Problem **1**(a):

A point $\mathbf{y}$ on a rotating sphere moves at velocity $\mathbf{v} = \vec{\omega} \times \mathbf{y}$, which for a uniformly charged sphere sets up electric current of surface density

$$\mathbf{K}(\mathbf{y}) \;=\; \sigma\mathbf{v} \;=\; \sigma\vec{\omega} \times \mathbf{y}. \tag{S.1}$$

Or in terms of the 3D current density $\mathbf{J}$,

$$\mathbf{J}(\mathbf{y}) \;=\; \rho(\mathbf{y})\mathbf{v}(\mathbf{y}) \;=\; \sigma\delta(|\mathbf{y}| - R)\,\vec{\omega} \times \mathbf{y}. \tag{S.2}$$

Problem **1**(b):

Let

$$F(\mathbf{x}) \;\stackrel{\text{def}}{=}\; \iint\limits_{\text{sphere}} d^2\mathbf{y}\,\frac{\mathbf{y}}{|\mathbf{y} - \mathbf{x}|}\,. \tag{S.3}$$

This integral is a function of a single vector $\mathbf{x}$, so by the rotational symmetry it must point in the direction of $\mathbf{x}$ (or perhaps the opposite direction) while its magnitude depends on the magnitude of $\mathbf{x}$ and the radius R of the sphere we integrate over, thus

$$F(\mathbf{x}) \;=\; f(r_x, R)\,\mathbf{n}_x\,. \tag{S.4}$$

To calculate the magnitude here, let's use the spherical coordinates for the $\mathbf{y}$ where the North pole $\theta = 0$ points in the direction of the $\mathbf{x}$. Then

$$\text{component of } \mathbf{y} \text{ in the direction of } \mathbf{x} \;=\; \mathbf{y} \cdot \mathbf{n}_x \;=\; R\cos\theta \tag{S.5}$$

while

$$d^2\mathbf{y} \;\stackrel{\text{def}}{=}\; d^2\text{area}(\mathbf{y}) \;=\; R^2 \sin\theta\, d\theta\, d\phi \;\to\; 2\pi R^2\, d(-\cos\theta) \tag{S.6}$$

and

$$|\mathbf{x}-\mathbf{y}| \;=\; \sqrt{R^2+r_x^2-2Rr_x\cos\theta}\,. \tag{S.7}$$

Consequently,

$$\begin{aligned}
f(r_x,R) \;&=\; 2\pi R^2\int\limits_{-1}^{+1} d(-\cos\theta)\,\frac{R\cos\theta}{\sqrt{R^2+r_x^2-2Rr_x\cos\theta}}\\
&\quad\langle\!\langle\,\text{changing variables from } -\cos\theta \text{ to } t=r_x^2+R^2-2Rr_x\cos\theta\,\rangle\!\rangle\\
&=\; 2\pi R^3\int\limits_{(r_x-R)^2}^{(r_x+R)^2}\frac{dt}{2Rr_x}\,\frac{(r_x^2+R^2-t)/2Rr_x}{\sqrt{t}}\\
&=\; \frac{\pi R}{2r_x^2}\int\limits_{(r_x-R)^2}^{(r_x+R)^2} dt\,\frac{r_x^2+R^2-t}{\sqrt{t}} \;=\; \frac{\pi R}{2r_x^2}\left(2\sqrt{t}(R^2+r_x^2)\;-\;\tfrac{2}{3}t^{3/2}\right)\Big|_{(r_x-R)^2}^{(r_x+R)^2} \qquad\text{(S.8)}\\
&=\; \frac{\pi R}{r_x^2}\Big((r+R)\big((r^2+R^2)\;-\;\tfrac{1}{3}(r+R)^2\big)\\
&\qquad\qquad -\;|r_x-R|\big((r_x^2+R^2\;-\;\tfrac{1}{3}(r_x-R)^2)\big)\Big)\\
&\quad\langle\!\langle\,\text{after a bit of algebra}\,\rangle\!\rangle\\
&=\; \frac{\pi R}{r_x^2}\times\frac{4}{3}\min\big(r_x^3,R^3\big),
\end{aligned}$$

and therefore

$$F(\mathbf{x}) \;=\; \frac{4\pi R}{3}\,\min\big(r_x^3,R^3\big)\,\frac{\mathbf{n}_x}{r_x^2}\,. \tag{S.9}$$

Quod erat demonstrandum.

Problem **1**(c):

The vector potential (in the transverse gauge) of the current (S.2) obtains as a volume integral

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\iiint\frac{\mathbf{J}(\mathbf{y})\,d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,, \tag{S.10}$$

In light of the delta-function factor $\delta(|\mathbf{y}|-R)$ in the current (S.2) on the surface of the rotating sphere, this volume integral reduces to a surface integral

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi} \iint\limits_{\text{sphere}} \frac{(\sigma\vec{\omega}\times\mathbf{y})\, d^2\text{Area}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,. \tag{S.11}$$

After pulling out the constant factors from this integral — including the vector-product factor $\vec{\omega}\times$ — we arrive at

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0\sigma}{4\pi}\,\vec{\omega}\times \iint\limits_{\text{sphere}} \frac{\mathbf{y}\, d^2\text{Area}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \tag{S.12}$$

where the remaining integral no longer depends on the angular velocity $\vec{\omega}$ or even its direction; instead, it depends only on the $\mathbf{x}$ and the radius R of the sphere. In fact, it's precisely the integral (1) we have evaluated in part (b), thus

$$\begin{aligned} \mathbf{A}(\mathbf{x}) \;&=\; \frac{\mu_0\sigma}{4\pi}\,\vec{\omega}\times\left(\frac{4\pi R}{3}\,\min\bigl(r_x^3,R^3\bigr)\,\frac{\mathbf{n}_x}{r_x^2}\right) \\ &=\; \frac{\mu_0\sigma R}{3}\,\min\bigl(r_x^3,R^3\bigr)\,\frac{\vec{\omega}\times\mathbf{n}_x}{r_x^2}\,, \end{aligned} \tag{S.13}$$

or in terms of separate formulae for the two sides of the charged sphere,

$$\text{for } |\mathbf{x}|<R \;\; \text{(inside the sphere)}, \quad \mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0\sigma R}{3}\,(\vec{\omega}\times\mathbf{x}), \tag{S.14}$$

$$\text{for } |\mathbf{x}|>R \;\; \text{(outside the sphere)}, \quad \mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0\sigma R^4}{3}\,\frac{\vec{\omega}\times\mathbf{n}_x}{r_x^2}\,. \tag{S.15}$$

Note that outside the sphere, the vector potential (S.15) looks like the potential of a pure magnetic dipole with magnetic moment

$$\mathbf{m} \;=\; \frac{4\pi}{3}\,R^4\sigma\,\vec{\omega}. \tag{S.16}$$

Consequently, the magnetic field $\mathbf{B}$ outside the sphere looks like a pure dipole field

$$\mathbf{B}_{\text{outside}}(\mathbf{x}) \;=\; \frac{\mu_0\sigma R^4}{3}\,\frac{3(\mathbf{n}_x\cdot\vec{\omega})\mathbf{n}_x \,-\, \vec{\omega}}{|\mathbf{x}|^3}\,. \tag{S.17}$$

As to the inside the sphere, the vector potential (S.14) corresponds to a *uniform* magnetic field. Indeed,

$$\nabla\times(\vec{\omega}\times\mathbf{x}) \;=\; \vec{\omega}(\nabla\cdot\mathbf{x}) \;-\; (\vec{\omega}\cdot\nabla)\mathbf{x} \;=\; 3\vec{\omega} \;-\; \vec{\omega} \;=\; 2\vec{\omega} \;=\; \text{const}, \tag{S.18}$$

hence uniform

$$\mathbf{B}_{\text{inside}} \;=\; \frac{2\mu_0\sigma R}{3}\,\vec{\omega}. \tag{S.19}$$

Finally, the dipole moment of the rotating sphere. There are two ways to calculate it — and the students who use any of these methods should receive full grades. The first method is to look at the magnetic field we have already calculated and extract the magnetic dipole moment from the $\mathbf{B}(\mathbf{x})$ for $r_x\to\infty$. As it happens, the field (S.17) anywhere outside the sphere looks like a pure dipole field for a magnetic moment (S.16), so the sphere's dipole moment $\mathbf{m}$ must be precisely as in eq. (S.17), namely

$$\mathbf{m} \;=\; \frac{4\pi}{3}\,R^4\sigma\,\vec{\omega}. \tag{S.16}$$

The second method is to use the general formula we have derived in class for the volume currents,

$$\mathbf{m} \;=\; \frac{1}{2}\iiint \mathbf{y}\times\mathbf{J}(\mathbf{y})d^3\mathbf{y}. \tag{S.20}$$

For the surface current (S.2) on the sphere, this formula becomes

$$\begin{aligned}\mathbf{m} \;&=\; \tfrac{1}{2}\iint\limits_{\text{sphere}} \mathbf{y}\times(\sigma\vec{\omega}\times\mathbf{y})\,d^2\text{area}(\mathbf{y})\\ &=\; \frac{\sigma}{2}\iint\limits_{\text{sphere}} \big(\vec{\omega}(\mathbf{y}^2)-(\vec{\omega}\cdot\mathbf{y})\mathbf{y}\big)d^2\text{area}(\mathbf{y}),\end{aligned} \tag{S.21}$$

or in components

$$m_i \;=\; \frac{\sigma_2}{2}\,\omega_i\times\iint\limits_{\text{sphere}}(\delta_{ij}\mathbf{y}^2 \;-\; y_iy_j)\,d^2\text{area}(\mathbf{y}). \tag{S.22}$$

In the integral here

$$\iint\limits_{\text{sphere}} \mathbf{y}^2 \, d^2\text{area}(\mathbf{y}) \;=\; R^2 \times 4\pi R^2 \tag{S.23}$$

while

$$\iint\limits_{\text{sphere}} u_i y_j \, d^2\text{area}(\mathbf{y}) \;=\; \delta_{ij}\,\frac{R^2}{3} \times 4\pi R^2, \tag{S.24}$$

hence altogether

$$\iint\limits_{\text{sphere}} (\delta_{ij}\mathbf{y}^2 \;-\; y_i y_j) \, d^2\text{area}(\mathbf{y}) \;=\; \frac{8\pi R^4}{3}\,\delta_{ij} \tag{S.25}$$

and therefore

$$m_i \;=\; \frac{4\pi R^4 \sigma}{3}\,\omega_i \,. \tag{S.26}$$

In other words,

$$\mathbf{m} \;=\; \frac{4\pi}{3}\,\sigma R^4 \vec{\omega}, \tag{S.27}$$

exactly as in eq. (S.16).

<u>Problem **1**(d)</u>:

Finally, consider two concentric spheres: the outer sphere of radius R_1 and charge density σ_1 rotating with angular velocity ω_1, and the inner sphere of radius $R_2 < R_1$ and charge density σ_2 rotating at angular velocity ω_2. The two spheres rotate around different axes (although both go through the common center), and that's what causes the magnetic torque between the spheres.

By the angular version of the Newton's Third Law, we may calculate either the torque on the inner sphere due to magnetic field of the outer sphere, or the torque on the outer sphere due to magnetic field of the inner sphere: the two torques must be equal in magnitude and opposite in direction,

$$\vec{\tau}_{1\,\text{on}\,2} \;=\; -\vec{\tau}_{2\,\text{on}\,1}\,, \tag{S.28}$$

as long as both torques are taken WRT the same pivot, — the common center of the two spheres. Torque on the inner sphere happens to be *much* easier to calculate, so let's work it out.

In part (c) we saw that the magnetic field $\mathbf{B}_1(\mathbf{x})$ due to the outer sphere is uniform inside that sphere. So the torque on the inner sphere is due to a uniform magnetic field $\mathbf{B}_1$, which obtains from a simple formula

$$\vec{\tau}_{1\,\mathrm{on}\,2} \;=\; \mathbf{m}_2 \times \mathbf{B}1\,, \tag{S.29}$$

without any further integration. Thus,

$$\begin{aligned} \vec{\tau}_{1\,\mathrm{on}\,2} \;&=\; \mathbf{m}_2 \times \mathbf{B}_1 \;=\; \left(\frac{4\pi}{3}\,\sigma_2 R_2^4 \vec{\omega}_2\right) \times \left(\frac{2}{3}\mu_0 \sigma_1 R_1 \vec{\omega}_1\right) \\ &=\; \frac{8\pi}{9}\,\mu_0 \sigma_1 \sigma_2 R_1 R_2^4\,(\vec{\omega}_2 \times \vec{\omega}_1). \end{aligned} \tag{S.30}$$

As to the torque from the inner sphere on the outer sphere, the direct calculation involves non-trivial integration and a messy vector algebra. So let me skip the direct calculation and simply use the Third Law (S.28) for the torques, thus

$$\vec{\tau}_{2\,\mathrm{on}\,1} \;=\; -\frac{8\pi}{9}\,\mu_0 \sigma_1 \sigma_2 R_1 R_2^4\,(\vec{\omega}_2 \times \vec{\omega}_1) \;=\; +\frac{8\pi}{9}\,\mu_0 \sigma_1 \sigma_2 R_1 R_2^4\,(\vec{\omega}_1 \times \vec{\omega}_2). \tag{S.31}$$

Problem **2**(a):

First of all, by the rotational symmetry of the solenoid — and hence of the magnetic field — around the solenoid's axis, the magnetic field's components in the cylindrical coordinates $B_z(z,s,\phi)$, $B_s(z,s,\phi)$, and $B_\phi(z,s,\phi)$ do not depend on the angular coordinate ϕ so they are functions of z and s only. Second, by the Ampere's Law $B_\phi = 0$; indeed, take a coaxial circle of radius $s < R$ centered at some axis point z, then

$$\oint\limits_{\mathrm{circle}} \mathbf{B} \cdot d\vec{\ell} \;=\; 2\pi s \times B_\phi(s,z) \;=\; \mu_0 \times I[\text{through the circle}] \;=\; 0. \tag{S.32}$$

Thus, B_ϕ is zero everywhere while B_z and B_s are some functions of z and s only, and since there are no wires inside the solenoid, they should be *analytic functions* of z and s.

This means, we may expand them into convergent power series in s and z, although for our purpose it's more convenient to expand them into power series in s with z–dependent coefficients,

$$B_z(z,s) \;=\; \sum_{k=0}^{\infty} a_k(z)\times s^k, \qquad B_s(s,z) \;=\; \sum_{k=0}^{\infty} b_k(z)\times s^k. \tag{S.33}$$

To bring these series into the form (2–4), we need to show that the series for the $B_z(z,s)$ comprises only the even powers of s while the series for the $B_s(z,s)$ comprises only the odd powers. This follows from the requirement of the magnetic field being analytic not only in the cylindrical (z,s,ϕ) coordinates but also to in the Cartesian coordinates (x,y,z). Using

$$s \;=\; \sqrt{x^2+y^2}\,, \quad B_x \;=\; \frac{x}{s}\,B_s\,, \quad B_y \;=\; \frac{y}{s}\,B_s \quad [\text{for } B_\phi=0], \tag{S.34}$$

we may convert the series (S.33) into

$$B_z(x,y,z) \;=\; \sum_{k=0}^{\infty} a_k(z)\times(x^2+y^2)^{k/2}, \tag{S.35}$$

$$B_x(x,y,z) \;=\; \sum_{k=0}^{\infty} b_k(z)\times x(x^2+y^2)^{(k-1)/2}, \tag{S.36}$$

$$B_y(x,y,z) \;=\; \sum_{k=0}^{\infty} b_k(z)\times y(x^2+y^2)^{(k-1)/2}. \tag{S.37}$$

To make these series analytic in the Cartesian x and y coordinates, each term in each series must be a polynomial in x and y; the half-integral or negative powers of (x^2+y^2) are not allowed. Thus, we must have

$$\begin{aligned} a_k &\;=\; 0 \quad \text{unless } k \text{ is even},\\ b_k &\;=\; 0 \quad \text{unless } k \text{ is odd}. \end{aligned} \tag{S.38}$$

Going back to the cylindrical coordinates and to the series (S.33), we see that the first series has only even powers of s while the second series has only odd powers. In other words,

$$B_z(z,s) \;=\; \sum_{n=0}^{\infty} \alpha_n(z)\times s^{2n}, \qquad B_s(s,z) \;=\; \sum_{n=0}^{\infty} \beta_n(z)\times s^{2n+1} \tag{S.39}$$

where I have renamed $a_{2n}\to\alpha_n$ and $b_{2n+1}\to\beta_n$ for consistency with eqs. (2–3).

Problem **2**(b):

In the cylindrical coordinates, the divergence and the curl of an axially symmetric magnetic field with $B_\phi = 0$ obtain as

$$\nabla \cdot \mathbf{B} \;=\; \frac{\partial B_z}{\partial z} \;+\; \frac{\partial B_s}{\partial s} \;+\; \frac{B_s}{s}\,, \qquad \nabla \times \mathbf{B} \;=\; \left(\frac{\partial B_z}{\partial s} \;-\; \frac{\partial B_s}{\partial z}\right)\mathbf{n}_\phi\,. \tag{S.40}$$

To make these divergence and curl vanish, we should have

$$\frac{\partial B_s}{\partial z} \;=\; \frac{\partial B_z}{\partial s}\,, \tag{S.41}$$

$$\frac{\partial B_z}{\partial z} \;=\; -\left(\frac{\partial}{\partial s} \;+\; \frac{1}{s}\right) B_s. \tag{S.42}$$

In terms of the power series (2) and (3), we have

$$\frac{\partial B_s}{\partial z} \;=\; \frac{\partial}{\partial z}\sum_{n=0}^{\infty} \beta_n(z) \times s^{2n+1} \;=\; \sum_{n=0}^{\infty} \beta_n'(z) \times s^{2n+1}, \tag{S.43}$$

$$\frac{\partial B_z}{\partial s} \;=\; \frac{\partial}{\partial s}\sum_{n=0}^{\infty} \alpha_n(z) \times s^{2n} \;=\; \sum_{n=1}^{\infty} \alpha_n(z) \times 2ns^{2n-1} \;=\; \sum_{n=0}^{\infty} \alpha_{n+1} \times (2n+2)s^{2n+1}, \tag{S.44}$$

thus eq. (S.41) translates to

$$\sum_{n=0}^{\infty} \beta_n'(z) \times s^{2n+1} \;=\; \sum_{n=0}^{\infty} \alpha_{n+1} \times (2n+2)s^{2n+1} \tag{S.45}$$

and therefore

$$\text{for each } n: \quad \beta_n'(z) \;=\; (2n+2)\alpha_{n+1}(z). \tag{S.46}$$

Likewise, in terms of the series (2) and (3),

$$\frac{\partial B_z}{\partial z} \;=\; \frac{\partial}{\partial z}\sum_{n=0}^{\infty} \alpha_n(z) \times s^{2n} \;=\; \sum_{n=0}^{\infty} \alpha_n'(z) \times s^{2n}, \tag{S.47}$$

$$-\left(\frac{\partial}{\partial s}+\frac{1}{s}\right)B_s \;=\; -\left(\frac{\partial}{\partial s}+\frac{1}{s}\right)\sum_{n=0}^{\infty}\beta_n(z)\times s^{2n+1}$$

$$=\; -\sum_{n=0}^{\infty}\beta_n(z)\times\bigl((2n+1)s^{2n}\;+\;s^{2n}\;=\;(2n+2)s^{2n}\bigr), \tag{S.48}$$

hence eq. (S.42) translates to

$$\sum_{n=0}^{\infty}\alpha'_n(z)\times s^{2n} \;=\; -\sum_{n=0}^{\infty}\beta_n(z)\times(2n+2)s^{2n} \tag{S.49}$$

and therefore

$$\text{for each } n: \quad \alpha'_n(z) \;=\; -(2n+2)\beta_n(z). \tag{S.50}$$

Together, eqs. (S.46) and (S.50) provide recursive relations between the coefficients of the powers of s in the series (1) and (2) and the z-derivatives of the coefficients of lower powers. Solving these recursion relations, we have

$$\alpha_n(z) \;=\; \frac{1}{2n}\,\beta'_{n-1}(z) \;=\; \frac{1}{2n}\,\frac{d}{dz}\left(-\frac{\alpha'_{n-1}(z)}{2n}\right) \;=\; -\frac{1}{(2n)^2}\,\alpha''_{n-1}(z) \tag{S.51}$$

and hence

$$\begin{aligned}\alpha_n(z) \;&=\; \frac{-1}{(2n)^2}\,\frac{d^2}{dz^2}\left(\alpha_{n-1}(z) \;=\; \frac{-1}{(2n-2)^2}\,\alpha''_{n-2}\right) \;=\; \frac{+1}{(2n)^2(2n-2)^2}\,\alpha''''_{n-2}(z)\\ &=\; \frac{-1}{(2n)^2(2n-2)^2(2n-4)^2}\,\frac{d^6}{dz^6}\,\alpha_{n-3}(z) \;=\; \cdots\\ &=\; \frac{(-1)^n}{(2n)^2(2n-2)^2\cdots(2)^2}\,\frac{d^{2n}}{dz^{2n}}\alpha_0(z)\,.\end{aligned} \tag{S.52}$$

Or in a more compact form

$$\alpha_n(z) \;=\; \frac{(-1)^n}{[2^n n!]^2}\,\frac{d^{2n}}{dz^{2n}}\,\alpha_0(z), \tag{5.a}$$

and consequently,

$$\beta_n(z) \;=\; \frac{-1}{2n+2}\,\alpha'_n(z) \;=\; \frac{(-1)^{n+1}}{2^{2n+1}n!(n+1)!}\,\frac{d^{2n+1}}{dz^{2n+1}}\,\alpha_0(z). \tag{5.b}$$

Finally, eqs. (6) and (7) follow by plugging these $\alpha_n(z)$ and $\beta_n(z)$ into the series (2) and (3).

Problem $\mathbf{2}$(c):

A tightly-would solenoid can be approximated by a continuous sequence of current loops of linear density N/L, each carrying the solenoid's current I. In class, we had calculated the magnetic field of on such ring on the axis through the ring's center and $\perp$ to the ring's plane — *i.e.*, on the solenoid's symmetry axis — as

$$B_z^{\rm ring}(\delta z) \;=\; \frac{\mu_0 I}{2}\times\frac{R^2}{[R^2+\Delta z^2]^{3/2}} \tag{S.53}$$

where Δz is the distance between the on-axis point in question and the ring's center. For the solenoid whose turns run from $z=-(L/2)$ to $z=+(L/2)$, this gives us

$$B_z(z,0) \;=\; \frac{N}{L}\int\limits_{-L/2}^{+L/2} dz'\,B_z^{\rm ring}(z'-z) \tag{S.54}$$

and hence

$$B_z(z,0) \;=\; \frac{N}{L}\times\frac{\mu_0 IR^2}{2}\times\int\limits_{-(L/2)-z}^{+(L/2)-z}\frac{d(\Delta z)}{[R^2+\Delta z^2]^{3/2}}\,. \tag{S.55}$$

To evaluate the integral here, let's change variables according to

$$\Delta z \;=\; R\times\tan\lambda, \tag{S.56}$$

thus

$$R^2\,+\,\Delta z^2 \;=\; 1\,+\,\tan^2\lambda \;=\; \frac{1}{\cos^2\lambda}\,,\qquad d(\Delta z)\;=\;\frac{R\,d\lambda}{\cos^2\lambda}\,,$$

hence

$$\frac{d(\Delta z)}{[R^2+\Delta z^2]^{3/2}} \;=\; \frac{R\,d\lambda}{\cos^2\lambda}\times\frac{\cos^3\lambda}{R^3} \;=\; \frac{\cos\lambda\,d\lambda}{R^2} \;=\; \frac{d(\sin\lambda)}{R^2} \tag{S.57}$$

where

$$\sin\lambda \;=\; \frac{\tan\lambda}{\sqrt{1+\tan^2\lambda}} \;=\; \frac{\Delta z/R}{\sqrt{1+(\Delta z/R)^2}} \;=\; \frac{\Delta z}{\sqrt{R^2+\Delta z^2}}\,. \tag{S.58}$$

Consequently,

$$\int\limits_{-(L/2)_z}^{+(L/2)-z} \frac{d(\Delta z)}{[R^2+\Delta z^2]^{3/2}} \;=\; \frac{1}{R^2}\left(\frac{(L/2)-z}{\sqrt{R^2+((L/2)-z)^2}} \,-\, \frac{-(L/2)-z}{\sqrt{R^2+((L/2)+z)^2}}\right) \qquad \text{(S.59)}$$

and therefore

$$B_z(0) \;=\; \frac{\mu_0 IN}{L}\times\frac{1}{2}\left(\frac{(L/2)-z}{\sqrt{R^2+((L/2)-z)^2}} \,+\, \frac{(L/2)+z}{\sqrt{R^2+((L/2)+z)^2}}\right), \qquad \text{(S.60)}$$

exactly as in eq. (8).

Problem **2**(d):

In the middle part of a long thin solenoid, both $x+ = (L/2)\pm z$ and $x_- = (L/2)-z$ are much larger than the solenoid's radius R, so we may approximate eq. (8) for the the on-axis magnetic field using

$$\frac{x}{\sqrt{x^2+R^2}} \;\approx\; 1 \;-\; \frac{R^2}{2x^2} \;+\; O(R^4/x^2) \quad \text{for } x\gg R, \qquad \text{(S.61)}$$

hence

$$\begin{aligned} \frac{(L/2)-z}{\sqrt{R^2+((L/2)-z)^2}} + \frac{(L/2)+z}{\sqrt{R^2+((L/2)+z)^2}} \;\approx\; & \\ \approx\; 2 \;-\; \frac{R^2}{2}\left(\frac{1}{((L/2)-z)^2} + \frac{1}{((L/2)+z)^2}\right) \;+\; O(R^4/L^4) & \qquad \text{(S.62)} \\ =\; 2 \;-\; R^2\times\frac{(L/2)^2+z^2}{[(L/2)^2-z^2]^2} \;+\; O(R^4/L^4) & \end{aligned}$$

and therefore

$$B_z(z,0) \;=\; \alpha_0(z) \;\approx\; \frac{\mu_0 IN}{L}\times\left(1 \,-\, 2R^2\times\frac{L^2+4z^2}{[L^2-4z^2]^2} \,+\, O(R^4/L^4)\right). \qquad \text{(S.63)}$$

Given this formula, we may estimate the derivatives of the α_0 at $|z|\ll L$ (the central region of the solenoid) as

$$(d/dz)^{2n}\alpha_0(z) \;\approx\; (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^2}{L^{2n+2}} \tag{S.64}$$

for the even-order derivatives, and

$$(d/dz)^{2n+1}\alpha_0(z) \;\approx\; (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^2 z}{L^{2n+4}} \tag{S.65}$$

for the odd-numbered derivatives.

In light of eqs. (4), this means

$$\begin{aligned}\alpha_n(z) &\approx (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^2}{L^{2n+2}}\,,\\ \beta_n(r) &\approx (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^2 z}{L^{2n+4}}\,.\end{aligned} \tag{S.66}$$

Consequently, the n^{th} term in the series (5) for the $B_z(z,s)$ can be estimated as

$$\alpha_n(z)\times s^{2n} \;\approx\; (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^2\,s^{2n}}{L^{2n+2}}\,, \tag{S.67}$$

and since $s\le R$ inside the solenoid,

$$\alpha_n\times s^{2n} \;\lesssim\; (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^{2+2n}}{L^{2n+2}} \;=\; (\text{number})\times\alpha_0\times\left(\frac{R}{L}\right)^{2n+2}. \tag{S.68}$$

Thus, the subsequent terms in the series (5) are suppressed by larger and larger powers of the small ratio R/L, so for a long thin solenoid we need only the first few terms. In particular, to the accuracy of $(R/L)^4$, we need just the α_0 and the α_1 terms thus

$$B_z(z,s) \;\approx\; \alpha_0(z) \;+\; \alpha_1(z)\times s^2. \tag{S.69}$$

Likewise, n^{th} term in the series (5) for the $B_s(z,s)$ can be estimated as

$$\beta_n(z)\times s^{2n+1} \;\approx\; (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{R^2\,z\,s^{2n+1}}{L^{2n+4}}\,, \tag{S.70}$$

and since $s\le R$ inside the solenoid,

$$\beta_n(z)\times s^{2n+1} \;\lesssim\; (\text{number})\times\frac{\mu_0 IN}{L}\times\frac{z\,R^{2n+3}}{L^{2n+4}} \;=\; (\text{number})\times\alpha_0\times\frac{z}{L}\left(\frac{R}{L}\right)^{2n+3}. \tag{S.71}$$

Again, these terms rapidly diminish with n, so the leading term here dominates the series.

In particular, to get the $B_s(z,s)$ to the accuracy of $B_z \times (R/L)^4$, we need only the leading term, thus

$$B_s(z,s) \;\approx\; \beta_1(z) \times s. \tag{S.72}$$

Problem **2**(e):

Let's focus on the very central region $|z| = O(R) \ll L$ of the solenoid. In this region, eq. (8) for the $\alpha_0(z)$ may be approximated as

$$\alpha_0(z) \;=\; \frac{\mu_0 IN}{L}\left(1 \,-\, \frac{2R^2}{L^2} \,-\, \frac{24R^2z^2}{L^4} \,+\, \frac{6R^4}{L^4}\,, + O(R^6/L^6)\right), \tag{S.73}$$

hence

$$\begin{aligned}
\alpha_0'(z) \;&=\; \frac{\mu_0 IN}{L}\left(0 \,-\, \frac{48R^2z}{L^4} \,+\, O(R^5/L^6)\right), && \text{(S.74)}\\
\alpha_0''(z) \;&=\; \frac{\mu_0 IN}{L}\left(0 \,-\, \frac{48R^2}{L^4} \,+\, O(R^4/L^6)\right), && \text{(S.75)}\\
\alpha_0'''(z) \;&=\; \frac{\mu_0 IN}{L}\Bigl(0 \,+\, O(R^3/L^6)\Bigr), && \text{(S.76)}\\
&\textit{etc., etc.}
\end{aligned}$$

Plugging these formulae into eqs. (6) and (7) for the off-axis magnetic field, we immediately see that in the central region of the solenoid

$$\begin{aligned}
B_z(z,s) \;&=\; \frac{\mu_0 IN}{L}\left(1 \,-\, \frac{2R^2}{L^2} \,+\, \frac{6R^2(R^2-4z^2+2s^2)}{L^4} \,+\, O(R^6/L^6)\right), && \text{(9.a)}\\
B_s(z,s) \;&=\; \frac{\mu_0 IN}{L}\left(0 \,+\, \frac{24R^2zs}{L^4} \,+\, O(R^6/L^6)\right). && \text{(9.b)}
\end{aligned}$$

Quod erat demonstrandum.

Problem **3**(a):

Let's start with a simpler problem: two concentric metal spheres, and the space between the spheres is either completely empty or completely filled with a uniform dielectric. By spherical symmetry of this setup, the electric field between the spheres must point in the radial direction while its magnitude depends only on the radius; by Gauss Law,

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{A}{r^2}\,\mathbf{n} \tag{S.77}$$

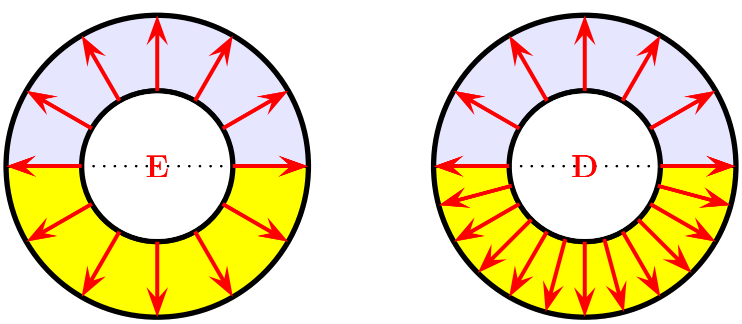
for some constant A. The value of this constant follows from the voltage between the plates:

$$V \;=\; \Phi(a) - \Phi(b) \;=\; A\left(\frac{1}{a}-\frac{1}{b}\right) \quad\Longrightarrow\quad A \;=\; \frac{Vab}{b-a}\,. \tag{S.78}$$

For the problem at hand, the space between the spheres is half-filled with a dielectric while the other half is vacuum. Fortunately, each material occupies a hemisphere, so the boundary between them lies in the equatorial plane. Consequently, for this geometry the electric field is exactly as in eq. (S.77), while the displacement field $\mathbf{D}$ is

$$\begin{aligned} \mathbf{D}_{\text{in vacuum}} \;&=\; \epsilon_0\mathbf{E} \;=\; \frac{\epsilon_0 A}{r^2}\,\mathbf{n},\\ \mathbf{D}_{\text{in dielectric}} \;&=\; \epsilon\epsilon_0\mathbf{E} \;=\; \frac{\epsilon\epsilon_0 A}{r^2}\,\mathbf{n}. \end{aligned} \tag{S.79}$$

Graphically,



(S.80)

Indeed, the field (S.77) and (S.79) obey the boundary conditions at the dielectric-boundary interface

$$\mathbf{E}^{\|}_{\text{vac}} \;=\; \mathbf{E}^{\|}_{\text{diel}}\,, \quad \mathbf{D}^{\perp}_{\text{vac}} \;=\; \mathbf{D}_{\text{diel}} \tag{S.81}$$

since (1) the $\mathbf{E}$ field is completely continuous across the boundary, while (2) the $\mathbf{D}$ field at both sides of the boundary points in the radial direction which happens to be parallel to the boundary, thus

$$\mathbf{D}^{\perp}_{\text{vac}} \;=\; 0 \;=\; \mathbf{D}^{\perp}_{\text{diel}} \,. \tag{S.82}$$

Besides these boundary conditions, the fields (S.77) and (S.79) obviously obey the electrostatic field equations

$$\nabla\times\mathbf{E} \;=\; 0 \quad\text{and}\quad \nabla\cdot\mathbf{D} \;=\; \rho \;=\; 0 \tag{S.83}$$

in both halves of the space between the spheres, as well as the boundary conditions on the metal spheres,

$$\begin{aligned} \text{for } \mathbf{x}\in\text{inner sphere},\quad \Phi(\mathbf{x}) \;&=\; \text{const},\\ \text{for } \mathbf{x}\in\text{outer sphere},\quad \Phi(\mathbf{x}) \;&=\; \text{const},\\ \Phi(\text{inner sphere}) \;-\; \Phi(\text{outer sphere}) \;&=\; V. \end{aligned} \tag{S.84}$$

Thus, the electric tension and displacement fields for this problem are indeed as in eqs. (S.77) and (S.79).

Problem **3**(b):

Inside the metal of each sphere $\mathbf{E}=0$ and hence $\mathbf{D}=0$. This makes the $\mathbf{D}$ field discontinuous at the outer surface of the inner sphere and at the inner surface of the outer sphere, and the physical reason for such a discontinuity is the surface density σ of macroscopic charges. By the Gauss Law,

$$\sigma \;=\; \mathbf{D}^{\perp}(\text{just outside the metal}) \;=\; \mathbf{D}\cdot\mathbf{n}^{\perp} \tag{S.85}$$

where $\mathbf{n}^{\perp}$ is the unit vector normal to the metal's surface and point out from the metal. For the outer surface of the inner sphere this makes $\mathbf{n}^{\perp}=+\mathbf{n}_r$ but for the inner surface of the outer sphere $\mathbf{n}^{\perp}=-\mathbf{n}_r$. Consequently,

$$\sigma(\text{inner sphere, vacuum side}) \;=\; +\frac{\epsilon_0 A}{a^2}\,, \tag{S.86}$$

$$\sigma(\text{inner sphere, dielectric side}) \;=\; +\frac{\epsilon\epsilon_0 A}{a^2}\,, \tag{S.87}$$

$$\sigma(\text{outer sphere, vacuum side}) \;=\; -\frac{\epsilon_0 A}{b^2}\,, \tag{S.88}$$

$$\sigma(\text{outer sphere, dielectric side}) \;=\; -\frac{\epsilon\epsilon_0 A}{b^2}\,, \tag{S.89}$$

where A is as in eq. (S.78).

Given these surface charge densities, the net charge on the inner sphere is

$$Q_{\text{inner}} \;=\; +\frac{\epsilon_0 A}{a^2}\times 2\pi a^2 \;+\; \frac{\epsilon\epsilon_0 A}{a^2}\times 2\pi a^2 \;=\; 2\pi(\epsilon+1)\epsilon_0 A, \tag{S.90}$$

while the net charge on the outer sphere is

$$Q_{\text{outer}} \;=\; -\frac{\epsilon_0 A}{b^2}\times 2\pi b^2 \;-\; \frac{\epsilon\epsilon_0 A}{b^2}\times 2\pi b^2 \;=\; -2\pi(\epsilon+1)\epsilon_0 A \;=\; -Q_{\text{inner}}\,. \tag{S.91}$$

Treating these two metal spheres as plates of a capacitor with charges $\pm Q$, the capacitance of this capacitor is

$$C \;=\; \frac{Q}{V} \;=\; 2\pi(\epsilon+1)\epsilon_0\times\frac{A}{V} \;=\; 2\pi(\epsilon+1)\epsilon_0\times\frac{ab}{b-a}\,. \tag{S.92}$$