LOCAL CONSERVATION LAWS of ELECTROMAGNETIC ENERGY AND MOMENTUM

Earlier in class we saw that the electric charge is not only conserved but *locally conserved.* That is, the net charge inside some volume of space can charge with time *only* due to the net electric current through the surface of that volume. In the differential form, this local conservation law becomes the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{1}$$

Similar local conservation law applies to the net energy of a system, provided we include all forms of the energy, including the EM energy. Thus, if $u_{\mathrm{net}}(\mathbf{x}, t)$ is the net energy density of the system and $\mathbf{S}_{\mathrm{net}}(\mathbf{x}, t)$ is the density of the net energy flow, then

$$\frac{\partial u_{\mathrm{net}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{net}} = 0. \tag{2}$$

Likewise, the net momentum of a system — including both the mechanical momentum and the momentum of the EM fields — is locally conserved. Specifically, let $g^i_{\mathrm{net}}(\mathbf{x}, t)$ be the density of the i^{th} component of the net momentum while $-T^{ij}_{\mathrm{net}}(\mathbf{x}, t)$ is the density of that component's flow in the j^{th} direction; then

$$\frac{\partial g^i_{\mathrm{net}}}{\partial t} - \nabla^j T^{ij}_{\mathrm{net}} = 0. \tag{3}$$

Later in these notes we shall see that T^{ij} is the *stress tensor*, and that this tensor must be symmetric, $T^{ij} = T^{ji}$. But first, let's take care of the energy conservation, and specifically the electromagnetic energy conservation.

Electromagnetic Energy and the Poynting Theorem

While the net energy is always conserved, the electromagnetic energy can be converted to another form; by the work-energy theorem, the rate at which the EM energy is converted to other forms is the electric power,

$$\frac{d}{dt}U_{\text{em}} = -\text{Power}_{\text{em}}\,. \tag{4}$$

For a circuit element like a piece of resistive wire, Power $= IV$, or in terms of fields and currents,

$$\begin{aligned} I &= \mathbf{J}\cdot\textbf{Area}[\text{cross-section}], \\ V &= \mathbf{E}\cdot\textbf{Length}, \\ \text{hence}\quad \text{Power} &= (\mathbf{J}\cdot\mathbf{E})*(\text{Volume}), \end{aligned} \tag{5}$$

which generalizes to a conductor carrying non-uniform $\mathbf{J}$ and $\mathbf{E}$ as

$$\text{Power}_{\text{em}} = \int \mathbf{J}\cdot\mathbf{E}\,d^3\mathbf{x}. \tag{6}$$

In other words

$$P_{\text{em}}(\mathbf{x},t) = \mathbf{J}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\quad\text{is the EM power } \textit{density.} \tag{7}$$

Consequently, we may restate the work-energy theorem (4) for the electromagnetic energy in the local form as

$$\frac{\partial u_{\text{em}}}{\partial t} + \nabla\cdot\mathbf{S}_{\text{em}} + P_{\text{em}} = 0. \tag{8}$$

Our next task is to spell out the EM energy density u_{em} and the flow density $\mathbf{S}_{\text{em}}$ in terms of the EM fields, and then to verify that eq. (8) indeed holds true. For simplicity, let me focus on the EM fields in the vacuum and leave the dielectric and/or magnetic media for your next homework, set#6, problem#3.

We saw earlier in class that the net EM energy of charges and current in the vacuum is

$$U_{\text{em}} = \iiint\left(\frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right)d^3\mathbf{x}, \tag{9}$$

so we may identify the integrand

$$u_{\text{em}} = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \tag{10}$$

as the EM energy density. So let's take the time derivative of this energy density and try to re-cast it in the form of eq. (8). Let's start with the electric term:

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}^2\right) = \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} \tag{11}$$

where by the Ampere–Maxwell Law

$$\frac{\partial\mathbf{D}}{\partial t} = \nabla\times\mathbf{H} - \mathbf{J}, \tag{12}$$

thus

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}^2\right) = -\mathbf{E}\cdot\mathbf{J} + \mathbf{E}\cdot(\nabla\times\mathbf{H}). \tag{13}$$

Likewise, for the magnetic term we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}\mathbf{B}^2\right) = \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{14}$$

where by the Induction Law

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \tag{15}$$

thus

$$\frac{\partial}{\partial t}\left(\frac{1}{2\mu_0}\mathbf{B}^2\right) = -\mathbf{H}\cdot(\nabla\times\mathbf{E}). \tag{16}$$

Altogether, this gives us

$$\begin{aligned} P_{\text{em}} + \frac{\partial u_{\text{em}}}{\partial t} &= \mathbf{J}\cdot\mathbf{E} + (-\mathbf{E}\cdot\mathbf{J}) + \mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{H}\cdot(\nabla\times\mathbf{E}) \\ &= 0 + \epsilon^{jik}E^j\left(\nabla^i H^k\right) - \epsilon^{kij}H^k\left(\nabla^i E^j\right) \\ &= -\epsilon^{ijk}\Big(E^j\left(\nabla^i H^k\right) + H^k\left(\nabla^i E^j\right) = \nabla^i\left(E^j H^k\right)\Big) \\ &= -\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\nabla\cdot\mathbf{S}_{\text{em}} \\ \text{for}\quad \mathbf{S}_{\text{em}} &\stackrel{\text{def}}{=} \mathbf{E}\times\mathbf{H}. \end{aligned} \tag{17}$$

In other words, we have verified the *Poynting theorem*

$$\frac{\partial u_{\text{em}}}{\partial t} + \nabla\cdot\mathbf{S}_{\text{em}} + P_{\text{em}} = 0 \tag{8}$$

in which we identify the flow density of the EM energy with the *Poynting vector* $\mathbf{S}_{\text{em}} = \mathbf{E}\times\mathbf{H}$.

As an example of such energy flow density, consider a plane EM wave moving at the speed of light c in some direction $\mathbf{n}$; for such a wave, we should have $\mathbf{S}_{\text{em}} = u_{\text{em}} * c\mathbf{n}$. And indeed, when we study the plane waves later in class, we shall see that this relation is precisely correct. For example, consider a vertically polarized plane wave moving in the direction $+x_1$,

$$\begin{aligned} \mathbf{E}(x_1,t) &= E_0 \cos(kx_1 - \omega t) * (0,0,1), \\ \mathbf{H}(x_1,t) &= H_0 \cos(kx_1 - \omega t) * (0,-1,0), \\ \text{for } \omega &= ck \quad \text{and} \quad H_0 = \sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0 = c\epsilon_0\, E_0\,. \end{aligned} \tag{18}$$

For this wave

$$u_{\text{em}} = \cos^2(kx_1 - \omega t) * \left(\frac{\epsilon_0}{2}E_0^2 + \frac{\mu_0}{2}H_0^2 = \epsilon_0 E_0^2\right), \tag{19}$$

$$\begin{aligned} |\mathbf{S}_{\text{em}}| &= \cos^2(kx_1 - \omega t) * \left(E_0 H_0 = c\epsilon_0\, E_0^2\right) \\ &= c * u_{\text{em}}\,, \end{aligned} \tag{20}$$

and the direction of the Poynting vector is

$$(0,0,1) \times (0,-1,0) = (+1,0,0), \quad \text{the direction of the wave.} \tag{21}$$

More generally, for any plane wave the directions of the electric and magnetic fields are always $\perp$ to the wave direction $\mathbf{n}$ and to each other. Specifically, if one looks at the EM wave from behind (*i.e.*, the wave moves away the observer), then the direction of $\mathbf{H}$ is 90° to the *right* from the direction of $\mathbf{E}$, so that

$$\text{direction}(\mathbf{S} = \mathbf{E} \times \mathbf{H}) = \mathbf{n} = \text{away from the observer.} \tag{22}$$

Stress Tensor and Momentum Flow

Before we deal with the momentum of the electromagnetic fields, let's consider the mechanical momentum of some continuous material, fluid or solid. In general, there are internal forces in the material, and they transfer the momentum from one piece of material to or from its neighbors. For example, consider the hydrostatic pressure $P(\mathbf{x})$ in a fluid, which exerts a force $F = P \times \text{Area}$ on any boundary in the direction $\perp$ to that boundary. If the pressure is non-uniform, then there is a net force on each volume element $\mathcal{V}$ of the fluid,

$$\mathbf{F} \;=\; \iint\limits_{\text{boundary}} P(\mathbf{x})\,(-d^2\mathbf{area}) \;=\; -\iiint\limits_{\mathcal{V}} (\nabla P)\,d^3\mathbf{x}, \tag{23}$$

thus **force density**

$$\frac{d\mathbf{F}}{d\mathcal{V}} \;=\; -\nabla P. \tag{24}$$

In a solid material, the internal forces do not have to be perpendicular to the boundary; instead, there are all kinds of stresses: tension, compression, sheer, *etc.*. Most generally, the force on a $d\mathbf{a}$ infinitesimal boundary area is

$$dF^i \;=\; T^{ij}\,da^j \tag{25}$$

for some *stress tensor* T^{ij}. The boundary here could be external or internal, with the positive direction being inward rather than outward,⋆ hence a hydrostatic pressure P corresponds to stress tensor

$$T^{ij} \;=\; -P\delta^{ij}. \tag{26}$$

Similar to a non-uniform pressure, a non-uniform stress tensor $T^{ij}(\mathbf{x})$ leads to non-zero net internal force of density

$$\frac{dF^i}{d\mathcal{V}} \;=\; +\nabla^j T^{ij}. \tag{27}$$

In equilibrium, these internal forces must be balanced by some kind of external forces of

⋆ This sign convention came from mechanical engineering where the tension is considered a positive stress while compression a negative stress.

opposite density $\mathbf{f}_{\rm ext}(\mathbf{x})$,

$$\nabla^j T^{ij} \ +\ f^i_{\rm ext}\ =\ 0. \tag{28}$$

Out of equilibrium, the net internal + external forces would generate — or change — the mechanical momentum momentum of the material in question. In terms of the local momentum density $\mathbf{g}(\mathbf{x},t)$,

$$\frac{\partial\mathbf{g}}{\partial t}\ =\ \mathbf{f}_{\rm ext}\ +\ \mathbf{f}_{\rm int}\,, \tag{29}$$

or in terms of the stress tensor,

$$\frac{\partial g^i}{\partial t}\ =\ \nabla^j T^{ij}\ +\ f^i_{\rm ext}\,. \tag{30}$$

And that's why we may identify the T^{ij} — or rather $-T^{ij}$ — as the flow density in the $j^{\rm th}$ direction of the $i^{\rm th}$ component of the momentum.

The stress tensor must be symmetric $T^{ij}=T^{ji}$, because otherwise the internal forces (27) would generate a non-zero net torque and hence break the Law of Angular Momentum Conservation. Indeed, the net torque of the internal forces is

$$\vec{\tau}_{\rm int}\ =\iiint\mathbf{x}\times\mathbf{f}_{\rm int}(\mathbf{x})\,d^3\mathbf{x}, \tag{31}$$

or in components

$$\tau^i_{\rm int}\ =\iiint\epsilon^{ijk}x^j\big(f^k_{\rm int}\ =\ \nabla^\ell T^{k\ell}\big)\,d^3\mathbf{x} \tag{32}$$

where

$$\epsilon^{ijk}x^j\left(\nabla^\ell T^{k\ell}\right)\ =\ \nabla^\ell\big(\epsilon^{ijk}x^jT^{k\ell}\big)-\epsilon^{ijk}T^{k\ell}\big(\nabla^\ell x^j\ =\ \delta^{\ell j}\big)\ =\ \nabla^\ell\big(\epsilon^{ijk}x^jT^{k\ell}\big)-\epsilon^{ijk}T^{kj}, \tag{33}$$

hence

$$\tau^i_{\rm int}\ =\ \iint\limits_{\rm boundary}\epsilon^{ijk}x^jT^{k\ell}\,d^2{\rm area}^\ell\ -\iiint\epsilon^{ijk}T^{kj}\,d^3\mathbf{x}. \tag{34}$$

Moreover, if a continuous body has a boundary not subject to any external forces, then the stress tensor on that boundary must vanish, so the boundary term in the above formula actually vanishes and we are left with

$$\tau_{\text{int}}^i = -\iiint \epsilon^{ijk} T^{kj} d^3\mathbf{x} = \iiint \epsilon^{ijk} T^{jk} d^3\mathbf{x}. \tag{35}$$

To make sure this self-torque is zero and the angular momentum is conserved, we need

$$\epsilon^{ijk} T^{jk} = 0 \iff T^{jk} = T^{kj}, \tag{36}$$

quod erat demonstrandum.

Although the above argument was aimed at mechanical forces in some continuous material, the result that the stress tensor should always be symmetric extends to the non-mechanical systems such as the electromagnetic fields — as we shall see in the next section — and even to the quantum fields of QED or other quantum field theories.

Electromagnetic Momentum and Stress Tensor

Similar to the mechanical momentum of some continuous material, the momentum density $\mathbf{g}_{\text{em}}$ of the EM fields, the EM stress tensor T_{em}^{ij}, and the EM force density $\mathbf{f}_{\text{em}}$ should obey the local-momentum-conservation equation

$$\frac{\partial g_{\text{em}}^i}{\partial t} - \nabla^j T_{\text{em}}^{ij} + f_{\text{em}}^i = 0. \tag{37}$$

Note the opposite sign of the EM force term here relative to the mechanical equation (30): that's because in the mechanical case $\mathbf{f}_{\text{ext}}$ was the external force density **on** the mechanical medium in question, while in eq. (37) $\mathbf{f}_{\text{em}}$ is the force density **by** the EM fields on whatever charges and currents might be present. Specifically, the net electric + magnetic force is

$$\mathbf{F}_{\text{electric}} = \iiint d^3\mathbf{x} \Big(\rho(\mathbf{x})\mathbf{E}(\mathbf{x}) + \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) \Big), \tag{38}$$

so the EM force *density* is

$$\mathbf{f}_{\text{em}} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{E}. \tag{39}$$

The EM momentum density $\mathbf{g}$ obtains from the following heuristic argument: A plane EM wave propagating in the direction $\mathbf{n}$ is equivalent to a stream of photons moving at velocity $c\mathbf{n}$; the energy and the momentum of a massless photon are related by the relativistic formula $\mathbf{p} = (E/c)\mathbf{n}$, so the momentum density and the energy density should have a similar relation,

$$\mathbf{g}_{\text{em}} = \frac{u_{\text{em}}}{c}\,\mathbf{n}. \tag{40}$$

The same wave has Poynting vector $\mathbf{S}_{\text{em}} = cu_{\text{em}}\mathbf{n}$, so we may identify

$$\mathbf{g}_{\text{em}} = \frac{1}{c^2}\mathbf{S}_{\text{em}} = \mu_0\epsilon_0\,\mathbf{E}\times\mathbf{H} = \mathbf{D}\times\mathbf{B}. \tag{41}$$

Although we have derived eq. (41) just for the plane waves, it is generally true for any configuration of the EM fields. Indeed, let's verify that this momentum density of a most general EM field configuration obeys the local momentum conservation formula (37). By Maxwell equations,

$$\frac{\partial\mathbf{D}}{\partial t} = \nabla\times\mathbf{H} - \mathbf{J}, \quad \frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \tag{42}$$

hence for $\mathbf{g}_{\text{em}}$ as in eq. (41),

$$\frac{\partial\mathbf{g}_{\text{em}}}{\partial t} = \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} + \mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t} = (\nabla\times\mathbf{H})\times\mathbf{B} - \mathbf{J}\times\mathbf{B} - \mathbf{D}\times(\nabla\times\mathbf{E}). \tag{43}$$

At the same time,

$$\mathbf{f}_{\text{em}} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = (\nabla\cdot\mathbf{D})\mathbf{E} + \mathbf{J}\times\mathbf{B}, \tag{44}$$

so together

$$\begin{aligned}\frac{\partial\mathbf{g}_{\text{em}}}{\partial t} + \mathbf{f}_{\text{em}} &= (\nabla\times\mathbf{H})\times\mathbf{B} + (\nabla\cdot\mathbf{D})\mathbf{E} - \mathbf{D}\times(\nabla\times\mathbf{E})\\ \text{in vacuum} &\to \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \epsilon_0\Big((\nabla\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E})\Big).\end{aligned} \tag{45}$$

where the RHS should be the total space derivative $\nabla^j T^{ij}$ of some stress tensor T^{ij}. To see that this is indeed the case, let's start with the red electric terms:

$$\begin{aligned}\big[\mathbf{E}\times(\nabla\times\mathbf{E})\big]^i &= \epsilon^{ijk}E^j\epsilon^{k\ell m}\nabla^\ell E^m \\ &= \big(\delta^{i\ell}\delta^{jm}-\delta^{im}\delta^{j\ell}\big)E^j\nabla^\ell E^m \\ &= E^j\nabla^i E^j \ - \ E^j\nabla^j E^i,\end{aligned} \tag{46}$$

hence

$$\begin{aligned}\big[(\nabla\cdot\mathbf{E})\mathbf{E} \ - \ \mathbf{E}\times(\nabla\times\mathbf{E})\big]^i &= E^i\nabla^j E^j \ + E^j\nabla^j E^i \ - \ E^j\nabla^i E^j \\ &= \nabla^j\big(E^iE^j\big) \ - \ \nabla^i\big(\tfrac{1}{2}\mathbf{E}^2\big) \\ &= \nabla^j\big(E^iE^j \ - \ \tfrac{1}{2}\mathbf{E}^2\delta^{ij}\big).\end{aligned} \tag{47}$$

As to the magnetic blue term on the bottom line of eq. (45), we have

$$(\nabla\times\mathbf{B})\times\mathbf{B} \ = \ (\nabla\cdot\mathbf{B}=0)\mathbf{B} \ - \ \mathbf{B}\times(\nabla\times\mathbf{B}) \tag{48}$$

which looks exactly like the electric terms but with **B** instead of **E**, hence in exactly similar fashion

$$\big[(\nabla\times\mathbf{B})\times\mathbf{B}\big]^i \ = \ \nabla^j\big(B^iB^j \ - \ \tfrac{1}{2}\mathbf{B}^2\delta^{ij}\big). \tag{49}$$

Altogether, the RHS of eq. (45) indeed has the desired form $\nabla^j T^{ij}$ for the *Maxwell stress tensor*

$$\begin{aligned}T^{ij}_{\rm em} &= \epsilon_0\big(E^iE^j \ - \ \tfrac{1}{2}\mathbf{E}^2\delta^{ij}\big) \ + \ \frac{1}{\mu_0}\big(B^iB^j \ - \ \tfrac{1}{2}\mathbf{B}^2\delta^{ij}\big) \\ &= \epsilon_0E^iE^j \ + \ \frac{1}{\mu_0}B^iB^j \ - \ u_{\rm em}\delta^{ij} \\ &= \big(E^iD^j = D^iE^j\big) \ + \ \big(H^iB^j = B^iH^j\big) - \ u_{\rm em}\delta^{ij}.\end{aligned} \tag{50}$$

Together, this stress tensor, the momentum density $\mathbf{g}_{\rm em} = \mathbf{D}\times\mathbf{B}$, and the EM force density $\mathbf{f}_{\rm em} = \rho\mathbf{E}+\mathbf{J}\times\mathbf{B}$ obey the local momentum conservation equation

$$\frac{\partial g^i_{\rm em}}{\partial t} \ - \ \nabla^j T^{ij}_{\rm em} \ + \ f^i_{\rm em} \ = \ 0. \tag{37}$$

Comments about Maxwell Stress Tensor

I would like to conclude this notes with a few comments about the Maxwell stress tensor (50). First of all, for most field configurations this tensor is highly anisotropic, $T^{ij} \neq -P\delta^{ij}$. For a good example, consider a uniform magnetic field $\mathbf{B}$ in x_3 direction, and no electric field. In this case,

$$T^{ij}_{\text{em}} = \frac{1}{2\mu_0}\Big(2B^iB^j - \mathbf{B}^2\delta^{ij}\Big) = \frac{B^2}{2\mu_0}\Big(2\delta^{i3}\delta^{j3} - \delta^{ij}\Big), \tag{51}$$

or in the matrix form

$$T_{\text{em}} = \frac{B^2}{2\mu_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}. \tag{52}$$

In other words, we have *tension in the direction of the magnetic field but compression in the $\perp$ directions.* Also, both the tension and the compression have magnitudes

$$|P| = \frac{B^2}{2\mu_0} \approx (B \text{ in Tesla})^2 \times 4 \text{ atm}. \tag{53}$$

In plasmas, such magnetic pressure/tension are often stronger than the gas pressure, and that's why the magnetic field plays such important role in plasma physics.

Next, consider a hot cavity filled with EM radiation emitted by the walls (and maybe also by the gas inside the plasma). In thermal equilibrium, there are EM waves moving in all direction at once, so the electric and the magnetic fields have zero means but positive mean squares,

$$\langle\mathbf{E}\rangle = \langle\mathbf{B}\rangle = 0 \quad \text{but} \quad \big\langle\mathbf{E}^2\big\rangle > 0 \quad \text{and} \quad \big\langle\mathbf{B}^2\big\rangle > 0. \tag{54}$$

Furthermore, at any particular point $\mathbf{x}$ and instance of time t, the momentary electric field $\mathbf{E}(\mathbf{x}, t)$ is equally likely to point in any direction, so averaging over such directions we get

$$\big\langle E^iE^j\big\rangle = \mathbf{E}^2 \iint d^2\Omega(\mathbf{n})\, n^in^j = \mathbf{E}^2\,\frac{\delta^{ij}}{3}\,, \tag{55}$$

and hence after averaging over the field's magnitudes as well as its directions

$$\left\langle E^i E^j \right\rangle \;=\; \tfrac{1}{3}\delta^{ij}\left\langle \mathbf{E}^2 \right\rangle . \tag{56}$$

Likewise, for the magnetic field we also have

$$\left\langle B^i B^j \right\rangle \;=\; \tfrac{1}{3}\delta^{ij}\left\langle \mathbf{B}^2 \right\rangle . \tag{57}$$

Consequently, the mean EM stress tensor becomes

$$\begin{aligned}
\left\langle T^{ij}_{\rm em} \right\rangle &= \epsilon_0\Big(\left\langle E^i E^j \right\rangle - \tfrac{1}{2}\delta^{ij}\left\langle \mathbf{E}^2 \right\rangle\Big) + \frac{1}{\mu_0}\Big(\left\langle B^i B^j \right\rangle - \tfrac{1}{2}\delta^{ij}\left\langle \mathbf{B}^2 \right\rangle\Big) \\
&= \epsilon_0\Big(\tfrac{1}{3}\delta^{ij}\left\langle \mathbf{E} \right\rangle^2 - \tfrac{1}{2}\delta^{ij}\left\langle \mathbf{E}^2 \right\rangle\Big) + \frac{1}{\mu_0}\Big(\tfrac{1}{3}\delta^{ij}\left\langle \mathbf{B}^2 \right\rangle - \tfrac{1}{2}\delta^{ij}\left\langle \mathbf{B}^2 \right\rangle\Big) \\
&= \epsilon_0\Big(-\tfrac{1}{6}\delta^{ij}\left\langle \mathbf{E}^2 \right\rangle\Big) + \frac{1}{\mu_0}\Big(-\tfrac{1}{6}\delta^{ij}\left\langle \mathbf{B}^2 \right\rangle\Big) \\
&= -\frac{\delta^{ij}}{3}\Big(\frac{\epsilon_0}{2}\left\langle \mathbf{E}^2 \right\rangle + \frac{1}{2\mu_0}\left\langle \mathbf{B}^2 \right\rangle\Big) \\
&= -\frac{\delta^{ij}}{3} \times \left\langle u_{\rm em} \right\rangle .
\end{aligned} \tag{58}$$

Or in macroscopic terms, the thermal radiation has isotropic stress tensor related to the energy density as

$$T^{ij}_{\rm rad} \;=\; -\tfrac{1}{3}\delta^{ij} u_{\rm rad} . \tag{59}$$

Such isotropic (and negative) stress tensor corresponds to the *radiation pressure*

$$P_{\rm rad} \;=\; \tfrac{1}{3}u_{\rm rad} , \quad T^{ij}_{\rm rad} \;=\; -\delta^{ij} P_{\rm rad} . \tag{60}$$

Note: we have obtained the relation pressure $= \frac{1}{3}$energy_density for the classical EM radiation. But the quantum theory yields exactly the same result: The ideal gas of relativistic photons has $P = \frac{1}{3}u$. Indeed, the ideal gas of any massless or ultra-relativistic particles has $P = \frac{1}{3}u$.