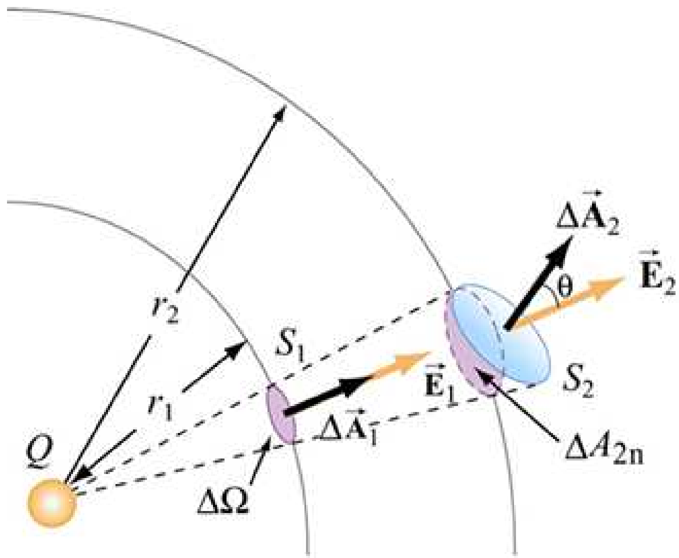
Problem **2**(a):

Let's span the current-carrying wire loop $\mathcal{L}$ with some surface $\mathcal{S}$. To find the solid angle occupied by the image of $\mathcal{S}$ as viewed from point $\mathbf{x}$, we project $\mathcal{S}$ onto a unit sphere centered on $\mathbf{x}$, and then measure the area of the image. For an infinitesimal piece of $\mathcal{S}$ of vector area $d\mathbf{a}$, we first project this piece onto a line of sight from $\mathbf{x}$ and then further project it onto the unit sphere:



On this picture

$$\Delta A_{2n} \;=\; \Delta A_2 \times \cos\theta, \quad \Delta\Omega \;=\; \frac{\Delta A_1}{r_1^2} \;=\; \frac{\Delta A_2}{r_2^2}\,, \tag{S.1}$$

which in our notations corresponds to

$$d\Omega \;=\; \frac{\mathbf{n}\cdot d^2\mathbf{area}}{R^2} \tag{S.2}$$

where R is the distance from the observation point $\mathbf{x}$ and $\mathbf{n}$ is the unit vector along the line of sight. For the infinitesimal piece of $\mathcal{S}$ located at $\mathbf{y}$,

$$R \;=\; |\mathbf{y}-\mathbf{x}|, \quad \mathbf{n} \;=\; \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\,, \tag{S.3}$$

hence

$$d\Omega \;=\; \frac{(\mathbf{y}-\mathbf{x})\cdot d^2\mathbf{area}(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^3}\,. \tag{S.4}$$

Integrating this formula over the whole surface $\mathcal{S}$ spanning the loop $\mathcal{L}$, we arrive at

$$\Omega(\mathbf{x}) \;=\; \iint\limits_{\mathcal{S}} \frac{(\mathbf{y}-\mathbf{x})\cdot d^2\mathbf{area}(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^3}\,. \tag{2}$$

Quod erat demonstrandum.

Problem **2**(b):

The sign convention for the $\Omega(\mathbf{x})$ follow from eq. (2) and the standard convention for the direction of the area vector. For simplicity, consider a flat loop $\mathcal{L}$ spanned by a flat surface $\mathcal{S}$. The area vector $\mathbf{a}$ of this surface is perpendicular to the surface itself, but which perpendicular? To make the Stokes' theorem work without an extra sign, the direction of $\mathbf{a}$ should follow from the sense of the loop $\mathcal{L}$ by the right hand rule: if you see the loop (or rather the current in the loop) running clockwise, then the area vector $\mathbf{a}$ points away from you, *i.e.*, makes angle $< 90^\circ$ with the line of sight; but if you see the loop $\mathcal{L}$ running counterclockwise, then $\mathbf{a}$ points towards you, *i.e.*, makes angle $> 90^\circ$ with the line of site. The same rule applies to the area vector of any infinitesimal piece of $\mathcal{S}$, so the integrand in eq. (2) is positive for a clockwise loop $\mathcal{L}$ and negative for a counterclockwise $\mathcal{L}$.

For a non-flat surface, the rule for for the direction of the $d\mathbf{a}$ vector is topological. The surface $\mathcal{S}$ spanning the loop $\mathcal{L}$ must be orientable, *i.e.*, have two well defined sides; Möbius strips and similar non-orientable surfaces are not allowed. Depending on the sense of the loop $\mathcal{L}$, we call one side 'inner' and the other side 'outer' according at the right hand rule, and then the direction of $d\mathbf{a}$ is the $\perp$ to the surface (at the point in question) and pointing from the 'inside' to the 'outside'. Consequently, if the loop $\mathcal{L}$ and the surface $\mathcal{S}$ are not too twisted and lie largely to one side of $\mathbf{x}$, then the sign of $\Omega(\mathbf{x})$ obtaining from eq. (2) follows from the sense of the loop as viewed from $\mathbf{x}$ similarly to the flat-surface case.

The problem with eq. (2) is that different surfaces spanning the same loop $\mathcal{L}$ may yield different values of $\Omega(\mathbf{x})$, although all the different values for the same point $\mathbf{x}$ differ by 4π, or at worst by $4\pi \times$ an integer. To see how this works, let two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ span $\mathcal{L}$ and consider the space $\mathcal{V}$ trapped between these surfaces. Together, $\mathcal{S}_1$ and $\mathcal{S}_2$ form the complete surface of the volume $\mathcal{V}$, but one of the the two surfaces — say, $\mathcal{S}_2$ — has a wrong orientation — its infinitesimal area vectors point inside $\mathcal{V}$ rather than outside. So properly speaking, the complete surface of $\mathcal{V}$ is $\mathcal{S}_1 - \mathcal{S}_2$. By Gauss theorem, this means that for any vector field $\mathbf{f}(\mathbf{y})$

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{f}\, d^3\mathbf{y} \;=\; \iint_{\mathcal{S}_1} \mathbf{f} \cdot d^2\mathbf{a} - \iint_{\mathcal{S}_2} \mathbf{f} \cdot d^2\mathbf{a}\,. \tag{S.5}$$

Now let

$$\mathbf{f}(\mathbf{y}) \;=\; \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \tag{S.6}$$

for any fixed point $\mathbf{x}$. Then calculating $\Omega(\mathbf{x})$ using the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ and taking the difference, we obtain

$$\Omega_1(\mathbf{x}) - \Omega_2(\mathbf{x}) \;=\; \iint_{\mathcal{S}_1} \frac{(\mathbf{y}-\mathbf{x}) \cdot d^2\mathbf{a}(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^3} - \iint_{\mathcal{S}_2} \frac{(\mathbf{y}-\mathbf{x}) \cdot d^2\mathbf{a}(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^3} \;=\; \iiint_{\mathcal{V}} \nabla_y \cdot \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\right) d^3\mathbf{y}. \tag{S.7}$$

But

$$\nabla_y \cdot \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\right) \;=\; 4\pi\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{S.8}$$

hence

$$\Omega_1(\mathbf{x}) \;-\; \Omega_2(\mathbf{x}) \;=\; \begin{cases} 4\pi & \text{if } \mathbf{x} \text{ lies inside } \mathcal{V},\ \textit{i.e.} \text{ between } \mathcal{S}_1 \text{ and } \mathcal{S}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{S.9}$$

In other words, if we take two surfaces spanning the same loop $\mathcal{L}$ but on different sides from point $\mathbf{x}$, then the corresponding angles $\Omega_1(\mathbf{x})$ and $\Omega_2(\mathbf{x})$ differ by 4π.

A qualitative way to see this multivaluedness is to project both surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ and the loop $\mathcal{L}$ onto the the unit sphere centered at $\mathbf{x}$. The image of the loop $\mathcal{L}$ divides the sphere into two parts, and if the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ lie on different sides of $\mathbf{x}$, then their images are precisely the two parts of the sphere divided by the image of $\mathcal{L}$. Together, these two images complete the sphere, so their solid angles must add up to 4π. But one of the two images has a wrong orientation, so the solid angle it occupies should be taken with a minus sign, hence

$$\text{either } \Omega_1(\mathbf{x}) \;-\; \Omega_2(\mathbf{x}) \;=\; 4\pi \quad \text{or } \Omega_2(\mathbf{x}) \;-\; \Omega_1(\mathbf{x}) \;=\; 4\pi. \tag{S.10}$$

Finally, when the wire loop $\mathcal{L}$ is a coil of many turns, a surface spanning it must span every turn, which calls for some kind of a helicoid. Projecting such a helicoid onto a sphere creates many overlapping patches, and their solid angles must be added up to produce the correct $\Omega(\mathbf{x})$. Consequently, for an $\mathbf{x}$ close to a coil of many turns we may get $\Omega(\mathbf{x}) \gg 4\pi$. Also, when $\mathbf{x}$ is in the middle of the coil, then different helicoid-like surfaces spanning the same coil may have several turns on different side of $\mathbf{x}$. Consequently, the values of $\Omega(\mathbf{x})$ for these two surfaces may differ not just by 4π but by $4\pi \times$ an integer, *i.e.*,

$$\Omega_1(\mathbf{x}) \;-\; \Omega_2(\mathbf{x}) \;=\; 0 \;\text{ or }\; \pm 4\pi \;\text{ or }\; \pm 8\pi \;\text{ or }\; \pm 12\pi \;\text{ or }\; \cdots. \tag{S.11}$$

However, since the differences between the values of $\Omega(\mathbf{x})$ for the same point $\mathbf{x}$ are always integer multiples of 4π, they cannot gradually change from $\mathbf{x}$ to $\mathbf{x} + \delta\mathbf{x}$. Therefore, *despite the multivaluedness of the* $\Omega(\mathbf{x})$, *the gradient* $\nabla\Omega(\mathbf{x})$ *is single-valued.*

Problem **2**(c):

First, let's derive eq. (3). Take any vector field $\mathbf{f}(\mathbf{y})$ and any constant vector $\mathbf{c}$. By the double vector product formula,

$$\nabla \times (\mathbf{f} \times \mathbf{c}) \;=\; (\mathbf{c} \cdot \nabla)\mathbf{f} \;-\; (\nabla \cdot \mathbf{f})\mathbf{c}. \tag{S.12}$$

In particular, let

$$\mathbf{f}(\mathbf{y}) \;=\; \frac{(\mathbf{y} - \mathbf{x})}{|\mathbf{y} - \mathbf{x}|^3} \;=\; \nabla_y \left(\frac{-1}{|\mathbf{y} - \mathbf{x}|} \right) \tag{S.13}$$

for a fixed $\mathbf{x}$. For this 'field', $\nabla_y \cdot \mathbf{f} = 0$ for $\mathbf{y} \neq \mathbf{x}$, so eq. (S.12) simplifies to

$$\nabla_y \times (\mathbf{f} \times \mathbf{c}) \;=\; (\mathbf{c} \cdot \nabla_y)\mathbf{f} \tag{S.14}$$

and hence

$$\nabla_y \times \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \times \mathbf{c} \right) \;=\; (\mathbf{c}\cdot\nabla_y)\, \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\,. \tag{3}$$

Now let's use this formula to calculate the gradient of $\Omega(\mathbf{x})$ as calculated in eq. (2). Let $\mathbf{c}$ be come constant vector, then

$$\begin{aligned}
\mathbf{c}\cdot\nabla\Omega(\mathbf{x}) \;&=\; (\mathbf{c}\cdot\nabla_x) \iint\limits_{\mathcal{S}} \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\cdot d^2\mathbf{a}(\mathbf{y}) \;=\; \iint\limits_{\mathcal{S}} (\mathbf{c}\cdot\nabla_x) \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \right)\cdot d^2\mathbf{a}(\mathbf{y}) \\
&\qquad \langle\!\langle \text{ using } \nabla_x f(\mathbf{y}-\mathbf{x}) = -\nabla_y f(\mathbf{y}-\mathbf{x}) \,\rangle\!\rangle \\
&=\; -\iint\limits_{\mathcal{S}} (\mathbf{c}\cdot\nabla_y) \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \right)\cdot d^2\mathbf{a}(\mathbf{y}) \\
&=\; -\iint\limits_{\mathcal{S}} \left(\nabla_y \times \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \times \mathbf{c} \right) \right)\cdot d^2\mathbf{a}(\mathbf{y}) \qquad \langle\!\langle \text{ using eq. (3) } \rangle\!\rangle \\
&=\; -\oint\limits_{\mathcal{L}} \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \times \mathbf{c} \right)\cdot d^2\mathbf{y} \qquad \langle\!\langle \text{ by the Stokes' theorem } \rangle\!\rangle \\
&=\; +\oint\limits_{\mathcal{L}} \left(\mathbf{c}\cdot \left(\frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{y} \right) \right) \qquad \langle\!\langle \text{ vector identity } \rangle\!\rangle \\
&=\; \mathbf{c}\cdot \oint\limits_{\mathcal{L}} \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{y}.
\end{aligned} \tag{S.15}$$

Since $\mathbf{c}$ on both sides of this equation is an arbitrary constant vector, this means

$$\nabla\Omega(\mathbf{x}) \;=\; \oint\limits_{\mathcal{L}} \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{y} = \;-\oint\limits_{\mathcal{L}} d\mathbf{y} \times \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\,. \tag{S.16}$$

Finally, let's see what all this math has to do with eq. (1) for the scalar magnetic potential $\Psi(\mathbf{x})$. The magnetic intensity field $\mathbf{H}$ follows from $\Psi(\mathbf{x})$ as $-\nabla\Psi$, hence according to eqs. (1) and (S.16),

$$\mathbf{H}(\mathbf{x}) \;=\; -\nabla\Psi(\mathbf{x}) \;=\; -\frac{I}{4\pi}\,\nabla\Omega(\mathbf{x}) \;=\; +\frac{1}{4\pi} \oint\limits_{\mathcal{L}} I\, d\mathbf{y} \times \frac{(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}\,. \tag{S.17}$$

But this is precisely the Biot–Savart–Laplace formula for the magnetic field of the current I flowing through the wire loop $\mathcal{L}$!

Quod erat demonstrandum.

Problem **3**:

Before addressing the problem at hand, let's consider the work and the energy for a variable-capacitance capacitor connected to a battery. As the capacitance changes, the charge stored in the capacitor changes, so a current flows through the battery, which performs electric work

$$W_{\rm el} \;=\; V\,\delta Q.$$

Also, changing the capacitance of a charged capacitor takes a mechanical work $W_{\rm mech}$, which can be calculated from the energy balance equation

$$\delta U \;=\; \delta W_{\rm el} \;+\; \delta W_{\rm mech} \tag{S.18}$$

where

$$U \;=\; \frac{Q^2}{2C} \;=\; \frac{CV^2}{2} \;=\; \frac{VQ}{2} \tag{S.19}$$

is the energy stored in the capacitor. Consequently

$$\delta U \;=\; \frac{Q\delta Q}{C} \;-\; \frac{Q^2}{2C^2}\,\delta C \;=\; V\,\delta Q \;-\; \frac{V^2}{2}\,\delta C\,, \tag{S.20}$$

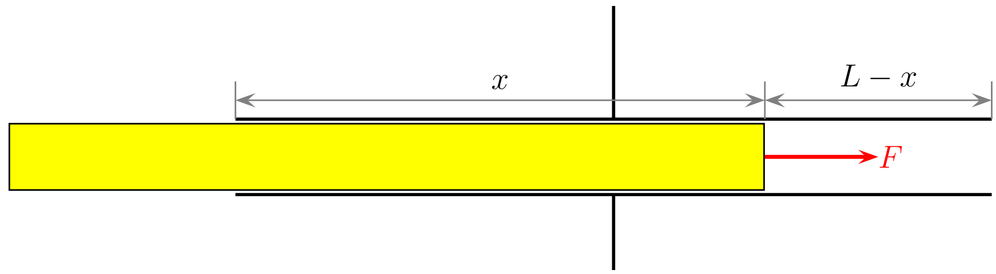
and hence

$$\delta W_{\rm mech} \;=\; \delta U \;-\; V\,\delta Q \;=\; -\frac{V^2}{2}\,\delta C. \tag{S.21}$$

BTW, the above calculation does not depend on the battery's voltage V being fixed. So the mechanical work involved in an infinitesimal change of capacitance is always given by eq. (S.21), regardless of whether the capacitor is hooked up to a fixed-voltage battery, or to more complicated power supply, or even charged and disconnected.

Now consider moving a piece of dielectric in or out from between the plates of a charged capacitor. Such movement changes the capacitance C, and according to eq. (S.21) this takes a mechanical work and hence mechanical forces. Specifically, there is a force pulling the dielectric inside the capacitor.

To see how this works, take a parallel plate capacitor, with rectangular plates of length L, width w, and distance d between the plates, $d \ll L, w$. The movable dielectric completely fills the gap between the plates and covers their whole width but not the length:



This capacitor can be thought as a parallel circuit of two capacitors, one vacuum-filled of length $L - z$ and the other dielectric-filled of length x, so altogether

$$C = \epsilon\epsilon_0 \times \frac{xw}{d} + \epsilon_0 \times \frac{(L-x)w}{d}. \tag{S.22}$$

Pulling the dielectric in through length δx changes the capacitance by

$$\delta C = (\epsilon - 1)\epsilon_0 \frac{w}{d} \times \delta x, \tag{S.23}$$

and according to eq. (S.21) this takes mechanical work *upon the capacitor*

$$W_{\text{mech}} = -\frac{V^2}{2} \times \frac{(\epsilon - 1)\epsilon_0 w}{d} \times \delta x. \tag{S.24}$$

The mechanical work done *by the capacitor* obtains by sign reversal, and equating this work to $F \times \delta x$, we find the force F pulling the dielectric inside the capacitor,

$$F = +\frac{V^2}{2} \times \frac{(\epsilon - 1)\epsilon_0 w}{d}. \tag{S.25}$$

Finally, let's turn the capacitor plates vertically and immerse them part-way into transformer oil. The oil is a dielectric, so the force (S.25) pulls it into the space between the plates, and that's what raises the oil level between the plates compared to its level outside. The height h through which the oil is raised follows from balancing the pulling force F against the weight of extra oil between the plates,

$$F = g\rho w d h, \tag{S.26}$$

and hence

$$h = \frac{F}{g\rho w d} = (\epsilon - 1)\epsilon_0 \frac{V^2}{2g\rho d^2}. \tag{4}$$

Note that the plates' width w cancels out from this formula.

For a numeric example, take transformer oil with dielectric constant $\epsilon = 1.34$ and mass density 882 kg/m^3, make the gap between the plates 1.00 mm wide, and charge the capacitor to $V = 3\underline{00}0$ Volts, then the oil in the gap will rise to $h = 4.6$ mm.

<u>Problem **4**</u>:

Let's start with the quasistatic magnetic field of the wire. At the moment the wire hangs along the z axis, we have

$$\mathbf{B}(s, \phi, z) = \frac{\mu_0 I}{2\pi s}\,\mathbf{n}_\phi\,, \tag{S.27}$$

or in Cartesian coordinates

$$\mathbf{B}(x, y, z) = \frac{\mu_0 I}{2\pi}\,\frac{(-y, +x, 0)}{x^2 + y^2}\,. \tag{S.28}$$

For the sake of definiteness, let the wire move in the x direction,

$$x_{\rm wire}(t) = vt, \quad y_{\rm wire}(t) = 0, \tag{S.29}$$

so the quasistatic magnetic field moving with the wire is

$$\mathbf{B}(x, y, z, t) = \frac{\mu_0 I}{2\pi}\,\frac{(-y, x - vt, 0)}{(x - vt)^2 + y^2}\,. \tag{S.30}$$

The electric field induced by this time-dependent magnetic field obeys the Induction Law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \tag{S.31}$$

but instead of solving this equation directly for the field (S.30), it's easier to find the time-dependent vector potential $\mathbf{A}(\mathbf{x},t)$ and then use

$$\mathbf{E} \;=\; -\frac{\partial}{\partial t}\mathbf{A} \;-\; \nabla\Phi. \tag{S.32}$$

Moreover, in the Coulomb gauge $\nabla\cdot\mathbf{A}=0$ for the potential, the $\Phi(\mathbf{x},t)$ here is the instantaneous Coulomb potential of the electric charges in the system. But the system at hand has no electric charges, in the Coulomb gauge $\Phi(\mathbf{x},t)\equiv 0$, hence

$$\mathbf{E} \;=\; -\frac{\partial}{\partial t}\mathbf{A}\,. \tag{S.33}$$

Quasistatically, the Coulomb-gauge vector potential for a current in a wire obtains as

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} \frac{I\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,. \tag{S.34}$$

For the wire hanging along the z axis, this integral evaluates to

$$\mathbf{A}(s,\phi,z) \;=\; \frac{\mu_0 I}{2\pi}\mathbf{n}_z\bigl(\text{const} \;-\; \log s\bigr), \tag{S.35}$$

or in Cartesian coordinates

$$\mathbf{A}(x,y,z) \;=\; \frac{\mu_0 I}{4\pi}\mathbf{n}_z\bigl(\text{const} \;-\; \log(x^2+y^2)\bigr). \tag{S.36}$$

Consequently, for a moving wire

$$\mathbf{A}(x,y,z,t) \;=\; \frac{\mu_0 I}{4\pi}\mathbf{n}_z\bigl(\text{const} \;-\; \log((x-vt)^2+y^2)\bigr), \tag{S.37}$$

and it is easy to check that

$$\nabla\cdot\mathbf{A} \;=\; 0, \quad \nabla\times\mathbf{A} \;=\; \mathbf{B}[\text{from eq. (S.30)}]. \tag{S.38}$$

Therefore, the induced electric field is

$$\begin{aligned}\mathbf{E}(x,y,z,t) \;&=\; -\frac{\partial\mathbf{A}}{\partial t} \;=\; +\frac{\mu_0 I}{4\pi}\mathbf{n}_z\left(\frac{\partial\log((x-vt)^2+y^2)}{\partial t}\right)\\ &=\; \frac{\mu_0 I}{4\pi}\,\frac{-2v(x-vt)}{(x-vt)^2+y^2}\,\mathbf{n}_z\\ &=\; -\mathbf{v}\times\mathbf{B}(x,y,z,t).\end{aligned} \tag{S.39}$$

PS: In fact, for any magnet or electromagnet moving as a rigid body, the induced electric field is

$$\mathbf{E}(\mathbf{x},t) \;=\; -\mathbf{v}\times\mathbf{B}(\mathbf{x},t). \tag{S.40}$$

Indeed, the magnetic field of a magnet moving as a rigid body moves with the magnet, thus

$$\mathbf{B}(\mathbf{x},t) \;=\; \mathbf{B}_0(\mathbf{x}-\mathbf{v}t). \tag{S.41}$$

Consequently, the time-derivative of this field at a fixed location $\mathbf{x}$ is related to its space derivatives as

$$\frac{\partial\mathbf{B}}{\partial t} \;=\; -(\mathbf{v}\cdot\nabla)\mathbf{B}. \tag{S.42}$$

Furthermore, by the double cross product formula

$$\nabla\times(\mathbf{v}\times\mathbf{B}) \;=\; (\nabla\cdot\mathbf{B})\mathbf{v} \;-\; (\mathbf{v}\cdot\nabla)\mathbf{B} \;=\; 0 \;-\; (\mathbf{v}\cdot\nabla)\mathbf{B} \tag{S.43}$$

where the second equality stems from $\nabla\cdot\mathbf{B}=0$, hence

$$\frac{\partial\mathbf{B}}{\partial t} \;=\; -(\mathbf{v}\cdot\nabla)\mathbf{B} \;=\; +\nabla\times(\mathbf{v}\times\mathbf{B}). \tag{S.44}$$

Consequently, according to Faraday's Induction Law

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t} \;=\; -\nabla\times(\mathbf{v}\times\mathbf{B}), \tag{S.45}$$

hence

$$\nabla\times(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \;=\; 0, \tag{S.46}$$

and therefore

$$\mathbf{E} \;+\; \mathbf{v}\times\mathbf{B} \;=\; -\nabla\Phi \tag{S.47}$$

for some scalar potential Φ. In the absence of electric charges we may set $\Phi(\mathbf{x},t)\equiv 0$, and therefore

$$\mathbf{E}(\mathbf{x},t) \;=\; -\mathbf{v}\times\mathbf{B}(\mathbf{x},t). \tag{S.48}$$

Quod erat demonstrandum.