MAGNETIC ENERGY

Before I get to the magnetic energy, let me remind you of the Faraday's Law of Induction. Take any closed loop of coil of wire and place it in presence of magnetic fields; let Φ be the net magnetic flux through this loop or coil. *If the flux Φ changes for any reason whosoever, this induces an electromotive force (EMF) in the loop/coil according to*

$$\mathcal{E} \;=\; -\frac{d\Phi}{dt}. \tag{1}$$

The minus sign in this formula reflects the *Lenz rule:* the EMF (1) leads to a current in the loop whose magnetic field has opposite direction to the $\Delta\Phi$.

Now consider an inductor coil. For simplicity, let's assume that the coil either does not have a ferromagnetic core or else the ferromagnetic material of the core is linear, which means inside the core $\mathbf{B} = \mu\mu_0\mathbf{H}$. Consequently, when we run a current I through the coil, the magnetic fields $\mathbf{H}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are proportional to the current and hence the magnetic flux Φ through the coil is also proportional to the current. Thus

$$\Phi \;=\; L \times I \tag{2}$$

for some constant coefficient L called the self-inductance of the coil. When the current through the coil changes for any reason, the flux also changes according to eq. (2), and according to the Faraday's law (1) this induces EMF in the coil,

$$\mathcal{E} \;=\; -L \times \frac{dI}{dt}\,. \tag{3}$$

Given these preliminaries, we may turn to the magnetic energy, and let's start with the magnetic energy stored in the inductor coil. Suppose we try to increase the current in the coil by an infinitesimal amount δI. Changing the current induces EMF in the coil, and to compensate for this negative EMF the power source providing the current must also provide voltage

$$V \;=\; -\mathcal{E} \;+\; \text{a bit extra for the ohmic losses in the coil,} \tag{4}$$

and hence power $P = I \times V$. Some of this power is dissipated by the ohmic losses, but that would happen even for a time-independent current and hence $\mathcal{E} = 0$. So let's focus on the extra power due to $-\mathcal{E} \times I$ and hence the extra work done by the power supply while the current increases by δI in time δt:

$$\delta W_{\text{extra}} = -\mathcal{E} \times I \times \delta t = +L \times \frac{\delta I}{\delta t} \times I \times \delta t = LI \times \delta I = \delta\left(\tfrac{1}{2} LI^2\right). \tag{5}$$

This extra work — which is independent on the time δt it takes to raise the current — goes to the *magnetic energy of the coil*

$$U = \tfrac{1}{2} LI^2 = \tfrac{1}{2} \Phi I = \frac{\Phi^2}{2L}. \tag{6}$$

Now let's express this magnetic energy in terms of the magnetic field $\mathbf{B}(\mathbf{x})$ inside the coil. The magnetic flux though the coil can be expressed in terms of the vector potential as

$$\Phi = \oint_{\text{coil}} d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}), \tag{7}$$

hence the magnetic energy

$$U = \tfrac{1}{2} I\Phi = \frac{1}{2} \oint_{\text{coil}} I \, d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}). \tag{8}$$

This formula assumes a coil made of thin wires; if we replace them with thicker conductors carrying some volume currents $\mathbf{J}(\mathbf{x})$, then in the integral (8) we replace $I \, d\mathbf{x} \to d^3\mathbf{x} \, \mathbf{J}(\mathbf{x})$, hence

$$U = \frac{1}{2} \iiint d^3\mathbf{x} \, \mathbf{J}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}). \tag{9}$$

Now let's integrate by parts using $\nabla \times \mathbf{A} = \mathbf{B}$ and the Ampere's Law $\nabla \times \mathbf{H} = \mathbf{J}$. For any vector fields $\mathbf{f}$ and $\mathbf{g}$, we have

$$\nabla \cdot (\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot (\nabla \times \mathbf{f}) - \mathbf{f} \cdot (\nabla \times \mathbf{g}), \tag{10}$$

hence

$$\nabla\cdot(\mathbf{H}\times\mathbf{A}) \;=\; \mathbf{A}\cdot(\nabla\times\mathbf{H}) \;-\; \mathbf{H}\cdot(\nabla\times\mathbf{A}) \;=\; \mathbf{A}\cdot\mathbf{J} \;-\; \mathbf{H}\cdot\mathbf{B} \tag{11}$$

and therefore

$$\iiint\limits_{\mathcal{V}} d^3\mathbf{x}\,\mathbf{J}\cdot\mathbf{A} \;=\; \oiint\limits_{\mathcal{S}}(\mathbf{H}\times\mathbf{A})\cdot d^2\mathbf{a} \;+\; \iiint\limits_{\mathcal{V}} d^3\mathbf{x}\,\mathbf{H}\cdot\mathbf{B}. \tag{12}$$

for any integration volume $\mathcal{V}$ and its surface $\mathcal{S}$. In eq. (9), we integrate over the conductor's volume, but we may just as well extend the integration volume $\mathcal{V}$ to the whole space. Consequently, in eq. (12) the surface integral on the RHS goes away, and the magnetic energy (9) becomes

$$U \;=\; \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{x}\;\mathbf{H}\cdot\mathbf{B}. \tag{13}$$

Eq. (13) applies to magnetic energy in linear materials — the vacuum, diamagnetics, paramagnetics, and very soft ferromagnetics in weak fields, but we need a different approach to non-linear media. So let's go back to the inductor coil, but allow for a non-linear dependence between the current I and the magnetic flux Φ. Increasing the current in the coil — and hence the magnetic flux Φ — requires extra work by the power supply

$$\delta W_{\text{extra}} \;=\; -\mathcal{E}\times I\times\delta t \;=\; +\frac{\delta\Phi}{\delta t}\times I\times\delta t \;=\; \delta\Phi\times I, \tag{14}$$

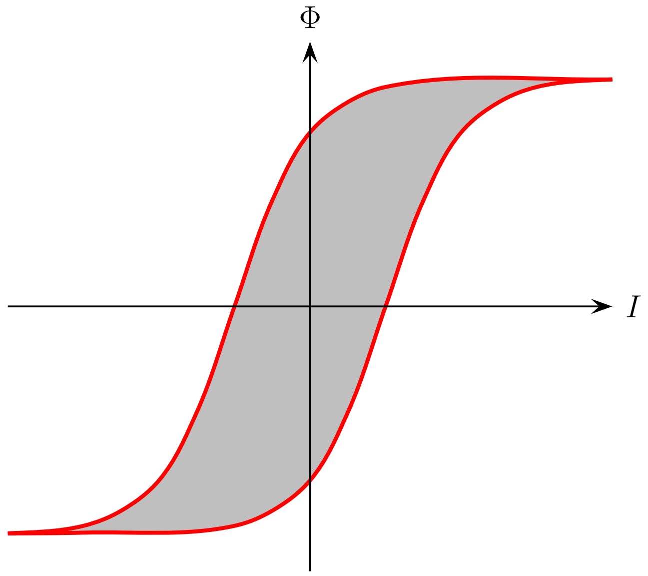
but this time we cannot re-express this work as δU without knowing the dependence of the flux Φ on the current I. If the relation between the current and the flux is non-linear but single-valued, then integrating eq. (12) yields magnetic energy

$$U(I) \;=\; \int\limits_0^I I'\times d\Phi(I'), \tag{15}$$

but if there is hysteresis and Φ depends on the past history of the current, then the magnetic work

$$W = \int I \, d\Phi \tag{16}$$

is irreversible and cannot be accounted by any magnetic energy as a function of the current. Indeed, if we change the current back and force and come back to its original value, then the net work (16) is the area of the hysteresis loop in the (I,Φ) plot,



and this net work is dissipated as heat rather than stored as magnetic energy.

Similar formulae obtains in terms of the magnetic fields $\mathbf{H}$ and $\mathbf{B}$:

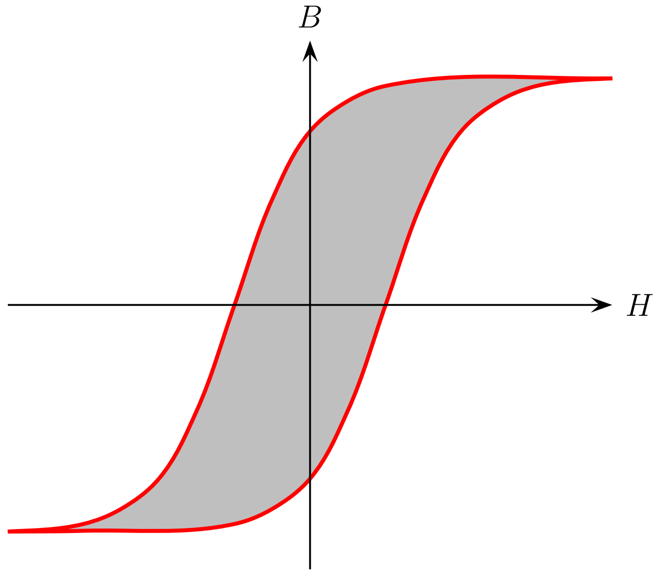
$$\begin{aligned} \delta W &= I \times \delta\Phi = \oint\limits_{\text{coil}} I \, d\mathbf{x} \cdot \delta\mathbf{A}(\mathbf{x}) \\ &\longrightarrow \iiint d^3\mathbf{x} \; \mathbf{J}(\mathbf{x}) \cdot \delta\mathbf{A}(\mathbf{x}) \\ &= \iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{x} \; \mathbf{H}(\mathbf{x}) \cdot \delta\mathbf{B}(\mathbf{x}). \end{aligned} \tag{17}$$

If the relation between the $\mathbf{H}$ and the $\mathbf{B}$ fields is linear, then this magnetic work is accounted by the magnetic energy (13). If the relation is non-linear but single-valued, then the magnetic energy becomes

$$U = \iiint d^3\mathbf{x}\, F(\mathbf{B}) \quad \text{where} \quad F(\mathbf{B}) = \int_0^{\mathbf{B}} \mathbf{H}(\mathbf{B}') \cdot d\mathbf{B}'. \tag{18}$$

But if there is hysteresis, then the magnetic work is irreversible and cannot be wholly accounted by the energy of the fields in the ferromagnetic involved. Instead, if we vary the magnetic field but eventually get back to the same field we have started from, we get a net energy loss

$$W_{\text{loss}} = \iiint d^3\mathbf{x}\ [\text{area of the hysteresis loop in the } (H,B) \text{ plane}] \tag{19}$$



Magnetic Forces

In this section I explain the forces magnetic fields exert on magnetic materials, be they diamagnetic, paramagnetic, or ferromagnetic. But for simplicity, I focus on the linear magnetic materials inside which $\mathbf{B} = \mu\mu_0\mathbf{H}$.

Let's start by considering a small displacement of a piece of magnetic material near a current-carrying coil. In the process, the coil's self-inductance L changes by δL, which affects the magnetic energy $U = \frac{1}{2}I^2L$ stored in the coil. In general, the current I through the coil may also change by δI, so the net change of the coil's magnetic energy is

$$\delta U \;=\; \delta\bigl(\tfrac{1}{2}I^2L\bigr) \;=\; IL\times\delta I \;+\; \tfrac{1}{2}I^2\times\delta L. \tag{20}$$

At the same time, the magnetic flux $\Phi = IL$ through the coil changes by

$$\delta\Phi \;=\; L\times\delta I \;+\; I\times\delta L, \tag{21}$$

which induces EMF $\mathcal{E} = -d\Phi/dt$ in the circuit supplying the current I to the coil. This EMF makes the battery ultimately supplying this current to perform extra electric work

$$\delta W_{\text{el}} \;=\; -\mathcal{E}\times I\delta t \;=\; +\frac{\delta}{\delta t}\bigl(L\delta I \;+\; I\delta L\bigr)\times I\delta t \;=\; IL\times\delta I \;+\; I^2\times\delta L. \tag{22}$$

Note the difference between this electric work and the change (20) of the coil's magnetic energy,

$$\delta W_{\text{el}} \;-\; \delta U \;=\; +\tfrac{1}{2}I^2\times\delta L. \tag{23}$$

This difference must come from the mechanical work involved in moving the magnetic material to change the self-inductance L of the coil. Defining the sign of the mechanical work as the work done by the coil on the magnetic material, we have

$$\delta W_{\text{mech}} \;=\; \delta W_{\text{el}} \;-\; \delta U \;=\; +\tfrac{1}{2}I^2\times\delta L. \tag{24}$$

In terms of the magnetic force on the magnetic material we move $\delta W_{\text{mech}} = F\delta x$, hence the force is

$$F \;=\; \tfrac{1}{2}I^2\times\frac{dL}{dx}\,. \tag{25}$$

The direction of this force is the direction of the motion which would increase the coil's self-inductance, and the stronger the increase, the stronger the force. Thus:

- the ferromagnetic materials are strongly pulled into the coil;

- the paramagnetic material are weakly pulled into the coil;
- the diamagnetic material are weakly pushed out from the coil.

Going beyond the inductor coil, consider inserting of a piece of magnetic material into a space where some currents $\mathbf{J}_0(\mathbf{x})$ create a magnetic field $\mathbf{H}_0(\mathbf{x})$. For simplicity, let's assume we somehow maintain a fixed, time-independent conduction current $\mathbf{J}(\mathbf{x}) = \mathbf{J}_0(\mathbf{x})$ during the insertion, although the magnetic fields change from $\mathbf{H}_0(\mathbf{x})$ and $\mathbf{B}_0(\mathbf{x}) = \mu_0\mathbf{H}_0(\mathbf{x})$ to $\mathbf{H}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x}) = \mu_0(\mathbf{H}(\mathbf{x}) + \mathbf{M}(\mathbf{x}))$. Consequently, inserting the magnetic material changes the magnetic energy of the system by

$$\begin{aligned}\Delta U \;=\; U \;-\; U_0 \;&=\; \frac{1}{2}\iiint(\mathbf{J}\cdot\mathbf{A})d^3\mathbf{x} \;-\; \frac{1}{2}\iiint(\mathbf{J}_0\cdot\mathbf{A}_0)d^3\mathbf{x}\\ \langle\!\langle \text{ using } \mathbf{J}(\mathbf{x}) = \mathbf{J}_0(\mathbf{x}) \,\rangle\!\rangle \qquad & \\ &=\; \frac{1}{2}\iiint \mathbf{J}_0\cdot(\mathbf{A}-\mathbf{A}_0)\,d^3\mathbf{x}\end{aligned} \tag{26}$$

However, to keep the currents constant despite changing the magnetic fluxes, the power supply have to provide extra work. Generalizing the electric work equation for a coil

$$W_{\text{el}} \;=\; I\times\Delta\Phi \;=\; \oint_{\text{coil}} I\,d\mathbf{x}\cdot\Delta\mathbf{A}(\mathbf{x}) \tag{27}$$

to the volume currents, we get

$$\Delta W_{\text{el}} \;=\; \iiint d^3\mathbf{x}\;\mathbf{J}(\mathbf{x})\cdot\Delta\mathbf{A}(\mathbf{x}). \tag{28}$$

Consequently, the mechanical work of moving the magnetic material into the system is

$$W_{\text{mech}} \;=\; \Delta U \;-\; W_{\text{el}} \;=\; -\frac{1}{2}\iiint d^3\mathbf{x}\;\mathbf{J}_0\cdot(\mathbf{A}-\mathbf{A}_0). \tag{29}$$

Integrating by parts, we turn this formula into

$$W_{\text{mech}} \;=\; -\frac{1}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{x}\;\mathbf{H}_0\cdot(\mathbf{B}-\mathbf{B}_0), \tag{30}$$

where

$$\mathbf{B} \;-\; \mathbf{B}_0 \;=\; \mu_0(\mathbf{H}+\mathbf{M}-\mathbf{H}_0) \;=\; \mu_0(\mathbf{M}+\Delta\mathbf{H}). \tag{31}$$

Consequently, the mechanical work of inserting a piece of magnetic material into a preexisting magnetic field $\mathbf{H}_0$ evaluates to

$$W_{\text{mech}} = -\frac{\mu_0}{2} \iiint\limits_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{x}\; \mathbf{H}_0 \cdot (\mathbf{M} + \Delta\mathbf{H}). \tag{32}$$

This formula explains why ferromagnetic and paramagnetic materials are attracted to regions where the magnetic field is strongest while diamagnetic materials are repelled by them. Indeed, take a small ball of magnetic material, so small that over its size we mat approximate $\mathbf{H}_0(\mathbf{x}) \approx \text{const} = \mathbf{H}_0(\text{center})$. Then proceeding similarly to the dielectric case we find that the $\mathbf{H}$ field inside the ball is

$$\mathbf{H}_{\text{inside}} = \frac{3}{\mu + 2}\, \mathbf{H}_0\,, \tag{33}$$

hence

$$\mathbf{M} = (\mu - 1)\mathbf{H}_{\text{inside}} = \frac{3(\mu-1)}{\mu+2}\, \mathbf{H}_0\,, \tag{34}$$

while

$$\Delta\mathbf{H} = \mathbf{H} - \mathbf{H}_0 = \frac{1-\mu}{\mu+2}\, \mathbf{H}_0 = -\frac{1}{3}\, \mathbf{M}. \tag{35}$$

Consequently, integrating over the inside of the ball, we get

$$\begin{aligned} \iiint\limits_{\text{inside}} d^3\mathbf{x}\, \mathbf{H}_0 \cdot (\mathbf{M} + \Delta\mathbf{H}) &= \frac{2}{3}\, (\mathbf{H}_0 \cdot \mathbf{M}) * \text{ball's volume} \\ &= \frac{2(\mu-1)}{\mu+2}\, \mathbf{H}_0^2 * \frac{4\pi R^3}{3}\,. \end{aligned} \tag{36}$$

But the space integral in eq. (32) is over the whole space, so we must add the integral over the outside of the ball, where

$$\mathbf{M} = 0 \quad \text{but } \Delta\mathbf{H}(\mathbf{x}) = \text{dipole field} = \frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}\,. \tag{37}$$

The $\mathbf{m}$ here is the net dipole moment of the ball; since its direction is the same as the $\mathbf{H}_0$

magnetic field, we have

$$\mathbf{H}_0\cdot(\Delta\mathbf{H}+\mathbf{M}) \;=\; \mathbf{H}_0\cdot\Delta\mathbf{H} \;=\; \frac{3(\mathbf{H}_0\cdot\mathbf{n})(\mathbf{m}\cdot\mathbf{n})-(\mathbf{H}_0\cdot\mathbf{m})}{r^3} \;=\; \frac{m}{H_0}*\frac{3(\mathbf{H}_0\cdot\mathbf{n})^2-\mathbf{H}_0^2}{r^3}, \quad (38)$$

so the volume integral over the outside of the ball becomes

$$\iiint\limits_{\text{outside}} d^3\mathbf{x}\,\mathbf{H}_0\cdot(\mathbf{M}+\Delta\mathbf{H}) \;=\; \frac{m}{H_0}\int\limits_R^\infty\frac{dr\,r^2}{r^3}*\iint d^2\Omega(\mathbf{n})\,\big[3(\mathbf{H}_0\cdot\mathbf{n})^2-\mathbf{H}_0^2\big]. \quad (39)$$

The angular integral here vanishes, so the entire integral in eq. (32) comes from the inside of the ball. Altogether,

$$W_{\text{mech}} \;=\; -\mathbf{H}_0^2*\frac{(\mu-1)\mu_0}{\mu+2}*\frac{4\pi R^3}{3}\,. \quad (40)$$

The $\mathbf{H}_0^2$ factor in this formula should be evaluated at the ball's location, and the W_{mech} is the mechanical work of bringing the ball to that location from infinitely far away. Consequently, this mechanical work as a function of the ball's location acts as its potential energy

$$U_{\text{potential}}(\mathbf{x}) \;=\; -\frac{4\pi R^3\mu_0}{3(\mu+2)}*(\mu-1)*\mathbf{H}_0^2(\mathbf{x}). \quad (41)$$

This potential energy governs the magnetic force on the ball according to

$$\mathbf{F} \;=\; -\nabla U_{\text{potential}} \;=\; (\mu-1)*\begin{pmatrix}\text{positive}\\ \text{factor}\end{pmatrix}*\nabla(\mathbf{H}_0^2), \quad (42)$$

so the direction of this force on diamagnetic material is opposite from the force on paramagnetic or ferromagnetic materials. Specifically:

- the ferromagnetic materials with $\mu\gg 1$ are strongly pulled towards the locations where $\mathbf{H}_0^2$ is strongest such as magnet's poles;
- the paramagnetic materials with $\mu=1+\text{small}$ are pulled in the same direction but with a weaker force;

- the diamagnetic materials with $\mu = 1 -$ small are pushed in the opposite direction, away from the magnet.

Thus, the diamagnetic materials — such as water — can be levitated above a strong magnet. Here is the famous levitating frog demo illustrating this effect.