1. This problem is about *birefringence* in anisotropic materials. The dielectric "constant" of anisotropic dielectric is a tensor ϵ_{ij} rather than a scalar, thus

$$D_i = \epsilon_{ij}\epsilon_0 E_j \,. \tag{1}$$

For simplicity, let's assume that at the optical frequencies $\epsilon_{ij}(\omega)$ is a real symmetric tensor, and that the material in question is non-conducting and non-magnetic, $\sigma = 0$ and $\mu = 1$.

Consider a plane EM wave

$$\mathbf{E}(\mathbf{x},t) = \vec{\mathcal{E}}\,\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) \tag{2}$$

propagating through such anisotropic material.

(a) Show that the electromagnetic fields of this wave obey

$$-\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = \omega^2\mu_0\mathbf{D}, \qquad \mathbf{H} = \frac{\mathbf{k}}{\omega\mu_0}\times\mathbf{E}, \tag{3}$$

where the magnetic field $\mathbf{H}$ and the electric displacement field $\mathbf{D}$ are transverse to the wave direction, but the electric tension field $\mathbf{E}$ is generally not transverse.

(b) A plane EM wave in an isotropic medium has its energy moving in the same direction as the wavefront, *i.e.* the direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ of the wave vector. But this is generally not true in an anisotropic medium: Show that for a plane wave with $\mathbf{E} \not\perp \mathbf{k}$, the wave's energy and the wavefront move in somewhat different directions.

In an anisotropic medium, the refraction index $n = c|\mathbf{k}|/\omega$ depends on the direction $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ of the wave vector. Moreover, for a given direction $\mathbf{k}$ the two independent polarizations of the wave generally have different refraction indices $n_1 \neq n_2$.

(c) Use eq. (3) to show that the refraction indices and the polarization vectors of the two independent polarizations obtain from the generalized eigenvalue problem

$$\Big(\epsilon_{ij} \;-\; n^2\big(\delta_{ij} \;-\; \hat{k}_i\hat{k}_j\big)\Big)\mathcal{E}_j \;=\; 0. \tag{4}$$

In particular, the (squares of the) refraction indices obtain as zeros of the determinant

$$\chi(n^2) \;=\; \det\Big(\epsilon_{ij} \;-\; n^2\big(\delta_{ij} \;-\; \hat{k}_i\hat{k}_j\big)\Big). \tag{5}$$

From now on, let's work in a Cartesian coordinate system where the ϵ_{ij} tensor is diagonal, $\epsilon_{ij} = \epsilon_i\delta_{ij}$.

(d) Calculate the determinant (5) in this basis and show that

$$\chi(n^2) \;=\; \sum_{i=1}^{3} \hat{k}_i{}^2\epsilon_i \times \prod_{j\neq i}(n^2 - \epsilon_j). \tag{6}$$

If you get bogged down in algebra, use Mathematica.

(e) Suppose the three eigenvalues of the ϵ_{ij} tensor are different, say $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$. Show that in this case, the square of one of the refraction indices lies between ϵ_1 and ϵ_2 while the square of the other lies between ϵ_2 and ϵ_3,

$$\epsilon_1 \;\geq\; n_1^2 \;\geq\; \epsilon_2 \;\geq\; n_2^2 \;\geq\; \epsilon_3\,. \tag{7}$$

Moreover, all these inequalities become strict when all 3 of the $\hat{k}_1^2$, $\hat{k}_2^2$, $\hat{k}_3^2$ are positive, *i.e.* when the wave vector $\mathbf{k}$ is not parallel to any principal axis of the ϵ_{ij} tensor. Also, in this case, the $\chi(n^2) = 0$ equation is equivalent to the *Fresnel equation*

$$\sum_{i=1}^{3} \frac{\epsilon_i\hat{k}_i^2}{n^2 - \epsilon_i} \;=\; 0. \tag{8}$$

Now suppose $\epsilon_1 = \epsilon_2 \neq \epsilon_3$; birefringence like this is called *uniaxial*, and the direction of the non-degenerate eigenvector is called the *optical axis*. For the waves traveling in the direction of that optical axis, there is no birefringence — both polarizations have the same $n = \sqrt{\epsilon_1 = \epsilon_2}$.

(f) Check this statement.

(g) Show that for the waves in all other directions $\mathbf{k} \neq \pm\mathbf{m}_3$, there are two independent polarizations with different refraction indices. Specifically:

($\perp$) $\vec{\mathcal{E}}$ is normal to both the optical axis and the wave direction $\hat{\mathbf{k}}$; for this wave, $n = \sqrt{\epsilon_1}$ regardless of the ϵ_3 or the angle θ.

($\|$) $\vec{\mathcal{E}}$ lies in the same plane as $\hat{\mathbf{k}}$ and the optical axis; for this wave,

$$n \;=\; \left(\frac{\sin^2\theta}{\epsilon_3} + \frac{\cos^2\theta}{\epsilon_1}\right)^{-1/2} \tag{9}$$

where θ is the angle between the wave direction $\mathbf{k}$ and the optical axis.

(h) Finally, show that for the ($\perp$) polarization, the wave's energy moves in the same direction $\hat{\mathbf{k}}$ as the wavefront; but for the ($\|$) polarization, the energy moves in a different direction from $\hat{\mathbf{k}}$. Also, calculate the angle between the directions of the energy's and the wave-front's motion for the ($\|$) polarization.

2. Now consider plasma in a uniform magnetic field $\mathbf{B}$. For simplicity, ignore the ions in the plasma and focus on the effect of the free electrons.

(a) Show that for a radio wave of frequency ω propagating through this plasma, the effective permittivity tensor is

$$\epsilon_{ij} \;=\; \delta_{ij} \;-\; \frac{\omega_p^2}{\omega^2(\omega^2 - \Omega^2)}\Big(\omega^2\delta_{ij} \;-\; \Omega^2\hat{b}_i\hat{b}_j \;-\; i\omega\Omega\epsilon_{ijk}\hat{b}_k\Big) \tag{10}$$

where $\omega_p = \sqrt{e^2 n_e/\epsilon_0 m_e}$ is the plasma frequency, $\Omega = (e/m_e)B$ is the cyclotron frequency of an electron in the magnetic field B, and $\hat{\mathbf{b}} = (\hat{b}_x, \hat{b}_y, \hat{b}_z)$ is the unit vector in the magnetic field's direction.

The tensor (10) is complex rather than real, but its matrix is Hermitian, $\epsilon_{ij}^* = \epsilon_{ji}$, so it has real eigenvalues $\epsilon_1, \epsilon_2, \epsilon_3$, although the corresponding eigenvectors $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ are complex rather than real.

(b) Find these eigenvalues and eigenvectors. For simplicity, work in the coordinate system where the z axis points in the direction of the magnetic field, thus $\hat{\mathbf{b}} = (0,0,1)$.

(c) Go back to the previous problem and show that for a complex orthonormal basis $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ for vectors' components,

$$\mathbf{m}_i^* \cdot \mathbf{m}_j \;=\; \delta_{ij} \quad \forall i,j = 1,2,3, \tag{11}$$

$$\text{any vector } \mathbf{v} \;=\; \sum_i v_i \mathbf{m}_i \quad \text{for } v_i \;=\; m_i^* \cdot \mathbf{v}, \tag{12}$$

eq. (4) becomes

$$\Big(\epsilon_{ij} \,-\, n^2\big(\delta_{ij} \,-\, \hat{k}_i \hat{k}_j^*\big)\Big)\mathcal{E}_j \;=\; 0. \tag{13}$$

Also show that for the $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ being complex eigenvectors of an Hermitian permittivity tensor, the Fresnel equation for the refraction indices[2] becomes

$$\sum_{i=1}^{3} \frac{\epsilon_i \times \big(|\hat{k}_i|^2 = |\mathbf{m}_i^* \cdot \hat{\mathbf{k}}|^2\big)}{n^2 - \epsilon_i} \;=\; 0. \tag{14}$$

Now return to the plasma in a magnetic field, and consider a wave propagating in a direction at angle θ from the direction of $\mathbf{B}$.

(d) Solve the Fresnel equation (14) for the plasma in the the high-frequency limit $\omega \gg \omega_p$.

- For simplicity, you may assume that $\omega \gg \Omega$ as well as $\omega \gg \omega_p$; in this limit, you should get

$$n_{1,2}^2 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2} \;\pm\; \frac{\omega_p^2 \Omega \cos\theta}{\omega^3} + \; O(1/\omega^4). \tag{15}$$

⋆ For extra credit, assume $\omega(\omega - \Omega) \gg \omega_p^2$ but do not assume that $\omega \gg \Omega$.