PLANE ELECTROMAGNETIC WAVES

In general, a plane wave is a solution of the free wave equation

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Psi(\mathbf{x},t) = 0 \tag{1}$$

of a specific form

$$\Psi(\mathbf{x},t) = \Psi_0 \cos\bigl(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi\bigr), \tag{2}$$

where $\omega = 2\pi f$ is the (angular) frequency and

$$\mathbf{k} = \frac{\omega}{v}\hat{\mathbf{k}} \tag{3}$$

is the *wave vector*; the plane wave propagates in its direction $\hat{\mathbf{k}}$. The amplitude Ψ_0 and the phase ϕ of the wave can be combined into the complex amplitude

$$\psi_0 = \Psi_0 e^{i\phi}, \tag{4}$$

hence

$$\Psi(\mathbf{x},t) = \mathrm{Re}\left(\psi_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\right). \tag{5}$$

In linear formulae, I shall often write this formula as simply

$$\Psi(\mathbf{x},t) = \psi_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{6}$$

and leave implicit taking the real part of the RHS.

Now let's specialize to the electromagnetic wave in some medium. For simplicity, let's assume a uniform linear medium where

$$\mathbf{D} = \epsilon\epsilon_0\mathbf{E}, \quad \mathbf{B} = \mu\mu_0\mathbf{H}, \tag{7}$$

for some *real* electric permittivity $\epsilon(\omega)$ and magnetic permeability $\mu(\omega)$, although they might depend on the frequency ω. Let us also assume there are no free charges or conduction currents, so the Maxwell equations for the EM fields in this medium become simply

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= 0, &(8)\\
\nabla\cdot\mathbf{B} &= 0, &(9)\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, &(10)\\
\nabla\times\mathbf{H} &= +\frac{\partial\mathbf{D}}{\partial t}. &(11)
\end{aligned}$$

As we saw earlier in class (*cf.* my notes on Maxwell equations), these equations lead to the wave equations for all 6 components of the EM fields $\mathbf{E}$ and $\mathbf{H}$:

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\begin{pmatrix}\mathbf{E}(\mathbf{x},t)\\ \mathbf{H}(\mathbf{x},t)\end{pmatrix} = 0 \tag{12}$$

for the wave speed

$$v = \frac{1}{\sqrt{\epsilon\epsilon_0\times\mu\mu_0}} = \frac{c}{\sqrt{\epsilon\mu}}, \tag{13}$$

and the solutions of the wave equation (12) includes the plane EM waves

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t),\\
\mathbf{H}(\mathbf{x},t) &= \vec{\mathcal{H}}\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t).
\end{aligned} \tag{14}$$

However, there is more to the Maxwell equations (8)through (11) than the wave equation (12), and this leads to several constraints on the 6 complex amplitudes $\mathcal{E}_{x,y,z}$ and $\mathcal{H}_{x,y,z}$ of a plane EM wave (14). Indeed, for a plane wave, the space derivative ∇ acts by multiplying by $i\mathbf{k}$ while the time derivative $\partial/\partial t$ acts by multiplying by $-i\omega$. Consequently, Maxwell eqs. (8) and (9) become

$$\begin{aligned}
i\mathbf{k}\cdot(\epsilon\epsilon_0\vec{\mathcal{E}})\,\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) &= 0,\\
i\mathbf{k}\cdot(\mu\mu_0\vec{\mathcal{H}})\,\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) &= 0,
\end{aligned} \tag{15}$$

which means that both the electric and the magnetic amplitude vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ must be perpendicular to the wave direction $\hat{\mathbf{k}}$. Thus, the electromagnetic waves are transverse waves.

Next, the other two Maxwell equations (10) and (11) become

$$i\mathbf{k}\times\vec{\mathcal{E}}\,\exp(i\mathbf{k}\cdot\mathbf{x}\,-\,i\omega t)\;=\;+i\omega(\mu\mu_0\vec{\mathcal{H}})\,\exp(i\mathbf{k}\cdot\mathbf{x}\,-\,i\omega t),\tag{16}$$

$$i\mathbf{k}\times\vec{\mathcal{H}}\,\exp(i\mathbf{k}\cdot\mathbf{x}\,-\,i\omega t)\;=\;-i\omega(\epsilon\epsilon_0\vec{\mathcal{E}})\,\exp(i\mathbf{k}\cdot\mathbf{x}\,-\,i\omega t),\tag{17}$$

and hence

$$\hat{\mathbf{k}}\times\vec{\mathcal{E}} = \;+\frac{\omega\mu\mu_0}{|\mathbf{k}|}\,\vec{\mathcal{H}},\qquad \hat{\mathbf{k}}\times\vec{\mathcal{H}}\;=\;-\frac{\omega\epsilon\epsilon_0}{|\mathbf{k}|}\,\vec{\mathcal{E}}.\tag{18}$$

Note that in both of these equations

$$\frac{\omega}{|\mathbf{k}|}\;=\;\text{wave speed }v\;=\;\frac{1}{\sqrt{\epsilon\epsilon_0\,\mu\mu_0}}\,,\tag{19}$$

so eqs. (18) become

$$\hat{\mathbf{k}}\times\vec{\mathcal{E}}\;=\;+Z\vec{\mathcal{H}},\qquad \hat{\mathbf{k}}\times\vec{\mathcal{H}}\;=\;-\frac{1}{Z}\,\vec{\mathcal{E}}\tag{20}$$

for the same *wave impedance*

$$Z\;=\;\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}\,.\tag{21}$$

Consequently, the two eqs. (20) are consistent with each other, provided both amplitude vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are transverse to the wave's direction $\hat{\mathbf{k}}$. Indeed, for a unit vector $\hat{\mathbf{k}}$ and the two amplitude vectors $\perp\hat{\mathbf{k}}$ we have

$$\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\vec{\mathcal{E}})\;=\;-\vec{\mathcal{E}},\qquad \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\vec{\mathcal{H}})\;=\;-\vec{\mathcal{H}},\tag{22}$$

and hence

$$\begin{aligned}\hat{\mathbf{k}}\times\vec{\mathcal{E}}\;&=\;+Z\,\vec{\mathcal{H}}\\ &\Updownarrow\\ +\vec{\mathcal{E}}\;=\;-\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\vec{\mathcal{E}})\;&=\;-Z\,\hat{\mathbf{k}}\times\vec{\mathcal{H}}\\ &\Updownarrow\\ \hat{\mathbf{k}}\times\vec{\mathcal{H}}\;&=\;-\frac{1}{Z}\,\vec{\mathcal{E}}.\end{aligned}\tag{23}$$

The bottom line is, the independent amplitudes of a plane EM wave comprise a complex vector $\vec{\mathcal{E}}$ in the 2D plane transverse to the wave direction $\hat{\mathbf{k}}$. Given such an electric amplitude vector, the magnetic amplitude vector obtains as

$$\vec{\mathcal{H}} = +\frac{1}{Z}\hat{\mathbf{k}} \times \vec{\mathcal{E}} \tag{24}$$

where Z is the wave impedance (21) of the medium in which the wave propagates. In particular, the vacuum has wave impedance

$$Z_{\text{vac}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = c\mu_0 = \frac{1}{c\epsilon_0} \approx 377\ \Omega. \tag{25}$$

Note units: in MKSA unit system, the electric tension field $\mathbf{E}$ — and hence the electric amplitude $\vec{\mathcal{E}}$ — is measured in V/m (Volts per meter), while the magnetic intensity field $\mathbf{H}$ — and hence the magnetic amplitude $\vec{\mathcal{H}}$ — is measured in A/m (Amperes per meter). Consequently, the ratio Z of these amplitudes is measured in $V/A = \Omega$, Volts per Ampere AKA Ohms. Thus the wave impedance of a medium WRT to the EM waves has the same dimesionality as the electric resistance or impedance of a circuit. In particular, $Z(\text{vacuum}) \approx 377\ \Omega$.

Now consider the energy density of the wave,

$$u = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} + \tfrac{1}{2}\mathbf{H}\cdot\mathbf{B} = \frac{\epsilon\epsilon_0}{2}\mathbf{E}^2 + \frac{\mu\mu_0}{2}\mathbf{H}^2. \tag{26}$$

Earlier in class we saw that for a harmonic AC current and voltage

$$I(t) = \operatorname{Re}\left(I_0 e^{-i\omega t}\right), \qquad V(t) = \operatorname{Re}\left(V_0 e^{-i\omega t}\right), \tag{27}$$

the time-averaged electric power is

$$\langle P\rangle = \langle IV\rangle = \tfrac{1}{2}\operatorname{Re}(I_0^* V_0) = \tfrac{1}{2}\operatorname{Re}(I_0 V_0^*). \tag{28}$$

Likewise, in a harmonic plane wave

$$\left\langle \mathbf{E}^2\right\rangle = \tfrac{1}{2}\operatorname{Re}\left(\vec{\mathcal{E}}^*\cdot\vec{\mathcal{E}}\right) = \tfrac{1}{2}\left|\vec{\mathcal{E}}\right|^2 \quad\text{and}\quad \left\langle \mathbf{H}^2\right\rangle = \tfrac{1}{2}\operatorname{Re}\left(\vec{\mathcal{H}}^*\cdot\vec{\mathcal{H}}\right) = \tfrac{1}{2}\left|\vec{\mathcal{H}}\right|^2, \tag{29}$$

hence time-averaged energy density of the wave is

$$\langle u\rangle = \frac{\epsilon\epsilon_0}{4}\left|\vec{\mathcal{E}}\right|^2 + \frac{\mu\mu_0}{4}\left|\vec{\mathcal{H}}\right|^2. \tag{30}$$

Moreover, the two terms on the RHS here are equal to each other, just like in a mechanical wave the time-averaged kinetic energy is equal to the time-averaged potential energy. Indeed, in light of the relation (24) between the electric an the magnetic amplitudes,

$$\begin{aligned} \left|\vec{\mathcal{H}}\right|^2 &= \frac{1}{Z^2}\left|\hat{\mathbf{k}}\times\vec{\mathcal{E}}\right|^2 \\ &= \frac{1}{Z^2}\left|\vec{\mathcal{E}}\right|^2 \qquad \langle\!\langle\, \text{since } \vec{\mathcal{E}}\perp\hat{\mathbf{k}}\,\rangle\!\rangle \\ &= \frac{\epsilon\epsilon_0}{\mu\mu_0}\left|\vec{\mathcal{E}}\right|^2, \end{aligned} \tag{31}$$

hence

$$\langle u_{\rm mag}\rangle = \frac{\mu\mu_0}{4}\left|\vec{\mathcal{H}}\right|^2 = \frac{\epsilon\epsilon_0}{4}\left|\vec{\mathcal{E}}\right|^2 = \langle u_{\rm el}\rangle\,. \tag{32}$$

Consequently, in terms of the electric amplitude $\vec{\mathcal{E}}$, the net time-averaged energy density of the plane EM wave is

$$\langle u_{\rm net}\rangle = 2\times\langle u_{\rm el}\rangle = \frac{\epsilon\epsilon_0}{2}\left|\vec{\mathcal{E}}\right|^2. \tag{33}$$

Next, the energy *flow* density of the plane wave, which obtains from the Poynting vector

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{34}$$

For the harmonically oscillating electric and magnetic fields, the time-average of this Poynting vector is related to the electric and magnetic amplitudes as

$$\langle\mathbf{S}\rangle = \tfrac{1}{2}\,\mathrm{Re}\left(\vec{\mathcal{E}}^*\times\vec{\mathcal{H}}\right), \tag{35}$$

where in light of the relation (24) between the electric and the magnetic amplitudes of the wave

$$\vec{\mathcal{E}}^*\times\vec{\mathcal{H}} = \frac{1}{Z}\,\vec{\mathcal{E}}^*\times(\hat{\mathbf{k}}\times\vec{\mathcal{E}}) = \frac{1}{Z}\Big(\hat{\mathbf{k}}(\vec{\mathcal{E}}^*\cdot\vec{\mathcal{E}}) - \vec{\mathcal{E}}(\vec{\mathcal{E}}^*\cdot\hat{\mathbf{k}})\Big) = \frac{1}{Z}\Big(\hat{\mathbf{k}}\left|\vec{\mathcal{E}}\right|^2 - 0\Big), \tag{36}$$

where the last equality follows from $(\vec{\mathcal{E}}^*\cdot\hat{\mathbf{k}}) = (\vec{\mathcal{E}}\cdot\hat{\mathbf{k}})^* = 0$. Consequently, the time-averaged energy flow density of the plane EM wave is

$$\langle\mathbf{S}\rangle = \frac{|\vec{\mathcal{E}}|^2}{2Z}\,\hat{\mathbf{k}}. \tag{37}$$

Taking the ration of this energy flow density to the energy density (33), we get

$$\frac{|\langle\mathbf{S}\rangle|}{\langle u\rangle} = \frac{|\vec{\mathcal{E}}|^2/2Z}{(\epsilon\epsilon_0/2)|\vec{\mathcal{E}}|^2} = \frac{1}{Z\epsilon\epsilon_0} = \frac{1}{\sqrt{\epsilon\epsilon_0\,\mu\mu_0}} = v_{\text{wave}}\,, \tag{38}$$

and therefore

$$\langle\mathbf{S}\rangle = \langle u\rangle\,\mathbf{v}_{\text{wave}}\,. \tag{39}$$

In other words, the energy of the plane EM wave moves in space with exactly the same velocity vector $\mathbf{v}_{\text{wave}} = v\hat{\mathbf{k}}$ as the phase fronts

$$\mathbf{x}\text{–planes where } \mathbf{k}\cdot\mathbf{x} - \omega t = \text{const.} \tag{40}$$

Polarization

Both electric and magnetic fields of a plane EM wave are linearly related to the electric amplitude vector $\vec{\mathcal{E}}$, which is a complex vector in the 2D plane $\perp$ to the wave direction $\hat{\mathbf{k}}$. Hence, all superpositions of waves with the same frequency ω and wave vector $\mathbf{k}$ follow from the superpositions

$$\vec{\mathcal{E}}_{\text{net}} = \alpha_1\vec{\mathcal{E}}_1 + \alpha_2\vec{\mathcal{E}}_2 \tag{41}$$

of such 2D complex vectors. In this section, we shall see how it works, and how to decompose a general amplitude vector $\vec{\mathcal{E}}$ into two independent wave *polarizations*. But to simplify our notations, let's focus on the waves traveling in the positive z direction, $\hat{\mathbf{k}} = (0,0,+1)$. Consequently, the amplitude vectors of all such waves have form

$$\vec{\mathcal{E}} = (\mathcal{E}_x, \mathcal{E}_y, 0) \tag{42}$$

with 2 independent complex components $\mathcal{E}_x$ and $\mathcal{E}_y$. Depending on the relative phases — and also relative magnitudes — of these two components, we EM wave can be linearly polarized, circularly polarized, or elliptically polarized.

Linear polarizations

Linear polarizations (AKA planar polarizations) of the EM waves obtain when the complex amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ have the same phase (up to a sign). In general, the linearly polarized waves have

$$\mathcal{E}_x = \mathcal{E}_0 \times \cos\phi, \qquad \mathcal{E}_y = \mathcal{E}_0 \times \sin\phi \tag{43}$$

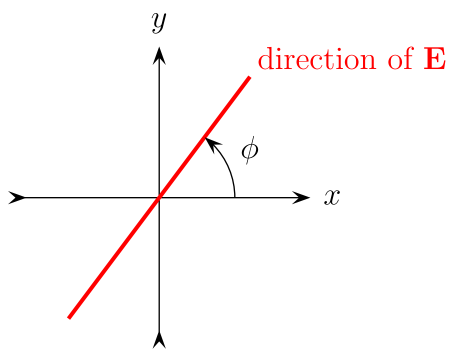
for some real angle ϕ, so when the electric field oscillates in time and space,

$$\mathbf{E}(z,t) = \mathrm{Re}\big(\vec{\mathcal{E}}\, e^{ikz-i\omega t}\big), \tag{44}$$

we get

$$\begin{aligned} \mathbf{E}(z,t) &= (\cos\phi, \sin\phi, 0) * \mathrm{Re}\big(\mathcal{E}_0\, e^{ikz-i\omega t}\big) \\ &= |\mathcal{E}_0| * (\cos\phi, \sin\phi, 0) * \cos(kz - \omega t + \delta) \\ &\quad \text{where } \delta = \arg(\mathcal{E}_0). \end{aligned} \tag{45}$$

Thus, in a linearly polarized wave, the electric field always points in the same direction $(\cos\phi, \sin\phi, 0)$ (modulo the overall sign), namely along the line in the (x, y) plane making angle ϕ with the x axis,



(46)

That's why such polarizations are called *linear*. The same polarizations are called *planar* after the shape of the 3D plot of the electric wave $\mathbf{E}(z)$ (for any fixed time t): Such a plot is restricted to a single 2D plane, spanning the z axis and the red line on the above diagram, for example this plot.

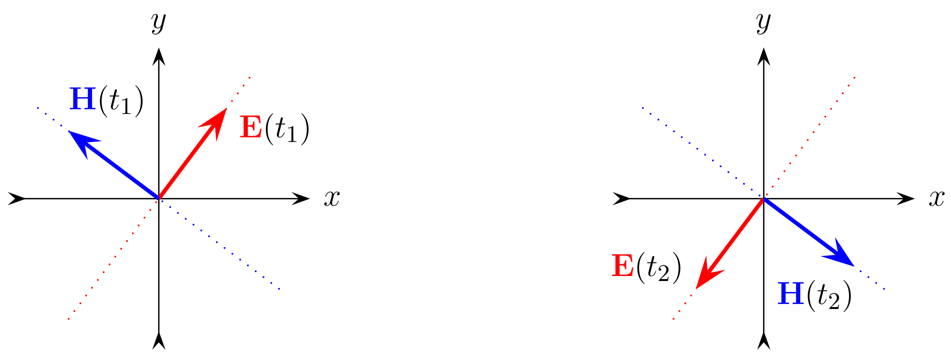
As to the magnetic field of a linearly polarized EM wave,

$$\vec{\mathcal{H}} \;=\; \frac{1}{Z}\,\hat{\mathbf{k}}\times\vec{\mathcal{E}} \;=\; \frac{\mathcal{E}_0}{Z}\,(0,0,1)\times(\cos\phi,\sin\phi,0) \;=\; \frac{\mathcal{E}_0}{Z}\,(-\sin\phi,+\cos\phi,0), \tag{47}$$

hence

$$\mathbf{H}(z,t) \;=\; \frac{|\mathcal{E}_0|}{Z} * (\sin\phi,+\cos\phi,0) * \cos(kz-\omega t+\delta). \tag{48}$$

In other words, the magnetic fields oscillates with the same phase as the electric field, but its direction is rotated 90° counterclockwise (in the (x,y) plane) from the electric field's direction. Here are snapshots of the electric and the magnetic fields at two instances of time:



(49)

Note: on this diagram, the $+z$ direction of the wave is towards your face, that's why the magnetic field points 90° to the left the electric field. If you were looking at the field from the opposite direction of the wave's source, the magnetic field would point 90° to the *right* of the electric field. This web page has an animated 3D diagram that clarifies the relative directions of the two fields.

Circular polarizations

In a circularly polarized wave, the complex amplitudes $\mathcal{E}_x$ and $\mathcal{E}_y$ have similar magnitudes but their phases differ by 90°, $\mathcal{E}_y = \pm i\mathcal{E}_x$ and hence

$$\vec{\mathcal{E}} \;=\; \frac{\mathcal{E}_0}{\sqrt{2}}\,(1,\pm i,0). \tag{50}$$

Consequently, the x and the y components of the electric field $\mathbf{E}(z,t)$ oscillate with phases differing by 90°:

$$E_x(z,t) \;=\; \frac{|\mathcal{E}_0|}{\sqrt{2}} \times \cos(kz-\omega t+\delta), \tag{51}$$

$$\begin{aligned} E_y(z,t) \;&=\; \frac{|\mathcal{E}_0|}{\sqrt{2}} \times \cos\bigl(kz-\omega t+\delta \pm \tfrac{\pi}{2}\bigr) \\ &=\; \mp\frac{|\mathcal{E}_0|}{\sqrt{2}} \times \sin(kz-\omega t+\delta). \end{aligned} \tag{52}$$

Thus, the electric field $\mathbf{E}$ keeps constant magnitude $|\mathbf{E}| = |\mathcal{E}_0|/\sqrt{2}$, but its direction moves in a circle in the (x,y) plane,

$$\text{direction}(\mathbf{E}) \;=\; \pm(\omega t - kz - \delta). \tag{53}$$

Here is a 3D animated illustration from wikipedia.

The two circular polarizations — one with $\vec{\mathcal{E}}_+ \propto (1,+i,0)$ and the other with $\vec{\mathcal{E}}_- \propto (1,-i,0)$ — correspond to the two opposite direction of the $\mathbf{E}$ field's rotation. But which direction of rotation we call 'right' and which we call 'left' depends on a convention:

- In both direction, we look at the electric field vector $\mathbf{E}(t)$ as a function of time at a fixed location $\mathbf{x}$.
- In the Optics convention, we look at the incoming wave — the unit wave vector $\hat{\mathbf{k}}$ points into your eye.
- But in the Particle Physics convention, we look at the outgoing wave, with the $\hat{\mathbf{k}}$ vector pointing away from you.

⋆ Consequently, the same physical direction of rotation that appears clockwise (right) in one convention would appear counterclockwise (left) in the other convention, and vice verse.

In particular, for the wave traveling in the $+z$ direction, looking at the (x,y) plane drawn on a horizontal piece of paper from above corresponds to the Optics convention: the wave travels up, towards your eyes. In this convention, the positive direction of angles in the (x,y) plane is counterclockwise (left), while eq. (53) tells us that the direction of $\mathbf{E}$ moves in the positive direction for $\vec{\mathcal{E}}_+ \propto (1, +i, 0)$ and in the negative direction for $\vec{\mathcal{E}}_- \propto (1, -i, 0)$. Consequently,

$$\begin{aligned}
&\text{In the Optical convention:}\\
&\vec{\mathcal{E}}_+ \ \propto\ (1,+i,0) \quad \text{is the left circular polarization,}\\
&\vec{\mathcal{E}}_- \ \propto\ (1,-i,0) \quad \text{is the right circular polarization.}
\end{aligned} \tag{54}$$

And of course,

$$\begin{aligned}
&\text{In the Particle Physics convention, it's the other way around:}\\
&\vec{\mathcal{E}}_+ \ \propto\ (1,+i,0) \quad \text{is the right circular polarization,}\\
&\vec{\mathcal{E}}_- \ \propto\ (1,-i,0) \quad \text{is the left circular polarization.}
\end{aligned} \tag{55}$$

The reason for this particular convention in the particle physics is that a circularly polarized plane EM wave corresponds to a beam of photons of definite helicity

$$\lambda \ \stackrel{\text{def}}{=}\ \hat{\mathbf{k}} \cdot \mathbf{Spin} \quad (\text{in units of } \hbar). \tag{56}$$

For a photon, the two allowed values of its helicity are $+1$ and -1 (but not 0), and its convenient to call a photon with $\lambda = +1$ as polarized right while a photon with $\lambda = -1$ as polarized left.

Altogether, we have the following correspondence table for the circular polarizations:

λ	Particle	Optics	Equation for $\vec{\mathcal{E}}$
$+1$	right	left	$i\hat{\mathbf{k}} \times \vec{\mathcal{E}}_+ = +\vec{\mathcal{E}}_+$
-1	left	right	$i\hat{\mathbf{k}} \times \vec{\mathcal{E}}_- = -\vec{\mathcal{E}}_-$

The last column here helps to right down the electric amplitudes $\vec{\mathcal{E}}$ for the 2 circular polarizations for a general direction $\hat{\mathbf{k}}$ of the plane wave: The two $\vec{\mathcal{E}}$'s are eigenvectors of a Hermitian linear operator $\vec{\mathcal{E}} \to i\hat{\mathbf{k}} \times \vec{\mathcal{E}}$ for its two non-zero eigenvalues $\lambda = \pm 1$. In particular, for $\hat{\mathbf{k}} = (0,0,1)$, the eigenvector equation becomes

$$i(0,0,1)\times(\mathcal{E}_x,\mathcal{E}_y,\mathcal{E}_z) \;=\; (-i\mathcal{E}_y,+i\mathcal{E}_x,0) \;=\; \pm(\mathcal{E}_x,\mathcal{E}_y,\mathcal{E}_z) \tag{57}$$

whose eigenvectors are indeed

$$\vec{\mathcal{E}}_\pm \;=\; \frac{\mathcal{E}_0}{\sqrt{2}}\,(1,\pm i,0). \tag{58}$$

Elliptic polarizations

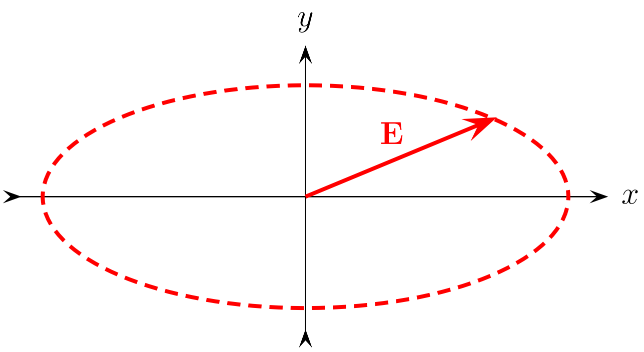
For generic $\mathcal{E}_x$ and $\mathcal{E}_y$ amplitudes of a plane wave — two complex numbers of different magnitudes and different phases, — the $\mathbf{E}(t)$ vector moves along an ellipse in the (x,y) plane, so such polarizations are called *elliptic*. For example, consider

$$\vec{\mathcal{E}} \;=\; \frac{\mathcal{E}_0}{\sqrt{2-r^2}}\big(1,\pm i\sqrt{1-r^2},0\big) \tag{59}$$

and hence

$$\begin{aligned} E_x(z,t) \;&=\; \frac{|\mathcal{E}_0|}{\sqrt{2-r^2}}\times\cos(\omega t-kz-\delta),\\ E_y(z,t) \;&=\; \pm\sqrt{1-r^2}\times\frac{|\mathcal{E}_0|}{\sqrt{2-r^2}}\times\sin(\omega t-kz-\delta). \end{aligned} \tag{60}$$

As a function of time (at a fixed z), the electric field vector with these components indeed follows an ellipse of eccentricity r in the (x,y) plane:



(61)

For this particular ellipse, its major axis is along the x axis while the minor axis is along the

y axis, but one may easily generalize this example to any other axis direction by taking

$$\vec{\mathcal{E}} = \frac{|\mathcal{E}_0|}{\sqrt{2-r^2}} * \Big((\cos\phi,\sin\phi,0) \pm i\sqrt{1-r^2} * (-\sin\phi,+\cos\phi,0)\Big). \tag{62}$$

Indeed, any complex 2D vector $\vec{\mathcal{E}} = (\mathcal{E}_x, \mathcal{E}_y, 0)$ can be written in the form (62) for some overall complex amplitude $\mathcal{E}_0$, a real angle ϕ, and a real eccentricity r between 0 and 1. For $r = 1$ we get a linear polarization in the direction ϕ, for $r = 0$ we get a circular polarization, and for any other $0 < r < 1$ we get an elliptic polarization.

Polarization bases

Consider superpositions of two (or several) EM waves of the same frequency ω traveling in the same direction $\hat{\mathbf{k}}$ (and hence having the same wave vector $\mathbf{k} = (\omega/v)\hat{\mathbf{k}}$). Since EM fields of a plane wave depend linearly on the electric amplitude vector $\vec{\mathcal{E}}$, superposition of all fields follow from superposition of these amplitude vectors:

$$\begin{aligned} \text{IF} \quad \vec{\mathcal{E}}_{\text{net}} &= \alpha_1\vec{\mathcal{E}}_1 + \alpha_2\vec{\mathcal{E}}_2 \\ \text{THEN} \quad \mathbf{E}_{\text{net}}(\mathbf{x},t) &= \alpha_1\mathbf{E}_1(\mathbf{x},t) + \alpha_2\mathbf{E}_2(\mathbf{x},t) \\ \text{AND} \quad \mathbf{H}_{\text{net}}(\mathbf{x},t) &= \alpha_1\mathbf{H}_1(\mathbf{x},t) + \alpha_2\mathbf{H}_2(\mathbf{x},t). \end{aligned} \tag{63}$$

So let's take a closer look at the linear space of the amplitude vectors $\vec{\mathcal{E}}$.

For a given direction $\hat{\mathbf{k}}$ of the plane waves, their electric amplitudes $\vec{\mathcal{E}}$ are complex two-dimensional vectors in the plane $\perp \hat{\mathbf{k}}$. Consequently, there are two independent polarizations $\mathbf{e}_1$ and $\mathbf{e}_2$, and all the amplitudes are linear combination of these polarizations,

$$\textbf{any} \quad \vec{\mathcal{E}} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \quad \text{for some complex } \alpha_1 \text{ and } \alpha_2\,. \tag{64}$$

For example, for a wave traveling in the $+z$ direction, we may use

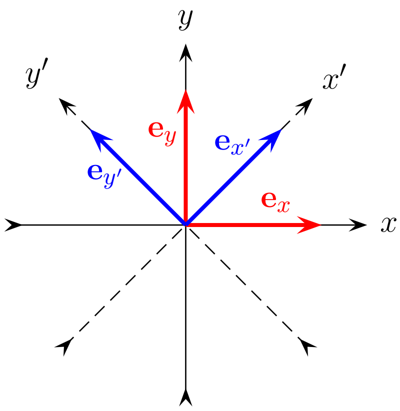
$$\mathbf{e}_x = (1,0,0),\ \mathbf{e}_y = (0,1,0) \quad \vec{\mathcal{E}} = (\mathcal{E}_x,\mathcal{E}_y,0) = \mathcal{E}_x\mathbf{e}_x + \mathcal{E}_y\mathbf{e}_y\,. \tag{65}$$

However, there are infinitely many other bases of a complex 2D vector space. Indeed, take any 2 complex unit vectors that are $\perp \hat{\mathbf{k}}$ and $\perp$ to each other, and they would form a basis:

$$\text{IF} \quad \mathbf{e}_1^*\cdot\mathbf{e}_1 = \mathbf{e}_2^*\cdot\mathbf{e}_2 = 1 \quad \text{AND} \quad \hat{\mathbf{k}}\cdot\mathbf{e}_1 = \hat{\mathbf{k}}\cdot\mathbf{e}_2 = \mathbf{e}_1^*\cdot\mathbf{e}_2 = 0$$

$$\text{THEN for any } \vec{\mathcal{E}} \perp \hat{\mathbf{k}}: \quad \vec{\mathcal{E}} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$$
$$\text{where } \alpha_1 = \mathbf{e}_1^* \cdot \vec{\mathcal{E}} \text{ and } \alpha_2 = \mathbf{e}_2^* \cdot \vec{\mathcal{E}}. \tag{66}$$

In particular, any pair of *real* unit vectors $\mathbf{e}_1 \perp \mathbf{e}_2$ (in the plane $\perp \hat{\mathbf{k}}$) forms a *basis of linear polarizations.* Indeed, a plane wave with amplitude $\alpha_1 \mathbf{e}_1$ for a real vector $\mathbf{e}_1$ is linearly polarized, and so is the wave with amplitude $\alpha_2 \mathbf{e}_2$, but their superposition may have any polarization we like, linear, circular, or elliptic, depending on the complex coefficients α_1 and α_2. Earlier in these notes, we have seen how this works in the $\mathbf{e}_x, \mathbf{e}_y$ basis (for wave moving in the $z+$ direction), but it would work in exactly the same way for any other pair of linear polarizations $\perp$ to each other. For example, for the same wave direction we may use a basis of $(\mathbf{e}_{x'}, \mathbf{e}_{y'})$ for some coordinate axes (x', y') rotated through some angle relative to (x, y):



$$\text{any } \vec{\mathcal{E}} = \mathcal{E}_x \mathbf{e}_x + \mathcal{E}_y \mathbf{e}_y = \mathcal{E}_{x'} \mathbf{e}_{x'} + \mathcal{E}_{y'} \mathbf{e}_{y'} . \tag{67}$$

The 2 circular polarizations

$$\mathbf{e}_+ = \frac{1}{\sqrt{2}}(1, +i, 0), \quad \mathbf{e}_- = \frac{1}{\sqrt{2}}(1, -i, 0), \tag{68}$$

also form a basis of all polarizations. For example, any linear polarization is a superposition of the two circular polarizations as

$$\vec{\mathcal{E}} = \mathcal{E}_0(\cos\phi, \sin\phi, 0) = \frac{\mathcal{E}_0}{\sqrt{2}} e^{-i\phi} \mathbf{e}_+ + \frac{\mathcal{E}_0}{\sqrt{2}} e^{+i\phi} \mathbf{e}_- , \tag{69}$$

— note coefficients of equal magnitudes but different phases, — while for any elliptic polarization one generally has

$$\vec{\mathcal{E}} = (\mathbf{e}_+^* \cdot \vec{\mathcal{E}}) \mathbf{e}_+ + (\mathbf{e}_-^* \cdot \vec{\mathcal{E}}) \mathbf{e}_- \tag{70}$$

where the two coefficient generally have both different magnitudes and different phases.

In principle, one may also use a pair of elliptic polarizations as a basis, but this is rarely done. In practice, one chooses a basis according to the available filters selecting a particular polarization or separating two polarizations from each other, and most such filters select either linear of circular polarizations.

Stokes parameters.

For EM waves other than radio waves or microwaves, the electric amplitude vectors $\vec{\mathcal{E}}$ cannot be directly measured. Instead, all we have is the wave's intensity

$$I = |\mathbf{S}| = \frac{\epsilon\epsilon_0}{2Z}|\vec{\mathcal{E}}|^2 = \frac{\epsilon\epsilon_0}{2Z}\langle\mathbf{E}^*\cdot\mathbf{E}\rangle\,. \tag{71}$$

However, by measuring intensities of an EM wave after letting it go through several polarizing filters, we may reconstruct the complex amplitude vector $\vec{\mathcal{E}}$ up to an overall phase. To see how this works, let's run the wave through 6 different polarizing filters — one filter at a time rather than a series of all 6 filters, — and measure the intensities of the filtered waves:

- Linear polarizing filters in x and y directions.
- Linear polarizing filters in x' and y' direction at 45° to the x and y axes.
- Left and Right circular polarizing filters.

Let $I_x, I_y, I_{x'}, I_{y'}, I_L, I_R$ denote the intensities of the wave after going through the respective filters, while

$$I_0 = I_x + I_y = I_{x'} + I_{y'} = I_L + I_R \tag{72}$$

is the intensity of the un-filtered wave. Now let's introduce the Stokes parameters of the wave:

$$\begin{aligned} S_0 &= \frac{2Z}{\epsilon\epsilon_0}\times I_0\,,\\ S_1 &= \frac{2Z}{\epsilon\epsilon_0}\times(I_x - I_y),\\ S_2 &= \frac{2Z}{\epsilon\epsilon_0}\times(I_{x'} - I_{y'}),\\ S_3 &= \frac{2Z}{\epsilon\epsilon_0}\times(I_L - I_R). \end{aligned} \tag{73}$$

In terms of the wave's electric amplitude vector

$$\vec{\mathcal{E}} \;=\; \mathcal{E}_x\mathbf{e}_x \;+\; \mathcal{E}_y\mathbf{e}_y \;=\; \mathcal{E}_{x'}\mathbf{e}_{x'} \;+\; \mathcal{E}_{y'}\mathbf{e}_{y'} \;=\; \mathcal{E}_L\mathbf{e}_L \;+\; \mathcal{E}_R\mathbf{e}_R\,, \tag{74}$$

— where

$$\mathcal{E}_{x',y'} \;=\; \frac{\mathcal{E}_x \pm \mathcal{E}_y}{\sqrt{2}}\,, \qquad \mathcal{E}_{L,R} \;=\; \frac{\mathcal{E}_x \mp i\mathcal{E}_y}{\sqrt{2}}\,, \tag{75}$$

— the Stokes parameters correspond to:

$$\begin{aligned}
S_0 &= |\vec{\mathcal{E}}|^2 \;=\; |\mathcal{E}_x|^2 \;+\; |\mathcal{E}_y|^2,\\
S_1 &= |\mathcal{E}_x|^2 \;-\; |\mathcal{E}_y|^2,\\
S_2 &= |\mathcal{E}_{x'}|^2 \;-\; |\mathcal{E}_{y'}|^2 \;=\; \tfrac{1}{2}|\mathcal{E}_x+\mathcal{E}_y|^2 \;-\; \tfrac{1}{2}|\mathcal{E}_x-\mathcal{E}_y|^2\\
&= 2\,\mathrm{Re}\big(\mathcal{E}_x^*\mathcal{E}_y\big),\\
S_3 &= |\mathcal{E}_L|^2 \;-\; |\mathcal{E}_R|^2 \;=\; \tfrac{1}{2}|\mathcal{E}_x-i\mathcal{E}_y|^2 \;-\; \tfrac{1}{2}|\mathcal{E}_x+i\mathcal{E}_y|^2\\
&= 2\,\mathrm{Im}\big(\mathcal{E}_x^*\mathcal{E}_y\big).
\end{aligned} \tag{76}$$

From these parameters, one may easily obtain the magnitudes $|\mathcal{E}_x|$ and $|\mathcal{E}_y|$ as well as the relative phase between the $\mathcal{E}_x$ and the $\mathcal{E}_y$, — and hence the whole complex electric amplitude vector $\vec{\mathcal{E}}$ up to an overall phase, $\vec{\mathcal{E}} \to e^{i\phi}\vec{\mathcal{E}}$.

In Quantum Mechanics, the spin states of a spin $=\frac{1}{2}$ particle or an atom work very similar to the polarization amplitudes of an EM wave. The quantum analogues of the polarizing filters are the Stern–Gerlach devices separating the atom beam according to the x, y, or z components of its magnetic moment being $+\mu_B$ or $-\mu_B$ (μ_B being the Bohr magneton $e\hbar/2m_e$), while the analogies of the Stokes parameters are the

$$\begin{aligned}
N_0 &= \text{net intensity of the atom beam},\\
N_1 &= N(m_z=+\mu_b) \;-\; N(m_z=-\mu_b),\\
N_2 &= N(m_x=+\mu_b) \;-\; N(m_x=-\mu_b),\\
N_3 &= N(m_y=+\mu_b) \;-\; N(m_y=-\mu_b).
\end{aligned} \tag{77}$$

Assuming all the atoms in the beam have the same spin state, we may reconstruct it from these 4 Stokes-like parameters up to an overall phase.