' = \omega \text{ when } \mathbf{k} + \mathbf{k}' = \mathbf{0}.
\end{aligned} \tag{15}$$

Similarly, $[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0$. Finally,

$$\begin{aligned}
[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;&=\; \frac{1}{\sqrt{4\omega\omega'}}\Big(\omega\omega'\,[\hat{\varphi}_{\mathbf{k}}, \hat{\varphi}^\dagger_{\mathbf{k}'}] \,+\, i\omega'\,[\hat{\pi}_{\mathbf{k}}, \hat{\varphi}^\dagger_{\mathbf{k}'}] \,-\, i\omega\,[\hat{\varphi}_{\mathbf{k}}, \hat{\pi}^\dagger_{\mathbf{k}'}] \,+\, [\hat{\pi}_{\mathbf{k}}, \hat{\pi}^\dagger_{\mathbf{k}'}]\Big)\\
&=\; \frac{1}{\sqrt{4\omega\omega'}}\left(0 \,+\, i\omega' \times -i\delta_{\mathbf{k},\mathbf{k}'} \,-\, i\omega \times +i\delta_{\mathbf{k},\mathbf{k}'} \,+\, 0\right)\\
&=\; \delta_{\mathbf{k},\mathbf{k}'} \times \frac{\omega + \omega'}{\sqrt{4\omega\omega'}}\\
&=\; \delta_{\mathbf{k},\mathbf{k}'} \qquad \text{because } \omega' = \omega \text{ when } \mathbf{k}' = \mathbf{k}.
\end{aligned} \tag{16}$$

To re-obtain the field mode operators $\hat{\varphi}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ from the creation and annihilation operators, let us combine the first and the last equations (13) for the $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{-\mathbf{k}}$. Adding and subtracting those equations, we find

$$\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}} = \sqrt{2\omega_{\mathbf{k}}} \times \hat{\varphi}_{\mathbf{k}}, \qquad -i\hat{a}_{\mathbf{k}} + i\hat{a}^{\dagger}_{-\mathbf{k}} = \sqrt{\frac{2}{\omega_{\mathbf{k}}}} \times \hat{\pi}_{\mathbf{k}}. \tag{17}$$

Consequently,

$$\begin{aligned} \omega_{\mathbf{k}}^2 \hat{\varphi}^{\dagger}_{\mathbf{k}} \hat{\varphi}_{\mathbf{k}} &= \frac{\omega_{\mathbf{k}}}{2} \times (\hat{a}^{\dagger}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}})(\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}) \\ &= \frac{\omega_{\mathbf{k}}}{2} \times \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} \right), \\ \hat{\pi}^{\dagger}_{\mathbf{k}} \hat{\pi}_{\mathbf{k}} &= \frac{\omega_{\mathbf{k}}}{2} \times (i\hat{a}^{\dagger}_{\mathbf{k}} - i\hat{a}_{-\mathbf{k}})(-i\hat{a}_{\mathbf{k}} + i\hat{a}^{\dagger}_{-\mathbf{k}}) \\ &= \frac{\omega_{\mathbf{k}}}{2} \times \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} - \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} \right), \end{aligned} \tag{18}$$

hence

$$\begin{aligned} \omega_{\mathbf{k}}^2 \hat{\varphi}^{\dagger}_{\mathbf{k}} \hat{\varphi}_{\mathbf{k}} + \hat{\pi}^{\dagger}_{\mathbf{k}} \hat{\pi}_{\mathbf{k}} &= \omega_{\mathbf{k}} \times \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{-\mathbf{k}} \hat{a}^{\dagger}_{-\mathbf{k}} \right) \\ &= \omega_{\mathbf{k}} \times \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}_{-\mathbf{k}} + 1 \right). \end{aligned} \tag{19}$$

Altogether, the Hamiltonian (12) becomes

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}} \left(\frac{1}{2} \hat{\pi}^{\dagger}_{\mathbf{k}} \hat{\pi}_{\mathbf{k}} + \frac{\omega_{\mathbf{k}}^2}{2} \hat{\varphi}^{\dagger}_{\mathbf{k}} \hat{\varphi}_{\mathbf{k}} \right) \\ &= \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2} \times \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}_{-\mathbf{k}} + 1 \right) \\ &= \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2} \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \tfrac{1}{2} \right) + \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{2} \left(\hat{a}^{\dagger}_{-\mathbf{k}} \hat{a}_{-\mathbf{k}} + \tfrac{1}{2} \right) \\ &\quad \langle\!\langle \text{ where the two sums are equal due to } \mathbf{k} \leftrightarrow -\mathbf{k} \text{ symmetry } \rangle\!\rangle \\ &= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\hat{a}^{\dagger}_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \tfrac{1}{2} \right). \end{aligned} \tag{20}$$

In light of the commutation relations (14), this Hamiltonian clearly describes an infinite family of harmonic oscillators, one oscillator for each plane-wave mode **k**.

Now consider the eigenvalues and the eigenstates of the multi-oscillator Hamiltonian (20). A single harmonic oscillator has eigenvalues $E_n = \omega(n + \frac{1}{2})$ where $n = 0, 1, 2, 3, \ldots$. For the multi-oscillator system at hand, each $\hat{n}_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}$ commutes with all the other $\hat{n}_{\mathbf{k}'}$, so we may diagonalize them all at the same time. This gives us eigenstates

$$|\{n_{\mathbf{k}} \text{ for all } \mathbf{k}\}\rangle \;=\; \bigotimes_{\mathbf{k}} |n_{\mathbf{k}}\rangle \qquad \text{of energy} \quad E_{\{n_{\mathbf{k}}\}} \;=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}}(n_{\mathbf{k}} + \tfrac{1}{2}). \tag{21}$$

where each $n_{\mathbf{k}}$ is an integer ≥ 0. Moreover, all combinations of the $n_{\mathbf{k}}$ are allowed because the $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ operators can change a particular $n_{\mathbf{k}} \to n_{\mathbf{k}} \pm 1$ without affecting any other $n_{\mathbf{k}'}$. (This follows from $[\hat{a}_{\mathbf{k}}, \hat{n}_{\mathbf{k}'}] = 0$ and $[\hat{a}^\dagger_{\mathbf{k}}, \hat{n}_{\mathbf{k}'}] = 0$ for $\mathbf{k}' \neq \mathbf{k}$.) Thus, the Hilbert space of the multi-oscillator system — and hence of the free quantum field theory — is a direct product of Hilbert spaces for each oscillator,

$$\mathcal{H}(\text{QFT}) \;=\; \bigotimes_{\mathbf{k}} \mathcal{H}(\text{harmonic oscillator for mode } \mathbf{k}). \tag{22}$$

From the Multi-Oscillator System to Identical Bosons

A constant term in the Hamiltonian of a quantum system does not affect its dynamics in any way, it simply shifts energies of all states by the same constant amount. So to simplify our analysis of the multi-oscillator system in particle terms, let's subtract the infinite zero-point energy $E_0 = \sum_{\mathbf{k}} \frac{1}{2}\omega_{\mathbf{k}}$ from the Hamiltonian (20), thus

$$\hat{H} \;\to\; \hat{H} \;-\; E_0 \;=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}\,. \tag{23}$$

I'll come back to the zero-point energy, but right now let's focus on other issues.

In the multi-oscillator Hilbert space (22) each *occupation number* $n_{\mathbf{k}}$ is independent from all others. However, *states of finite energy must have finite* $N = \sum_{\mathbf{k}} n_{\mathbf{k}}$, so let us re-organize the Hilbert space into eigenspaces of the $\hat{N} = \sum_{\mathbf{k}} \hat{n}_{\mathbf{k}}$ operator,

$$\mathcal{H}(\text{QFT}) \;=\; \bigotimes_{\mathbf{k}} \mathcal{H}(\text{mode } \mathbf{k}) \;=\; \bigoplus_{N=0}^{\infty} \mathcal{H}_N\,, \tag{24}$$

and consider what do those eigenspaces look like for different N. For $N = 0$, the $\mathcal{H}_0$ spans a single state, the *vacuum* $|0\rangle = |\text{all } n_{\mathbf{k}} = 0\rangle$. For $N = 1$, the $\mathcal{H}_1$ spans eigenstates with a single $n_{\mathbf{k}} = 1$ while all other $n_{\mathbf{k}'} = 0$. Renaming such eigenstates $|n_{\mathbf{k}} = 1, \text{other}\, n = 0\rangle \to |\mathbf{k}\rangle$ and noting their energies

$$E(|\mathbf{k}\rangle) \;=\; \omega_{\mathbf{k}} \;=\; \sqrt{\mathbf{k}^2 + m^2}\,, \tag{25}$$

we identify the $\mathcal{H}_1$ as a Hilbert space of a free relativistic particle with Hamiltonian

$$\hat{H}^{\text{particle}} \;=\; \sqrt{\hat{\mathbf{P}}^2 + m^2}\,. \tag{26}$$

For $N > 1$, we may have several modes with $n_{\mathbf{k}} > 0$, but for a finite N there can be only a finite number of such modes. So we rename such a state $|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle$ by listing only the modes $\mathbf{k}$ with $n_{\mathbf{k}} > 0$ and repeating each $\mathbf{k}$ $n_{\mathbf{k}}$ times. For example,

$$\big|3_{\mathbf{k}}, 2_{\mathbf{k}'}, 2_{\mathbf{k}''}, 1_{\mathbf{k}'''}, 0_{\text{everything else}}\big\rangle \;=\; \big|\mathbf{k}, \mathbf{k}, \mathbf{k}, \mathbf{k}', \mathbf{k}', \mathbf{k}'', \mathbf{k}'', \mathbf{k}'''\big\rangle\,. \tag{27}$$

In such notations, the $\mathcal{H}_N$ Hilbert space spans eigenstates $|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N\rangle$ labeled by N modes $\mathbf{k}_1, \ldots, \mathbf{k}_N$ (such modes may coincide but do not have to). The energy of such an eigenstate is

$$E(|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N\rangle) \;=\; \omega_{\mathbf{k}_1} \,+\, \omega_{\mathbf{k}_2} \,+\, \cdots \,+\, \omega_{\mathbf{k}_N}\,, \tag{28}$$

which allows us to identify the $\mathcal{H}_N$ as the Hilbert space of N free relativistic particles with the Hamiltonian

$$\hat{H}^{N\,\text{particles}} \;=\; \sum_{i=1}^{N} \sqrt{\hat{\mathbf{P}}^2(i^{\text{th}}) \,+\, m^2}\,. \tag{29}$$

However, treating the $\mathbf{k}_1, \ldots, \mathbf{k}_N$ momenta of N particles as independent over-counts the quantum states because the occupation numbers $n_{\mathbf{k}}$ do not specify the *order* in which we list the modes $\mathbf{k}_i$. For example, both $|\mathbf{k}_1, \mathbf{k}_2\rangle$ and $|\mathbf{k}_2, \mathbf{k}_1\rangle$ correspond to the same quantum state $\big|1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, 0_{\text{others}}\big\rangle$. More generally,

$$|\{n_{\mathbf{k}}\}\rangle \;=\; |\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle \;=\; |\text{any permutation of the } \mathbf{k}_1, \ldots, \mathbf{k}_N\rangle\,. \tag{30}$$

In other words, the N relativistic particles in the $\mathcal{H}_N$ are *identical bosons.*

Altogether, we have

$$\mathcal{H}(\text{QFT}) \;=\; \bigotimes_{\mathbf{k}} \mathcal{H}(\text{harmonic oscillator } \#\mathbf{k}) \;=\; \bigoplus_{N=0}^{\infty} \mathcal{H}(N \text{ identical bosons}). \tag{31}$$

Hilbert spaces of this kind — any number N of identical bosons (or fermions) are known as *Fock spaces.* So the Hilbert space of the quantum field is the same as the Fock space of particles, and the Hamiltonians are also the same:

$$\hat{H}[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x})] \;=\; \sum_{i=0}^{\hat{N}} \sqrt{\hat{\mathbf{P}}_i^2 \,+\, m^2}\,. \tag{32}$$

In other words, the quantum theory of the free field is identical to the quantum theory of (any number of) free identical bosons. For the theory in question, the field is a relativistic scalar $\varphi(x)$ and the bosons are spinless relativistic particles. But in exactly the same manner, the quantum theory of Maxwell fields $F^{\mu\nu}(x)$ is identical to the quantum theory of (any number of) photons — which are massless relativistic particles with two polarizations states (per photon) and obey Bose statistics.

Quantization of field theories with non-quadratic Hamiltonians (and hence non-linear classical equations of motion) also leads to theories equivalent to theories of quantum particles, but this time the particles are not free but interact with each other. In relativistic theories, the interactions also allow for creation and destruction of particles; such processes have to be described in terms of the Fock space rather than a fixed–N Hilbert space. In non-relativistic theories, the net particle number N is sometimes conserved, sometimes not, but even when it is conserved, the Fock-space formalism is often convenient.

Finally, a few words about the zero-point energy $E_0 = \sum_{\mathbf{k}} \frac{1}{2}\omega_{\mathbf{k}}$. From the particles' point of view, E_0 is the vacuum energy. It does not affect any properties of the individual particles or the way they interact with each other, so one usually simply ignores the E_0 and proceeds as if it was not there. However, in some situations the E_0 becomes important: (1) When one couples a quantum field theory to general relativity, the vacuum energy density becomes the cosmological constant. (2) When a QFT has some variable parameters, the vacuum energy acts as an effective potential for those parameters. This is important for cosmology of the early Universe, and also for the Casimir effect. Note that while the E_0 itself is infinite (except in supersymmetric theories where infinities cancel out between the bosonic and fermionic fields), it can be written as a sum of an infinite *constant* and a finite part which changes with parameters by a finite amount ΔE_0 — it's the finite part that's responsible for the effective potential and for the Casimir effect.

In this class, I shall discuss the effective potential in an extra lecture sometimes in the second semester. Meanwhile, you can read my notes on the subject. As to the Casimir effect, I have an optional exercise where I explain how to calculate the Casimir energy in few easy steps and you work them out on your own. When I have time, I'll write down the solutions to this exercise.