Problem **1**(a):

Eqs. (3) follow from Maxwell's curl equations,

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t} \;=\; -\mu_0\frac{\partial\mathbf{H}}{\partial t}\,, \qquad \nabla\times\mathbf{H} \;=\; \frac{\partial\mathbf{D}}{\partial t}\,, \tag{S.1}$$

assuming zero conduction current $\mathbf{J}_c$. For a wave where all fields depend on $\mathbf{x}$ and t as $\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$, the derivatives become

$$\nabla \;\to\; i\mathbf{k}, \qquad \frac{\partial}{\partial t} \;\to\; -i\omega, \tag{S.2}$$

so the curl equations (S.1) become

$$\mathbf{k}\times\mathbf{E} \;=\; +\mu_0\omega\mathbf{H}, \qquad \mathbf{k}\times\mathbf{H} \;=\; -\omega\mathbf{D}, \tag{S.3}$$

hence eqs. (3).

As to the transversality, Maxwell divergence equations

$$\nabla\cdot\mathbf{B} \;=\; 0, \qquad \nabla\cdot\mathbf{D} \;=\; \rho_{\text{free}} \;=\; 0, \tag{S.4}$$

immediately imply

$$\mathbf{k}\cdot\mathbf{B} \;=\; 0 \quad\Longrightarrow\quad \mathbf{k}\cdot\mathbf{H} \;=\; 0 \quad\text{and}\quad \mathbf{k}\cdot\mathbf{D} \;=\; 0. \tag{S.5}$$

Alternatively, eqs. (3) bring both $\mathbf{D}$ and $\mathbf{H}$ vectors to the form of $\mathbf{k}\times$ some vector, so both $\mathbf{H}$ and $\mathbf{D}$ must be perpendicular to the wave vector $\mathbf{k}$.

On the other hand, the electric tension field $\mathbf{E}$ — as opposed to the electric displacement field $\mathbf{D}$ — is not directly related to any curl, and there is no Gauss law $\nabla\cdot\mathbf{E}=0$, only

$\nabla\cdot\mathbf{D}=0$. Consequently, the **E** field does not have to be transverse WRT the wave vector **k**. Instead, the **E** vector is related to the **D** vector by eq. (1), or equivalently

$$E_i \;=\; \left(\epsilon_0^{-1}\epsilon^{-1}\right)_{ij} D_j\,, \tag{S.6}$$

so unless the **D** vector happens to be parallel to one of the principal axes[⋆] of the ϵ tensor, the **E** vector has a different direction from the **D**. Thus, while the electric displacement field **D** must be transverse to the wave vector **k**, the electric tension field **E** generally has both transverse and longitudinal components.

Problem **1**(b):

The motion of the electromagnetic energy is governed by the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$; for a harmonic EM wave this vector time-averages to

$$\langle\mathbf{S}\rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\vec{\mathcal{E}}\times\vec{\mathcal{H}}^*\big), \tag{S.7}$$

and its direction is the direction in which the wave's energy moves.

In light of the second eq. (3),

$$\vec{\mathcal{E}}\times\vec{\mathcal{H}}^* \;=\; \frac{1}{\omega\mu_0}\,\vec{\mathcal{E}}\times\big(\mathbf{k}\times\vec{\mathcal{E}}^*\big) \;=\; \frac{1}{\omega\mu_0}\Big(|\vec{\mathcal{E}}|^2\mathbf{k}\,-\,(\mathbf{k}\cdot\vec{\mathcal{E}}^*)\vec{\mathcal{E}}\Big), \tag{S.8}$$

where the first term inside $(\cdots)$ is purely longitudinal, but the second term has both longitudinal and transverse components when $\vec{\mathcal{E}}\not\perp\mathbf{k}$. Thus, the Poynting vector generally has both longitudinal and transverse components, specifically

$$\begin{aligned}\langle\mathbf{S}\rangle_\ell \;&=\; \frac{k}{2\omega\mu_0}\Big(|\vec{\mathcal{E}}|^2\,-\,\mathcal{E}_\ell^*\mathcal{E}_\ell \;=\; |\vec{\mathcal{E}}_t|^2\Big),\\ \langle\mathbf{S}\rangle_t \;&=\; \frac{k}{2\omega\mu_0}\Big(-\,\mathrm{Re}\big(\mathcal{E}_\ell^*\vec{\mathcal{E}}_t\big)\Big).\end{aligned} \tag{S.9}$$

In particular, for a linear polarization of the EM wave — meaning, a real amplitude vector

---

[⋆] A real symmetric 2-index tensor can be viewed as a real symmetric matrix. The directions of this matrix's eigenvectors are called the *principal axes* of the tensor.

$\vec{\mathcal{E}}$ up to an overall phase, — we have

$$\frac{|\langle\mathbf{S}\rangle_t|}{\langle\mathbf{S}\rangle_\ell} = \frac{|\mathcal{E}_\ell|}{|\vec{\mathcal{E}}_t|}. \tag{S.10}$$

Consequently, *the angle between the direction of the energy's motion and the wave vector* **k** *equals to the angle between the electric amplitude* $\vec{\mathbf{E}}$ *and the plane* ⊥ *to the wave vector* **k**.

Problem **1**(c):

Let's start with the first eq. (3). In components, its LHS becomes

$$\begin{aligned} \big[-\mathbf{k}\times(\mathbf{k}\times\mathbf{E})\big] &= -\epsilon_{ik\ell}k_k\epsilon_{\ell mj}k_mE_j = -(\delta_{im}\delta_{kj}-\delta_{ij}\delta_{km})k_kk_mE_j \\ &= -(k_ik_j-\delta_{ij}\mathbf{k}^2)E_j = +\mathbf{k}^2(\delta_{ij}-\hat{k}_i\hat{k}_j)E_j\,, \end{aligned} \tag{S.11}$$

while on the RHS

$$\omega^2\mu_0D_i = \omega^2\mu_0\epsilon_0\epsilon_{ij}E_j = \frac{\omega^2}{c^2}\,\epsilon_{ij}E_j\,, \tag{S.12}$$

hence altogether

$$\frac{\omega^2}{c^2}\,\epsilon_{ij}E_j = \mathbf{k}^2\big(\delta_{ij}-\hat{k}_i\hat{k}_j\big)E_j \tag{S.13}$$

Obviously, the electric amplitude vector $\vec{\mathcal{E}}$ must also obey this equation, thus

$$\left[\frac{\omega^2}{c^2}\,\epsilon_{ij} - \mathbf{k}^2\big(\delta_{ij}-\hat{k}_i\hat{k}_j\big)\right]\mathcal{E}_j = 0. \tag{S.14}$$

Finally, dividing this equation by ω^2/c^2 and identifying $\mathbf{k}^2c^2/\omega^2$ as n^2, we obtain

$$\Big(\epsilon_{ij} - n^2\big(\delta_{ij}-\hat{k}_i\hat{k}_j\big)\Big)\,\mathcal{E}_j = 0. \tag{4}$$

This equation has a form of a generalized eigenvalue problem. In particular, it has a non-zero solution for the $\vec{\mathcal{E}}$ when and only when the matrix on the LHS has a zero determinant, thus n^2 must obey

$$\chi(n^2) \overset{\text{def}}{=} \det\Big(\epsilon_{ij} - n^2\big(\delta_{ij}-\hat{k}_i\hat{k}_j\big)\Big) = 0. \tag{5}$$

Formally, this determinant is a polynomial of n^2 of degree = dimension of the matrix, which is 3 in a 3D space. But we shall see in the next part that the coefficient of $(n^2)^3$ in this polynomial happens to vanish, so $\chi(n^2)$ is actually a quadratic polynomial. And in later parts we shall see that both roots n_1^2 and n_2^2 of this quadratic polynomial are real and positive, thus two values of the refraction index for the two independent polarizations of the wave moving in a given direction $\hat{\mathbf{k}}$.

Problem **1**(d):

The three principal axis of the permittivity tensor ϵ_{ij} are $\perp$ to each other, so let's use them for the coordinate axes (x_1, x_2, x_3). In this coordinate system, the matrix of the permittivity tensor is diagonal,

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \tag{S.15}$$

hence the determinant in eq. (5) is

$$\chi(n^2) = \det \begin{pmatrix} \epsilon_1 - n^2(1-\hat{k}_1^2) & n^2\hat{k}_1\hat{k}_2 & n^2\hat{k}_1\hat{k}_3 \\ n^2\hat{k}_2\hat{k}_1 & \epsilon_2 - n^2(1-\hat{k}_2^2) & n^2\hat{k}_2\hat{k}_3 \\ n^2\hat{k}_3\hat{k}_1 & n^2\hat{k}_3\hat{k}_2 & \epsilon_3 - n^2(1-\hat{k}_3^2) \end{pmatrix}. \tag{S.16}$$

Every matrix element here is a linear polynomial in n^2, so expanding the whole determinant into powers of n^2, we obtain

$$\begin{aligned} \chi(n^2) = {} & \det \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \\ & + n^2 \times \left(\det \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & -1+\hat{k}_3^2 \end{pmatrix} + \text{two similar terms} \right) \\ & + n^4 \times \left(\det \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & -1+\hat{k}_2^2 & \hat{k}_2\hat{k}_3 \\ 0 & \hat{k}_3\hat{k}_2 & -1+\hat{k}_2^2 \end{pmatrix} + \text{two similar terms} \right) \\ & + n^6 \times \det \begin{pmatrix} -1+\hat{k}_1^2 & \hat{k}_1\hat{k}_2 & \hat{k}_1\hat{k}_3 \\ \hat{k}_2\hat{k}_1 & -1+\hat{k}_2^2 & \hat{k}_2\hat{k}_3 \\ \hat{k}_3\hat{k}_1 & \hat{k}_3\hat{k}_2 & -1+\hat{k}_3^2 \end{pmatrix} \end{aligned} \tag{S.17}$$

On the top line of this formula

$$\det\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \;=\; \epsilon_1\epsilon_2\epsilon_3\,, \tag{S.18}$$

on the second line

$$\det\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & -1+\hat{k}_3^2 \end{pmatrix} \;=\; \epsilon_1\epsilon_2(-1+\hat{k}_3^2) \;=\; -\epsilon_1\epsilon_2(\hat{k}_1^2+\hat{k}_2^2), \tag{S.19}$$

and likewise for the two similar terms. On the third line of eq. (S.17),

$$\begin{aligned} \det\begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & -1+\hat{k}_2^2 & \hat{k}_2\hat{k}_3 \\ 0 & \hat{k}_3\hat{k}_2 & -1+\hat{k}_2^2 \end{pmatrix} \;&=\; \epsilon_1\times\det\begin{pmatrix} -1+\hat{k}_2^2 & \hat{k}_2\hat{k}_3 \\ \hat{k}_3\hat{k}_2 & -1+\hat{k}_3^3 \end{pmatrix} \\ &=\; \epsilon_1\times\Big((-1+\hat{k}_2^2)(-1+\hat{k}_3^2)\;-\;\hat{k}_2^2\hat{k}_3^2\Big) \\ &=\; \epsilon_1\times\Big(1-\hat{k}_2^2-\hat{k}_3^2\Big) \;=\; \epsilon_1\times\hat{k}_1^2, \end{aligned} \tag{S.20}$$

and likewise for the two similar terms. Finally, the determinant on the last line of eq. (S.17) vanishes:

$$\begin{aligned} &\det\begin{pmatrix} -1+\hat{k}_1^2 & \hat{k}_1\hat{k}_2 & \hat{k}_1\hat{k}_3 \\ \hat{k}_2\hat{k}_1 & -1+\hat{k}_2^2 & \hat{k}_2\hat{k}_3 \\ \hat{k}_3\hat{k}_1 & \hat{k}_3\hat{k}_2 & -1+\hat{k}_3^2 \end{pmatrix} \;= \\ &=\; (-1+\hat{k}_1^2)(-1+\hat{k}_2^2)(-1+\hat{k}_3)^2 \;+\; 2\times\hat{k}_1^2\hat{k}_2^2\hat{k}_3^3 \\ &\qquad -\;(-1+\hat{k}_1^2)\times\hat{k}_2^2\hat{k}_3^2 \;-\;\text{two similar terms} \\ &=\; -1\;+\;(\hat{k}_1^2+\hat{k}_2^2+\hat{k}_3^2)\;-\;(\hat{k}_1^2\hat{k}_2^2+\hat{k}_1^2\hat{k}_3^2+\hat{k}_2^2\hat{k}_3^2)\;+\;\hat{k}_1^2\hat{k}_2^2\hat{k}_3^2 \\ &\qquad +\;2\times\hat{k}_1^2\hat{k}_2^2\hat{k}_3^3\;+\;(\hat{k}_1^2\hat{k}_2^2+\hat{k}_1^2\hat{k}_3^2+\hat{k}_2^2\hat{k}_3^2)\;-\;3\times\hat{k}_1^2\hat{k}_2^2\hat{k}_3^3 \\ &=\; -1\;+\;(\hat{k}_1^2+\hat{k}_2^2+\hat{k}_3^2)\;=\;0. \end{aligned} \tag{S.21}$$

Altogether,

$$\begin{aligned}
\chi(n^2) \;&=\; \epsilon_1\epsilon_2\epsilon_3 \;-\; n^2\times\Big(\epsilon_1\epsilon_2(\hat{k}_1^2+\hat{k}_2^2) \,+\, \epsilon_1\epsilon_3(\hat{k}_1^2+\hat{k}_3^2) \,+\, \epsilon_2\epsilon_3(\hat{k}_2^2+\hat{k}_3^2)\Big)\\
&\qquad +\; n^4\times\Big(\epsilon_1\hat{k}_1^2 \,+\, \epsilon_2\hat{k}_2^2 \,+\, \epsilon_3\hat{k}_3^2\Big)\\
&=\; (1=\hat{k}_1^2+\hat{k}_2^2+\hat{k}_3^2)\times\epsilon_1\epsilon_2\epsilon_3\\
&\qquad -\; n^2\times\Big(\hat{k}_1^2\epsilon_1(\epsilon_2+\epsilon_3) \,+\, \hat{k}_2^2\epsilon_2(\epsilon_1+\epsilon_3) \,+\, \hat{k}_3^2\epsilon_3(\epsilon_1+\epsilon_2)\Big)\\
&\qquad +\; n^4\times\Big(\hat{k}_1^2\epsilon_1 \,+\, \hat{k}_2^2\epsilon_2 \,+\, \hat{k}_3^2\epsilon_3\Big)\\
&=\; \hat{k}_1^2\epsilon_1\times\Big(\epsilon_2\epsilon_3 \,-\, (\epsilon_2+\epsilon_3)n^2 \,+\, n^4\Big) \;+\; \text{two similar terms}\\
&=\; \hat{k}_1^2\epsilon_1(n^2-\epsilon_2)(n^2-\epsilon_3) + \;\text{two similar terms}
\end{aligned} \tag{S.22}$$

or in a more compact form

$$\chi(n^2) \;=\; \sum_{i=1}^{3}\hat{k}_i^2\epsilon_i\times\prod_{j\neq i}(n^2-\epsilon_i). \tag{6}$$

Quod erat demonstrandum.

Problem **1**(e):

$\chi(n^2)$ is a quadratic polynomial of n^2, so it has at most two real roots. To bracket the locations of these roots, we note that at the 3 points — namely $n^2=\epsilon_1^2$, $n^2=\epsilon_2$, and $n^2=\epsilon_3$, — $\chi(n^2)$ has alternating signs. Specifically, for $\epsilon_1>\epsilon_2>\epsilon_3>0$,

$$\begin{aligned}
P(n^2=\epsilon_1) \;&=\; \hat{k}_1^2\epsilon_1(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3) \;\geq\; 0,\\
P(n^2=\epsilon_2) \;&=\; \hat{k}_2^2\epsilon_2(\epsilon_2-\epsilon_1)(\epsilon_2-\epsilon_2) \;\leq\; 0,\\
P(n^2=\epsilon_3) \;&=\; \hat{k}_3^2\epsilon_3(\epsilon_3-\epsilon_1)(\epsilon_3-\epsilon_2) \;\geq\; 0,
\end{aligned} \tag{S.23}$$

where each inequality is strict when the respective $\hat{k}_i^2$ does not vanish. Consequently, $\chi(n^2)$ must vanish for some value of n^2 between ϵ_1 and ϵ_2 and also for another value of n^2 between ϵ_2 and ϵ_3, thus inequalities (7) for the two refraction coefficients, with all the inequalities becoming strict when all 3 of $\hat{k}_1^2, \hat{k}_2^2, \hat{k}_3^2$ are non-zero.

As to the Fresnel equation (8), it follows from

$$\chi(n^2) \;=\; \prod_{i=1}^{3}(n^2-\epsilon_i)\times\sum_{i=1}^{3}\frac{\hat{k}_i^2\epsilon_i}{n^2-\epsilon_i}\,. \tag{S.24}$$

When the three eigenvalues $\epsilon_1, \epsilon_2, \epsilon_3$ are different from each other and $\mathbf{k}$ is not parallel to any of the principal axes — thus all three $\hat{k}_i^2 > 0$, — all the inequalities (S.23) become strict, which means that

$$\prod_{i=1}^{3}(n^2-\epsilon_i) \quad \text{does not vanish for } n^2=n_1^2 \text{ or } n^2=n_2^2. \tag{S.25}$$

Consequently, in eq. (S.24) it's the second factor which vanishes at either root of the $\chi(n^2)$,

$$\sum_{i=1}^{3}\frac{\hat{k}_i^2\epsilon_i}{n^2-\epsilon_i} \;=\; 0 \quad \text{for } n^2=n_1^2 \text{ or } n^2=n_2^2\,, \tag{S.26}$$

hence the Fresnel equation (8).

Problem **1**(f):

A uniaxial anisotropic material with $\epsilon_1=\epsilon_2\neq\epsilon_3$ has a rotational symmetry around its optical axis. For such a material, using the principal axes of the ϵ tensor for the 3 coordinate axes means using the optical axis for the x_3 axis, while the x_1 and x_2 axes can be any two axes we like as long as they are ⊥ to the x_3 axis and to each other. In any such coordinate system, the ϵ tensor has the same diagonal matrix

$$\epsilon \;=\; \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}. \tag{S.27}$$

Also, in any such coordinate system, the wave moving parallel to the optical axis means

$$\hat{k}_1 \;=\; \hat{k}_2 \;=\; 0, \quad \hat{k}_3 \;=\; \pm 1. \tag{S.28}$$

Consequently, eq. (4) for the refraction index and the polarization vector becomes

$$\begin{pmatrix} \epsilon_1 - n^2 & 0 & 0 \\ 0 & \epsilon_1 - n^2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix} = 0. \tag{S.29}$$

This generalized eigenvalue problem has two degenerate solutions, namely

$$n^2 = \epsilon_1 , \qquad \vec{\mathcal{E}} = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} . \tag{S.30}$$

In other words, the polarization vector $\vec{\mathcal{E}}$ may point in any direction perpendicular to the optical axis and hence to the wave direction $\hat{\mathbf{k}}$, and for any such polarization we have the same refraction index $n = \sqrt{\epsilon_1}$. *Quod erat demonstrandum.*

Problem **1**(g):

For $\epsilon_1 = \epsilon_2$, every term in the sum (6) has a factor of $(n^2 - \epsilon_1)$, thus

$$\begin{aligned} \chi(n^2) &= \hat{k}_1^2 \epsilon_1 (n^2 - \epsilon_1)(n^2 - \epsilon_3) + \hat{k}_2^2 \epsilon_1 (n^2 - \epsilon_1)(n^2 - \epsilon_3) + \hat{k}_3^2 \epsilon_3 (n^2 - \epsilon_1)^2 \\ &= (n^2 - \epsilon_1) \times \Big[(\hat{k}_1^2 + \hat{k}_2^2) \epsilon_1 (n^2 - \epsilon_3) + \hat{k}_3^3 \epsilon_3 (n^2 - \epsilon_1) \Big] . \end{aligned} \tag{S.31}$$

In terms of the angle θ between the wave vector and the optical axis,

$$\hat{k}_3^2 = \cos^2\theta, \qquad \hat{k}_1^2 + \hat{k}_2^2 = \sin^2\theta, \tag{S.32}$$

hence

$$\begin{aligned} \chi(n^2) &= (n^2 - \epsilon_1) \times \Big[\sin^2\theta \epsilon_1 (n^2 - \epsilon_2) + \cos^2\theta \epsilon_3 (n^2 - \epsilon_1) \Big] \\ &= (n^2 - \epsilon_1) \times \Big[(\sin^2\theta \epsilon_1 + \cos^2\theta \epsilon_2) n^2 - \epsilon_1 \epsilon_2 \Big] . \end{aligned} \tag{S.33}$$

The two roots of this quadratic polynomial gives us the refraction indices^2 of the two independent polarizations:

$$n_1^2 \;=\; \epsilon_1 \tag{S.34}$$

and

$$n_2^2 \;=\; \frac{\epsilon_1\epsilon_2}{\sin^2\theta\epsilon_1+\cos^2\theta\epsilon_2}\,,. \tag{S.35}$$

or equivalently

$$\frac{1}{n_2^2} \;=\; \frac{\sin^2\theta}{\epsilon_3} \;+\; \frac{\cos^2\theta}{\epsilon_1}\,. \tag{9}$$

Now consider the polarization vectors $\vec{\mathcal{E}}_1$ and $\vec{\mathcal{E}}_2$ corresponding to the waves with refraction indices n_1 and n_2. Thanks to the rotational symmetry around the optical axis (which we use as the x_3 axis), we may chose the x_1 and x_2 axes such that the wave vector $\mathbf{k}$ lies in the (x_1, x_3) plane. In this coordinate system

$$\hat{k}_1 \;=\; \sin\theta, \quad \hat{k}_2 \;=\; 0, \hat{k}_3 \;=\; \cos\theta, \tag{S.36}$$

so eq. (4) becomes

$$\begin{pmatrix} \epsilon_1 - n^2(1-\sin^2\theta) & 0 & n^2\sin\theta\cos\theta \\ 0 & \epsilon_1 - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & \epsilon_3 - n^2(1-\cos^2\theta) \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix} \;=\; 0. \tag{S.37}$$

The 3×3 matrix in this equation is block-diagonal: it has a 2×2 block for the x_1 and x_3 directions — which is the plane spanning the wave direction $\hat{\mathbf{k}}$ and the optical axis, — and a separate 1×1 block for the x_2 direction $\perp$ to that plane. Consequently, eq. (S.37) splits into 2 separate equations of the two blocks:

$$\begin{pmatrix} \epsilon_1 - n^2\cos^2\theta & n^2\sin\theta\cos\theta \\ n^2\sin\theta\cos\theta & \epsilon_3 - n^2\sin^2\theta \end{pmatrix} \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_3 \end{pmatrix} \;=\; 0, \tag{S.38}$$

$$\Big(\epsilon_1 - n^2\Big)\Big(\mathcal{E}_2\Big) \;=\; 0. \tag{S.39}$$

For $n^2 = n_1^2 = \epsilon_1$, the second equation here allows for $\mathcal{E}_2 \neq 0$ while the first equation keeps $\mathcal{E}_1 = \mathcal{E}_3 = 0$, so the polarization vector is $\vec{\mathcal{E}}^{(1)} = (0, \mathcal{E}, 0)$, normal to both the optical axis and the wave direction $\hat{\mathbf{k}}$. This obviously is the $(\perp)$ polarization.

On the other hand, for the other eigenvalue $n^2 = n_2^2$, we have eq. (S.39) requiring $\mathcal{E}_2 = 0$ while eq. (S.38) has a non-trivial solution in the (x_1, x_3) plane. Specifically, for $n^2 = n_2^2$ as in eq. (9),

$$\begin{aligned}
\epsilon_1 \;-\; n^2\cos^2\theta \;&=\; n_2^2\left(\frac{\epsilon_1}{n_2^2} \;-\; \cos^2\theta\right)\\
&=\; n_2^2\left(\epsilon_1\frac{\sin^2\theta}{\epsilon_3} \;+\; \epsilon_1\frac{\cos^2\theta}{\epsilon_1} \;-\; \cos^2\theta\right)\\
&=\; n_2^2\times\frac{\epsilon_1}{\epsilon_3}\sin^2\theta,\\
\epsilon_3 \;-\; n^2\sin^2\theta \;&=\; n_2^2\left(\frac{\epsilon_3}{n_2^2} \;-\; \sin^2\theta\right)\\
&=\; n_2^2\left(\epsilon_3\frac{\sin^2\theta}{\epsilon_3} \;+\; \epsilon_3\frac{\cos^2\theta}{\epsilon_1} \;-\; \sin^2\theta\right)\\
&=\; n_2^2\times\frac{\epsilon_3}{\epsilon_1}\cos^2\theta,
\end{aligned} \tag{S.40}$$

so eq. (S.38) becomes

$$n_2^2\begin{pmatrix} (\epsilon_1/\epsilon_3)\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & (\epsilon_3/\epsilon_1)\cos^2\theta \end{pmatrix}\begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_3\end{pmatrix} \;=\; 0, \tag{S.41}$$

which has a non-trivial solution with

$$\frac{\mathcal{E}_1}{\mathcal{E}_3} \;=\; -\frac{\epsilon_3\cos\theta}{\epsilon_1\sin\theta}\,. \tag{S.42}$$

Altogether, we have polarization vector

$$\vec{\mathcal{E}}^{(2)} \;=\; \frac{\mathcal{E}}{\sqrt{\epsilon_1^2\sin^2\theta + \epsilon_2^2\cos^2\theta}}\big(-\epsilon_3\cos\theta, 0, +\epsilon_1\sin\theta\big). \tag{S.43}$$

This vector lies in the same (x_1, x_3) plane as the optical axis and the wave's direction $\hat{\mathbf{k}}$, so this is the in-plane polarization ($\|$). *Quod erat demonstrandum.*

Problem $\mathbf{1}$(h):

For any plane wave in a non-magnetic material

$$\vec{\mathcal{H}} \;=\; \frac{\mathbf{k}}{\omega\mu_0}\times\vec{\mathcal{E}} \tag{S.44}$$

while the energy flows in the direction of the (time-averaged) Poynting vector

$$\langle\mathbf{S}\rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathcal{E}^*\times\vec{\mathcal{H}}\big). \tag{S.45}$$

For the ($\perp$) polarization in a uniaxial material of a wave in the direction $\hat{\mathbf{k}}=(\sin\theta,0,\cos\theta)$, we have

$$\vec{\mathcal{E}} \;=\; (0,\mathcal{E},0), \tag{S.46}$$

$$\vec{\mathcal{H}} \;=\; \frac{|\mathbf{k}|\mathcal{E}}{\omega\mu_0}\big(-\cos\theta,0,\sin\theta\big), \tag{S.47}$$

and hence

$$\langle\mathbf{S}\rangle \;=\; \frac{|\mathbf{k}|\,|\mathcal{E}|^2}{2\omega\mu_0}\big(\sin\theta,0,\cos\theta\big), \tag{S.48}$$

so the energy flows in the same direction as the wave vector $\mathbf{k}$.

On the other hand, for the in-plane polarization we have

$$\vec{\mathcal{E}} \;=\; \frac{\mathcal{E}}{\sqrt{\epsilon_1^2\sin^2\theta+\epsilon_2^2\cos^2\theta}}\big(-\epsilon_3\cos\theta,0,+\epsilon_1\sin\theta\big) \tag{S.43}$$

$$\;=\; \mathcal{E}\big(-\cos\alpha,0,+\sin\alpha\big) \tag{S.49}$$

$$\text{for}\quad \alpha \;=\; \arctan\left(\frac{\epsilon_1}{\epsilon_3}\,\tan\alpha\right), \tag{S.50}$$

hence

$$\vec{\mathcal{H}} \;=\; \frac{|\mathbf{k}|\mathcal{E}}{\omega\mu_0}\big(0,-\cos(\alpha-\theta),0\big), \tag{S.51}$$

and therefore

$$\langle \mathbf{S} \rangle \;=\; \frac{|\mathbf{k}|\,|\mathcal{E}|^2 \cos(\alpha-\theta)}{2\omega\mu_0}\,\bigl(\sin\alpha, 0, \cos\alpha\bigr). \tag{S.52}$$

This time, the direction of the Poynting vector is different from the wave direction $\mathbf{k}$. Specifically, both directions and the optical axis lie in the same (x_1, x_3) plane, but within that plane they differ by the angle

$$\Delta\phi \;=\; \alpha \;-\; \theta \;=\; \arctan\left(\frac{\epsilon_1}{\epsilon_3}\tan\theta\right) \;-\; \theta. \tag{S.53}$$

For your information, this angular difference disappears for $\theta = 0$ or $\theta = 90^\circ$, and reaches its maximum

$$\Delta\phi_{\max} \;=\; \arcsin\frac{|\epsilon_3 - \epsilon_1|}{\epsilon_3 + \epsilon_1} \tag{S.54}$$

for

$$\theta \;=\; \arctan\left(\sqrt{\frac{\epsilon_3}{\epsilon_1}}\right) \quad\Longrightarrow\quad \alpha \;=\; \arctan\left(\sqrt{\frac{\epsilon_1}{\epsilon_3}}\right). \tag{S.55}$$

However, this part of the angular calculation was not a part of your homework assignment.

Problem **2**(a):

A free electron in the constant magnetic field $\mathbf{B}$ and the electric field $\mathbf{E}$ of the wave moves according to

$$m\mathbf{a} \;+\; e\mathbf{v}\times\mathbf{B} \;=\; -e\mathbf{E}. \tag{S.56}$$

For a harmonic wave $\mathbf{E} = e^{-i\omega t}\mathbf{E}^0$ the electron also moves harmonically, $\mathbf{x}(t) = e^{-i\omega t}\mathbf{x}^0$, with the amplitude such that

$$-\omega^2 m\mathbf{x}^0 \;-\; i\omega\mathbf{x}^0\times\mathbf{B} \;=\; -e\mathbf{E}^0\,. \tag{S.57}$$

In components,

$$-m\omega^2 x_0^i \;-\; i\omega\epsilon_{ijk}x_0^j B_k \;=\; -eE_i^0\,, \tag{S.58}$$

or

$$\bigl(\omega^2\delta_{ij} + i\omega\Omega\epsilon_{ijk}\hat{b}_k\bigr)x_j^0 \;=\; \frac{e}{m}E_i^0 \tag{S.59}$$

where $\Omega = (eB/m)$ is the electron's cyclotron frequency in the magnetic field, and $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$

is the unit vector in the direction of the magnetic field. To solve the equation (S.58), we need the inverse of the Hermitian matrix

$$\mathcal{M}_{ij} \;=\; \omega^2\delta_{ij} \;+\; i\omega\Omega\epsilon_{ijk}\hat{b}_k\,, \tag{S.60}$$

and there are two simple methods of calculating this inverse:

<u>First method</u>:

Matrix $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ are tensors depending on a single vector $\hat{\mathbf{b}}$, so by the rotational symmetry

$$\left(\mathcal{M}^{-1}\right)_{ij} \;=\; \alpha\delta_{ij} \;+\; \beta\hat{b}_i\hat{b}_j \;+\; i\gamma\epsilon_{ijk}\hat{b}_k \tag{S.61}$$

for some scalars α, β, γ. To find these scalars, we simply demand that

$$\mathcal{M}\times\mathcal{M}^{-1} \;=\; 1 \quad\Longleftrightarrow\quad \mathcal{M}_{ij}\left(\mathcal{M}^{-1}\right)_{jk} \;=\; \delta_{jk}\,. \tag{S.62}$$

The explicit calculation of the LHS here yields

$$\begin{aligned}\mathcal{M}_{ij}\left(\mathcal{M}^{-1}\right)_{jk} \;&=\; \left(\omega^2\delta_{ij} + i\omega\Omega\epsilon_{ij\ell}\hat{b}_\ell\right)\times\left(\alpha\delta_{jk} + \beta\hat{b}_j\hat{b}_k + i\gamma\epsilon_{jkm}\hat{b}_m\right)\\ &=\; \omega^2\alpha\delta_{ik} \;+\; i\Omega\omega\alpha\epsilon_{ik\ell}\hat{b}_\ell \;+\; \omega^2\beta\hat{b}_i\hat{b}_k \;+\; i\Omega\omega\beta\times 0\\ &\qquad +\; i\omega^2\gamma\epsilon_{ikm}\hat{b}_m \;-\; \Omega\omega\gamma\left(\epsilon_{ij\ell}\epsilon_{jkm}\hat{b}_\ell\hat{b}_m \;=\; \hat{b}_i\hat{b}_k - \delta_{ik}\right)\\ &=\; (\omega^2\alpha + \Omega\omega\gamma)\delta_{ik} \;+\; (\omega^2\beta - \Omega\omega\gamma)\hat{b}_i\hat{b}_k \;+\; i(\omega^2\gamma + \Omega\omega\alpha)\epsilon_{ik\ell}\hat{b}_\ell\,,\end{aligned} \tag{S.63}$$

which calls for

$$\begin{aligned}\omega^2\times\alpha \;+\; \Omega\omega\times\gamma \;&=\; 1,\\ \omega^2\times\beta \;-\; \Omega\omega\times\gamma \;&=\; 0,\\ \omega^2\times\gamma \;+\; \Omega\omega\times\alpha \;&=\; 0.\end{aligned} \tag{S.64}$$

Solving these equations, we get

$$\alpha \;=\; \frac{1}{\omega^2 - \Omega^2}\,, \qquad \beta \;=\; -\frac{\Omega^2}{\omega^2(\omega^2 - \Omega^2)}\,, \qquad \gamma \;=\; -\frac{\Omega}{\omega(\omega^2 - \Omega^2)}\,, \tag{S.65}$$

and therefore

$$\left(\mathcal{M}^{-1}\right)_{ij} \;=\; \frac{1}{\omega^2(\omega^2 - \Omega^2)}\Big(\omega^2\delta_{ij} \;-\; \Omega^2\hat{b}_i\hat{b}_j \;-\; i\omega\Omega\epsilon_{ijk}\hat{b}_k\Big). \tag{S.66}$$

Second Method:

Let's use a coordinate system with the z axis pointing in the magnetic field direction, thus $\hat{\mathbf{b}} = (0, 0, 1)$. In this coordinate frame,

$$\mathcal{M} = \begin{pmatrix} \omega^2 & +i\Omega\omega & 0 \\ -i\Omega\omega & \omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{S.67}$$

and the inverse of this matrix is

$$\mathcal{M}^{-1} = \begin{pmatrix} \frac{1}{\omega^2-\Omega^2} & -i\frac{\Omega}{\omega(\omega^2-\Omega^2)} & 0 \\ +i\frac{\Omega}{\omega(\omega^2-\Omega^2)} & \frac{1}{\omega^2-\Omega^2} & 0 \\ 0 & 0 & \frac{1}{\omega^2} \end{pmatrix}. \tag{S.68}$$

Using

$$\frac{1}{\omega^2} = \frac{1}{\omega^2 - \Omega^2} - \frac{\Omega^2}{\omega^2(\omega^2 - \Omega^2)}, \tag{S.69}$$

we can bring this matrix to the form

$$\mathcal{M}^{-1} = \frac{1}{\omega^2(\omega^2 - \Omega^2)} \left[\omega^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \Omega^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \Omega\omega \begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right], \tag{S.70}$$

which in index notations becomes

$$\left(\mathcal{M}^{-1}\right)_{ij} = \frac{1}{\omega^2(\omega^2 - \Omega^2)} \Big(\omega^2 \delta_{ij} - \Omega^2 \hat{b}_i \hat{b}_j - i\omega\Omega\epsilon_{ijk}\hat{b}_k \Big). \tag{S.66}$$

Note: although we have used a specific coordinate system to derive this formula, once we have written it down in term of the unit vector $\hat{\mathbf{b}}$ — which is the only direction the matrices $\mathcal{M}$ and $\mathcal{M}^{-1}$ inverse care about — eq. (S.66) becomes valid in all coordinate systems.

Back to the electron:

However we calculate the inverse matrix (S.66), it gives the solution of eq. (S.59) for the electron's motion amplitude as

$$x_i^0 \;=\; \frac{e}{m}\bigl(\mathcal{M}^{-1}\bigr)_{ij} E_j^0 \tag{S.71}$$

and hence induced dipole moment amplitude

$$p_i^0 \;=\; -\frac{e^2}{m}\bigl(\mathcal{M}^{-1}\bigr)_{ij} E_j^0\,. \tag{S.72}$$

For the whole plasma, this gives the polarization

$$P_i \;=\; -\frac{e^2 n_e}{m}\bigl(\mathcal{M}^{-1}\bigr)_{ij} E_j\,, \tag{S.73}$$

and hence

$$D_i \;=\; \epsilon_0 E_i \;-\; \frac{e^2 n_e}{m}\bigl(\mathcal{M}^{-1}\bigr)_{ij} E_j\,. \tag{S.74}$$

In terms of the permittivity tensor, this means

$$\begin{aligned} \epsilon_{ij} \;&=\; \delta_{ij} \;-\; \frac{e^2 n_e}{m\epsilon_0}\bigl(\mathcal{M}^{-1}\bigr)_{ij} \\ &=\; \delta_{ij} \;-\; \frac{\omega_p^2}{\omega^2(\omega^2-\Omega^2)} \times \Bigl(\omega^2\delta_{ij} \;-\; \Omega^2 \hat{b}_i\hat{b}_j \;-\; i\omega\Omega\epsilon_{ijk}\hat{b}_k\Bigr), \end{aligned} \tag{S.75}$$

exactly as in eq. (7).

Problem **2**(b):

In the coordinate system whose z axis points in the direction of the magnetic field, the $\mathcal{M}^{-1}$

matrix is spelled out in eq. (S.68). Plugging it in into the top line of eq. (S.75), we get

$$\epsilon \;=\; \begin{pmatrix} 1-\frac{\omega_p^2}{\omega^2-\Omega^2} & +i\frac{\omega_p^2\Omega}{\omega(\omega^2-\Omega^2)} & 0 \\ -i\frac{\omega_p^2\Omega}{\omega(\omega^2-\Omega^2)} & 1-\frac{\omega_p^2}{\omega^2-\Omega^2} & 0 \\ 0 & 0 & 1-\frac{\omega_p^2}{\omega^2} \end{pmatrix}. \tag{S.76}$$

This is an Hermitian matrix of general form

$$\begin{pmatrix} A & +iC & 0 \\ -iC & A & 0 \\ 0 & 0 & B \end{pmatrix} \qquad \text{for real } A, B, C, \tag{S.77}$$

and it is easy to see that all such matrices have two eigenvalues $e_{1,2} = A \pm C$ corresponding to complex eigenvectors $\mathbf{m}_{1,2} = \sqrt{\frac{1}{2}}(1, \mp i, 0)$, while the third eigenvalue $e_3 = B$ corresponds to a real eigenvector $\mathbf{m}_3 = (0, 0, 1)$. For the ϵ_{ij} tensor (S.76) at hand, this means

$$\begin{aligned} \mathbf{m}_{1,2} \;&=\; \sqrt{\tfrac{1}{2}}(1, \mp i, 0) \qquad \text{for } \epsilon_{1,2} \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2-\Omega^2} \;\pm\; \frac{\omega_p^2\Omega}{\omega(\omega^2-\Omega^2)} \;=\; 1 \;-\; \frac{\omega_p^2}{\omega(\omega\pm\Omega)}\,, \\ \mathbf{m}_3 \;&=\; \quad (0, 0, 1) \qquad\quad \text{for } \epsilon_3 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2}\,. \end{aligned} \tag{S.78}$$

Problem **2**(c):

In vector notations, eq. (4) becomes

$$\overset{\leftrightarrow}{\epsilon} \cdot \vec{\mathcal{E}} \;-\; n^2\vec{\mathcal{E}} \;+\; n^2\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\vec{\mathcal{E}}) = \;0. \tag{S.79}$$

Taking the components of each term here in a complex but orthonormal basis $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$, we have

$$\left[\overset{\leftrightarrow}{\epsilon} \cdot \vec{\mathcal{E}}\right]_i \;=\; \mathbf{m}_i^* \cdot \overset{\leftrightarrow}{\epsilon} \cdot \vec{\mathcal{E}} \;=\; \sum_j (\mathbf{m}_i^* \cdot \overset{\leftrightarrow}{\epsilon} \cdot \mathbf{m}_j)(\mathbf{m}_j^* \cdot \vec{\mathcal{E}}) \;=\; \sum_j \epsilon_{ij}\mathcal{E}_j \tag{S.80}$$

where $\epsilon_{ij} = (\mathbf{m}_i^* \cdot \overset{\leftrightarrow}{\epsilon} \cdot \mathbf{m}_j)$ are the matrix elements of the $\overset{\leftrightarrow}{\epsilon}$ tensor in the complex basis at hand,

$$\big[\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\vec{\mathcal{E}})\big]_i \;=\; (\mathbf{m}_i^*\cdot\hat{\mathbf{k}})(\hat{\mathbf{k}}\cdot\vec{\mathcal{E}}) \;=\; (\mathbf{m}_i^*\cdot\hat{\mathbf{k}})\sum_j(\hat{\mathbf{k}}\cdot\mathbf{m}_j)(\mathbf{m}_j^*\cdot\vec{\mathcal{E}}) \;=\; \hat{k}_i\sum_j\hat{k}_j^*\mathcal{E}_j\,, \tag{S.81}$$

and therefore

$$\sum_j\Big(\epsilon_{ij}-n^2\delta_{ij}+n^2\hat{k}_i\hat{k}_j^{*}\Big)\mathcal{E}_j \;=\; 0. \tag{13}$$

From this point on, we may proceed exactly as in problem 1(d). To get a non-trivial solution of eq. (13) for the electric polarization vector $\vec{\mathcal{E}}$, the matrix $(\cdots)_{ij}$ must have a zero determinant, hence equation

$$\chi(n^2) \;\stackrel{\text{def}}{=}\; \det(\cdots) \;=\; 0 \tag{S.82}$$

for the refraction indices² for the two polarizations of the wave moving in a given direction $\hat{\mathbf{k}}$. To calculate this determinant, we use the basis $(\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3)$ made from the eigenvectors of the Hermitian tensor $\overset{\leftrightarrow}{\epsilon}$, hence

$$\begin{aligned}
\chi(n^2) \;&=\; \det\begin{pmatrix}\epsilon_1+n^2(|\hat{k}_1|^2-1) & n^2\hat{k}_1\hat{k}_2^* & n^2\hat{k}_1\hat{k}_3^*\\ n^2\hat{k}_2\hat{k}_1^* & \epsilon_2+n^2(|\hat{k}_2|^2-1) & n^2\hat{k}_2\hat{k}_3^*\\ n^2\hat{k}_3\hat{k}_1^* & n^2\hat{k}_3\hat{k}_2^* & \epsilon_3+n^2(|\hat{k}_3|^2-1)\end{pmatrix}\\
&=\; \epsilon_1\epsilon_2\epsilon_3 \;+\; n^2\Big(\epsilon_1\epsilon_2(|\hat{k}_3|^2-1)\;+\;\text{two similar terms}\Big)\\
&\qquad +\; n^4\epsilon_1\times\det\begin{pmatrix}|\hat{k}_2|^2-1 & \hat{k}_2\hat{k}_3^*\\ \hat{k}_3\hat{k}_2^* & |\hat{k}_3|^2-1\end{pmatrix}\;+\;\text{two similar terms}\\
&\qquad +\; n^6\times\det\begin{pmatrix}|\hat{k}_1|^2-1 & \hat{k}_1\hat{k}_2^* & \hat{k}_1\hat{k}_3^*\\ \hat{k}_2\hat{k}_1^* & |\hat{k}_2|^2-1 & \hat{k}_2\hat{k}_3^*\\ \hat{k}_3\hat{k}_1^* & \hat{k}_3\hat{k}_2^* & |\hat{k}_3|^2-1\end{pmatrix}
\end{aligned} \tag{S.83}$$

Similarly to what we had in problem 1(d), the determinant on the last line here evaluates to zero, while the determinants on the third line evaluate to

$$\det\begin{pmatrix} |\hat{k}_2|^2-1 & \hat{k}_2\hat{k}_3^* \\ \hat{k}_3\hat{k}_2^* & |\hat{k}_3|^2-1 \end{pmatrix} \;=\; (|\hat{k}_2|^2-1)(|\hat{k}_3|^2-1) \;-\; \hat{k}_2\hat{k}_3^*\hat{k}_3\hat{k}_2^*$$
$$=\; \cancel{|\hat{k}_2|^2\times|\hat{k}_3|^2} \;-\; |\hat{k}_2|^2 \;-\; |\hat{k}_3|^2 \;+\; 1 \;-\; \cancel{|\hat{k}_2|^2\times|\hat{k}_3|^3}$$
$$=\; 1 \;-\; |\hat{k}_2|^2 \;-\; |\hat{k}_3|^2 \;=\; |\hat{k}_1|^2, \tag{S.84}$$

and likewise for the two similar determinants. Consequently, assembling all the terms in eq. (S.83), we get exactly the same result as in problem 1(d), except that each $\hat{k}_i^2$ factor now becomes $|\hat{k}_i|^2 = |\mathbf{m}_i^*\cdot\hat{\mathbf{k}}|^2$. So at the end of the calculation we get eq. (6) with the $\hat{k}_i^2$ factors replaced with the $|\hat{k}_i|^2$, thus

$$\begin{aligned} \chi(n^2) \;&=\; \sum_{i=1}^{3}(|\hat{k}_i|^2 = |\mathbf{m}_i^*\cdot\hat{\mathbf{k}}|^2)\epsilon_i \times \prod_{j\neq i}(n^2-\epsilon_j) \\ &=\; \prod_{i=1}^{3}(n^2-\epsilon_i)\times\sum_{i=1}^{3}\frac{(|\hat{k}_i|^2 = |\mathbf{m}_i^*\cdot\hat{\mathbf{k}}|^2)\epsilon_i}{n^2-\epsilon_i}\,. \end{aligned} \tag{S.85}$$

Finally, similarly to problem 1(e) we assume 3 different eigenvalues of the ϵ tensor, hence the quadratic polynomial on the top line of this formula does not vanish for $n^2 =$ any of the ϵ_i. Instead, the roots of $\chi(n^2)$ follow from the zeros of the second factor on the second line of (S.85), hence the Fresnel equation

$$\sum_{i=1}^{3}\frac{(|\hat{k}_i|^2 = |\mathbf{m}_i^*\cdot\hat{\mathbf{k}}|^2)\epsilon_i}{n^2-\epsilon_i} \;=\; 0. \tag{14}$$

Quod erat demonstrandum.

Problem **2**(d):

The eigenvalues and the eigenvectors of the ϵ tensor for the plasma in a magnetic field were calculated in part (b) of this problem. In particular, in real coordinates with z axis along the magnetic field's direction, the eigenvectors are

$$\mathbf{m}_1 \;=\; \sqrt{\tfrac{1}{2}}(1,-i,0), \qquad \mathbf{m}_2 \;=\; \sqrt{\tfrac{1}{2}}(1,+i,0), \qquad \mathbf{m}_3 \;=\; (0,0,1). \tag{S.86}$$

For a wave propagating in a direction making angle θ with the magnetic field, we have

$$\hat{\mathbf{k}} \;=\; (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad \text{for some } \phi. \tag{S.87}$$

hence

$$\hat{k}_1 \;=\; \mathbf{m}_1^*\cdot\hat{\mathbf{k}} \;=\; \frac{\sin\theta}{\sqrt{2}}\,e^{+i\phi}, \qquad \hat{k}_2 \;=\; \mathbf{m}_2^*\cdot\hat{\mathbf{k}} \;=\; \frac{\sin\theta}{\sqrt{2}}\,e^{-i\phi}, \qquad \hat{k}_3 \;=\; \mathbf{m}_3^*\cdot\hat{\mathbf{k}} \;=\; \cos\theta, \tag{S.88}$$

and therefore

$$|\hat{k}_1|^2 \;=\; |\hat{k}_2|^2 = \tfrac{1}{2}\sin^2\theta, \qquad |\hat{k}_3|^2 \;=\; \cos^2\theta. \tag{S.89}$$

Plugging these values into the Fresnel equation (14), we arrive at

$$\frac{\sin^2\theta}{2} \times \left(\frac{\epsilon_1}{n^2-\epsilon_1} + \frac{\epsilon_2}{n^2-\epsilon_2}\right) \;+\; \cos^2\theta \times \frac{\epsilon_3}{n^2-\epsilon_3} \;=\; 0. \tag{S.90}$$

Next, the eigenvalues

$$\epsilon_{1,2} \;=\; 1 \;-\; \frac{\omega_p^2}{\omega(\omega\pm\Omega)}\,, \qquad \epsilon_3 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2}\,. \tag{S.91}$$

In the high-frequency limit $\omega(\omega-\Omega) \gg \omega_p$, all 3 eigenvalues are rather close to 1. In light of eq. (7) from the problem 1(e), — or rather

$$\epsilon_1 \;\geq\; n_1^2 \;\geq\; \epsilon_3 \;\geq\; n_2^2 \;\geq\; \epsilon_2 \tag{S.92}$$

since we now have $\epsilon_1 > \epsilon_3 > \epsilon_2$, — both solutions n_1^2 and n_2^2 of the Fresnel equation should also lie very close to 1, so let's zoom on this narrow range and let

$$n^2 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2} \;+\; \frac{\omega_p^2}{\omega^2}\times\nu. \tag{S.93}$$

Then

$$n^2 \;-\; \epsilon_3 \;=\; \frac{\omega_p^2}{\omega^2}\times\nu, \qquad n^2-\epsilon_{1,2} \;=\; \frac{\omega_p^2}{\omega^2}\times\left(\nu\mp\frac{\Omega}{\omega\pm\Omega}\right), \tag{S.94}$$

and we may rescale the Fresnel equation (S.90) as

$$\frac{\sin^2\theta}{2}\times\left(\frac{\epsilon_1}{\nu-\frac{\Omega}{\omega+\Omega}} \;+\; \frac{\epsilon_2}{\nu+\frac{\Omega}{\omega-\Omega}}\right) \;+\; \cos^2\theta\times\frac{\epsilon_3}{\nu} \;=\; 0. \tag{S.95}$$

Thus far, all the above calculations are exact. But now let's make use of the high-frequency limit in which all three $\epsilon_1,\epsilon_2,\epsilon_3\approx 1$ to replace the $\epsilon_{1,2,3}$ in the numerators of eq. (S.95) with ones, thus

$$\frac{\sin^2\theta}{2}\times\left(\frac{1}{\nu-\frac{\Omega}{\omega+\Omega}} \;+\; \frac{1}{\nu+\frac{\Omega}{\omega-\Omega}}\right) \;+\; \cos^2\theta\times\frac{1}{\nu} \;=\; 0. \tag{S.96}$$

This is the equation we are going to solve to understand the Faraday effect in plasma at high frequencies.

Simple version: the wave frequency ω is much higher than both the plasma frequency ω_p and the cyclotron frequency Ω.
In this limit, we may further approximate

$$\frac{\Omega}{\omega\pm\Omega} \;\approx\; \frac{\Omega}{\omega}, \tag{S.97}$$

hence in eq. (S.96)

$$\frac{1}{\nu-\frac{\Omega}{\omega+\Omega}} \;+\; \frac{1}{\nu+\frac{\Omega}{\omega-\Omega}} \;\approx\; \frac{1}{\nu-(\Omega/\omega)} \;+\; \frac{1}{\nu+(\Omega/\omega)} \;=\; \frac{2\nu}{\nu^2-(\Omega/\omega)^2)}, \tag{S.98}$$

and the whole (rescaled) Fresnel equation (S.96) becomes

$$\frac{\nu\sin^2\theta}{\nu^2-(\Omega/\omega)^2} \;+\; \frac{\cos^2\theta}{\nu} \;=\; 0. \tag{S.99}$$

Bringing the LHS here to a common denominator, we get

$$\text{numerator} \;=\; \nu^2\sin^2\theta \;+\; (\nu^2-(\Omega/\omega)^2)\cos^2\theta \;=\; \nu^2 \;-\; (\Omega/\omega)^2\cos^2\theta, \tag{S.100}$$

which vanishes for

$$\nu \;=\; \pm\frac{\Omega\cos\theta}{\omega}\,. \tag{S.101}$$

Or in terms of the refraction coefficients,

$$n_{1,2}^2 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2} \;\pm\; \frac{\omega_p^2\Omega}{\omega^3}\times\cos\theta\,. \tag{15}$$

<u>Harder version</u>: the limit of high frequency but also strong magnetic field, so that $\omega \gg \omega_p$ but not necessarily $\omega \gg \Omega$. Instead, we merely assume that $\omega > \Omega$ and $\omega(\omega-\Omega) \gg \omega_p^2$ to make sure that $1-\epsilon_2 \ll 1$ and hence $\forall i : (1-\epsilon_i) \ll 1$.

In this case, we need to solve eq. (S.96) without any further approximations, thus

$$\frac{1}{\nu-\frac{\Omega}{\omega+\Omega}}+\frac{1}{\nu+\frac{\Omega}{\omega-\Omega}} \;=\; \frac{\omega+\Omega}{\nu\omega+(\nu-1)\Omega}+\frac{\omega-\Omega}{\nu\omega-(\nu-1)\Omega} \;=\; \frac{2\nu\omega^2-2(\nu-1)\Omega^2}{\nu^2\omega^2-(\nu-1)^2\Omega^2}\,, \tag{S.102}$$

hence bringing eq. (S.96) to a common denominator yields

$$\begin{aligned} 0 \;=\; \text{numerator} \;&=\; \sin^2\theta\times\left[\nu\omega^2-(\nu-1)\Omega^2\right]\times\nu \;+\; \cos^2\theta\times\left[\nu^2\omega^2-(\nu-1)^2\Omega^2\right] \\ &=\; \nu^2\omega^2 \;-\; (\nu-1)\Omega^2(\nu-\cos^2\theta) \\ &=\; (\omega^2-\Omega^2)\times\nu^2 \;+\; \Omega^2(1+\cos^2\theta)\times\nu \;-\; \Omega^2\cos^2\theta. \end{aligned} \tag{S.103}$$

Solving this quadratic equation yields

$$\nu_{1,2} \;=\; \frac{-\Omega^2(1+\cos^2\theta)\pm\sqrt{4\omega^2\Omega^2\cos^2\theta+\Omega^4\sin^4\theta}}{2(\omega^2-\Omega^2)} \tag{S.104}$$

and hence

$$n_{1,2}^2 \;=\; 1 \;-\; \frac{\omega_p^2}{\omega^2} \;-\; \frac{\omega_p^2\Omega^2(1+\cos^2\theta)}{2\omega^2(\omega^2-\Omega^2)} \;\pm\; \frac{\omega_p^2\Omega}{\omega(\omega^2-\Omega^2)}\times\sqrt{\cos^2\theta+\frac{\Omega^2}{4\omega^2}\sin^4\theta}\,. \tag{S.105}$$

In the limit of $\omega \gg \Omega$, these solutions become (15).

FYI, the exact answer, without making any approximations at all. Although it helps to assume $\omega(\omega-\Omega) > \omega_p^2$ to make sure all 3 eigenvalues of the ϵ tensor are positive. In this general case

$$n_1^2 \;=\; \frac{\omega^2 \,-\, \omega_p^2}{\omega^2 \,-\, \nu_1 \times \omega_p^2}\,, \qquad n_2^2 \;=\; \frac{\omega^2 \,-\, \omega_p^2}{\omega^2 \,-\, \nu_2 \times \omega_p^2}\,, \tag{S.106}$$

$$\text{for}\quad \nu_{1,2} \;=\; \frac{\lambda}{1-\lambda^2}\left(-\frac{\lambda(1+\cos^2\theta)}{2} \,\pm\, \sqrt{\cos^2\theta \,+\, \tfrac{1}{4}\lambda^2\sin^4\theta}\right) \tag{S.107}$$

$$\text{where}\quad \lambda \;=\; \frac{\Omega\omega}{\omega^2-\omega_p^2}\,. \tag{S.108}$$