1. In conducting materials, the EM waves attenuate with distance. For a specific example, consider a uniform material with dielectric constant ϵ and conductivity σ; assume the frequency is low enough that ϵ and σ are real. Also assume negligible magnetism, $\mu = 1$.

   The attenuating plane wave propagating in $\mathbf{z}$ direction has general form

$$\mathbf{E}(x,y,z,t) \;=\; \vec{\mathcal{E}}\exp(ikz - \kappa z - i\omega t), \quad \mathbf{H}(x,y,z,t) \;=\; \vec{\mathcal{H}}\exp(ikz - \kappa z - i\omega t). \quad (1)$$

   (a) Write down formulae for k and κ as functions of ω. Also, relate the electric amplitude $\vec{\mathcal{E}}$ and the magnetic amplitude $\vec{\mathcal{H}}$ to each other.

   Now consider a boundary between a conducting material and the vacuum. Suppose an EM wave comes from the vacuum side and hits the boundary head-on.

   (b) Calculate the reflectivity $R = |r|^2$ of the boundary.

   (c) Show that for a good conductor

$$R \;\approx\; 1 \;-\; \frac{4\pi\delta}{\lambda_0} \quad (2)$$

   where λ_0 is the wavelength of the EM wave in the vacuum and δ is the skin-depth of the current of the same frequency in the conductor.

   (d) As an example, find the reflectivity of sea water ($\sigma \approx 5\mho$/m) at an FM radio frequency $\omega = 2\pi \times 100$ MHz.

2. Next, consider charge density perturbations $\rho(\mathbf{x}, t)$ in a metal.

   (a) Fourier transform time-dependence of macroscopic EM fields and charge/current densities to frequency dependence. Use the transformed Maxwell equations — as well as

$$\mathbf{D}(\mathbf{x},\omega) \;=\; \epsilon(\omega)\epsilon_0\mathbf{E}(\mathbf{x},\omega) \quad \text{and} \quad \mathbf{J}_{\text{cond}}(\mathbf{x},\omega) \;=\; \sigma(\omega)\mathbf{E}(\mathbf{x},\omega) \quad (3)$$

   where $\epsilon(\omega)$ and $\sigma(\omega)$ are the AC dielectric constant and conductivity — to show that

$\rho(\mathbf{x},\omega)$ obeys

$$\Big(\sigma(\omega)\;-\;i\omega\epsilon(\omega)\epsilon_0\Big)\rho(\omega,\mathbf{x})\;=\;0 \tag{4}$$

Hint: the net conduction + displacement current has zero divergence.

Drude–Lorentz formula tell us that in metals

$$\sigma(\omega)\;-\;i\omega\epsilon(\omega)\epsilon_0\;=\;\frac{ne^2 f_0}{m_e^*}\,\frac{1}{\gamma_0-i\omega}\;-\;i\omega\epsilon_b\epsilon_0\;\approx\;\epsilon_b\epsilon_0\left(\frac{\omega_p^2}{\gamma_0-i\omega}\;-\;i\omega\right) \tag{5}$$

where $\gamma_0=(1/\tau)$ is the rate at which the conduction electrons lose their average velocity vector to collisions with ions, and ω_p is the plasma frequency of the metal.

(b) Solve eq. (4) for density perturbations in a metal with $\omega_p\gg\gamma_0$. Show that as a function of time rather than frequency, $\rho(t,\mathbf{x})$ oscillates in place with the plasma frequency ω_p while oscillation amplitude decays as $\exp(-\gamma_0 t)$.

3. Now consider a 1D wave propagating through a linear and homogeneous but dispersive media with refraction index $n(\omega)$, *i.e.*, the phase velocity of a wave $v(\omega)=c/n(\omega)$. To allow for absorption, $n(\omega)$ may be complex rather than real.

(a) Show that the most general solution of the dispersive wave equation is

$$\psi(x,t)\;=\;\int\limits_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\Big(A(\omega)\times\exp(+i\omega n(\omega)x/c)\;+\;B(\omega)\times\exp(-i\omega n(\omega)x/c)\Big) \tag{6}$$

for some arbitrary complex functions $A(\omega)$ and $B(\omega)$.

(b) Show that a real wave $\psi(x,t)$ requires $n(-\omega)=n^*(+\omega)$ as well as $A(-\omega)=A^*(+\omega)$ and $B(-\omega)=B^*(+\omega)$.

(c) Suppose at $x=0$ we observe ψ and its x derivative as functions of time. Show that in terms of these data

$$\begin{aligned} A(\omega) &= \int_{-\infty}^{+\infty} dt\, e^{+i\omega t}\left[\frac{1}{2}\,\psi(0,t) - \frac{ic}{2\omega n(\omega)}\,\frac{\partial\psi}{\partial x}(0,t)\right], \\ B(\omega) &= \int_{-\infty}^{+\infty} dt\, e^{+i\omega t}\left[\frac{1}{2}\,\psi(0,t) + \frac{ic}{2\omega n(\omega)}\,\frac{\partial\psi}{\partial x}(0,t)\right]. \end{aligned} \tag{7}$$

4. Finally, show that in the regime of normal dispersion — *i.e.*, at frequencies not too close to any of the resonances — the group velocity of the EM wave is always less than c. For simplicity, assume negligible magnetism $\mu \approx 1$ and use the low-density approximation to the dielectric constant,

$$\epsilon(\omega) \approx 1 + \frac{ne^2}{\epsilon_0 m_e} \sum_i^{\text{resonances}} \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}\,. \tag{8}$$

Hint: show that $v_{\text{group}} < v_{\text{phase}}$ and $v_{\text{group}} \times v_{\text{phase}} < c^2$.