1. Let's start with a reading assignment. The Laplace equation for the electrostatic potential $\Phi(\mathbf{x})$ subject to many kinds of boundary conditions is often solved by the method of variable separation in appropriate coordinates. You should have learned this method back in an undergraduate Electrodynamics class, but read the following sections of the Jackson's textbook to your memory and skills:

   (a) §2.9–10 about separation of variables in rectangular coordinates.

   (b) §3.1–3 about separation of variables in spherical coordinates for axisymmetric problems.

   (c) §3.5–6 about spherical harmonics and separation of variables in spherical coordinates for problems without axial symmetry.

   (d) §3.7–8 about Bessel functions and separation of variables in cylindrical coordinates.

2. Next, a few *simple* problems you can use by separating the variables.

   (a) Consider a cube of size $L \times L \times L$. The 4 vertical sides of the cube and its bottom are made of a conducting metal and grounded, but the top square is open. There are no electric charges inside the cube, but the outside charges give rise to a nontrivial potential $\Phi(x, y, z)$. Specifically, at the top square of the cube the potential happens to be

$$\Phi(x, y, z = L) \ = \ V_0 \times \sin\frac{3\pi x}{L} \times \sin\frac{4\pi y}{L}\,. \tag{1}$$

   Find the potential $\Phi(x, y, z)$ inside the cube.

   (b) Now consider a hollow cylindrical pipe of radius R and much larger length; for simplicity, assume the pipe runs from $z = 0$ to $z \to +\infty$. The pipe's wall is conducting and grounded, and you may also assume $\Phi \to 0$ for $z \to +\infty$. There are no electric charges inside the pipe, but the outside charges give rise to $\Phi(r, \phi, z) \neq 0$. Specifically, at the opening of the pipe

$$\Phi(r, \phi, z = 0) \ = \ V_0 \times J_1(kr) \times \cos\phi \tag{2}$$

   for some k such that $J_1(kR) = 0$. Find the potential $\Phi(r, \phi, z)$ inside the pipe.

(c) Next, a conducting sphere of radius R placed between very distant (and very large) plates of a capacitor. For simplicity, assume these plates are infinitely far away, and there are no other electric charges at finite distances from the sphere. However, the remote capacitor plates give rise to an electric field $\mathbf{E}(\mathbf{x})$ which asymptotes to uniform $\mathbf{E}_0$ for $r \to \infty$ (in any direction). Note: this is the asymptotic field for $r \gg R$, but closer to the sphere the electric field becomes more complicated.

Find the potential $\Phi(\mathbf{x})$ — and hence the electric field — everywhere outside the sphere.

3. Finally, consider the potential $\Phi(\mathbf{x})$ outside a spherical surface of radius R. There are no electric charges outside this surface, but there are some unknown charges inside it. Suppose you are given the boundary potential $\Phi_b(r = R, \theta, \phi)$ on the spherical surface in question, and you need to find the potential $\Phi(r, \theta, \phi)$ everywhere outside the sphere.

(a) Show that

$$\Phi(\mathbf{x}) = \iint\limits_{\text{sphere}} d^2\text{Area}(\mathbf{y})\, F(\mathbf{x}, \mathbf{y}) \times \Phi_b(\mathbf{y}) \tag{3}$$

where

$$F(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \frac{R^{\ell-1}}{|\mathbf{x}|^{\ell+1}} \sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n}_x)\, Y^*_{\ell,m}(\mathbf{n}_y). \tag{4}$$

(b) Sum the series (4) and show that

$$F(\mathbf{x}, \mathbf{y}) = \frac{|\mathbf{x}|^2 - R^2}{4\pi R} \times \frac{1}{|\mathbf{x} - \mathbf{y}|^3} \quad \text{for } |\mathbf{y}| = R. \tag{5}$$

FYI, here are some useful formulae:

$$\sum_{m=-\ell}^{+\ell} Y_{\ell,m}(\mathbf{n}_x) Y^*_{\ell,m}(\mathbf{n}_y) = \frac{2\ell+1}{4\pi} \times P_\ell(\mathbf{n}_x \cdot \mathbf{n}_y) \tag{6}$$

where $P_\ell(c)$ is the ℓ^{th} Legendre polynomial of c,

$$\sum_{\ell=0}^{\infty} t^\ell P_\ell(c) = \frac{1}{\sqrt{1 - 2tc + t^2}} \quad \text{for } |t| < 1 \text{ and } |c| \le 1, \tag{7}$$

$$\sum_{\ell=0}^{\infty} (2\ell+1) t^\ell P_\ell(c) = \text{(find out from the previous formula).} \tag{8}$$