Problem **1**:

The electric dipole moment of a hydrogen atom is simply $\hat{\mathbf{p}} = -e\hat{\mathbf{x}}$ in the coordinate system with the nucleus at the origin, hence

$$\langle 2|\,\hat{\mathbf{p}}\,|1\rangle \;=\; -e\iiint d^3\mathbf{x}\,\Psi_2^*(\mathbf{x})\mathbf{x}\Psi_1(\mathbf{x}). \tag{S.1}$$

For the two states in questions, both wave functions (1) and (2) are invariant under rotations around the z axis, hence the matrix element (S.1) of the electric dipole moment must be parallel to that axis, thus

$$\langle 2|\,\hat{p}_x\,|1\rangle \;=\; \langle 2|\,\hat{p}_y\,|1\rangle \;=\; 0 \tag{S.2}$$

while

$$\langle 2|\,\hat{p}_z\,|1\rangle \;=\; -e\iiint d^3\mathbf{x}\,\Psi_2^*(\mathbf{x})(z = r\cos\theta)\Psi_1(\mathbf{x}). \tag{S.3}$$

Evaluating this integral in spherical coordinates, we have

$$\begin{aligned}\Psi_2^*(r,\theta,\phi)\times(z=r\cos\theta)\times\Psi_1(r,\theta,\phi) \;&=\; \frac{1}{\sqrt{32\pi a^5}}\,re^{-r/2a}\cos\theta\times r\cos\theta\times\frac{1}{\sqrt{\pi a^3}}\,e^{-r/a}\\ &=\; \frac{1}{\sqrt{32}\pi a^4}\times r^2e^{-3r/2a}\times\cos^2\theta,\end{aligned} \tag{S.4}$$

hence

$$\begin{aligned}\langle 2|\,\hat{p}_z\,|1\rangle \;&=\; -e\times\frac{1}{\sqrt{32}\pi a^4}\times\int_0^\infty dr\,r^2\times r^2e^{-3r/2a}\times\int_{-1}^{+1}d\cos\theta\,\cos^2\theta\times\int_0^{2\pi}d\phi\\ &=\; -e\times\frac{1}{\sqrt{32}\pi a^4}\times 24\left(\frac{2a}{3}\right)^5\times\frac{2}{3}\times 2\pi\\ &=\; -\left(\frac{8}{9}\right)^{5/2}ea.\end{aligned} \tag{S.5}$$

Classically, an oscillating electric dipole moment with amplitude $\mathbf{p}_0$ radiates net EM power

$$P = \frac{Z_0}{12\pi c^2}\,\omega^4 |\mathbf{p}_0|^2. \tag{S.6}$$

In quantum mechanics, this formula corresponds to

$$P \equiv \hbar\omega\Gamma = \frac{Z_0}{12\pi c^2}\,\omega^4 \, \left|2\,\langle 2|\,\hat{\mathbf{p}}\,|1\rangle\right|^2 , \tag{S.7}$$

hence the transition rate is

$$\Gamma = \frac{Z_0}{3\pi c^2\hbar}\,\omega^3\,|\langle 2|\,\hat{\mathbf{p}}\,|1\rangle|^2 . \tag{S.8}$$

For the $2p \to 1s$ transition in question,

$$|\langle 2|\,\hat{\mathbf{p}}\,|1\rangle|^2 = \left(\frac{8}{9}\right)^5 e^2 a^2, \tag{S.9}$$

cf. eq. (S.5), while

$$\hbar\omega = E_1 - E_2 = -\frac{m_e e^4}{2(4\pi\epsilon_0\hbar)^2}\left(\frac{1}{4} - 1\right) = +\frac{3}{8}\times\alpha^2 m_e c^2 = +\frac{3}{8}\,\frac{\alpha\hbar c}{a}\,, \tag{S.10}$$

hence

$$\Gamma = \frac{Z_0}{3\pi c^2\hbar^4}\times\frac{3^3}{2^9}\,(\alpha\hbar c/a)^3 \times\frac{2^{15}}{3^{10}}\,e^2 a^2 = \frac{2^6}{3^8\pi}\times\frac{Z_0 e^2 c\alpha^3}{\hbar a}\,. \tag{S.11}$$

Moreover,

$$Z_0 e^2 = \frac{e^2}{\epsilon_0 c} = 4\pi\alpha\hbar, \tag{S.12}$$

hence

$$\Gamma = \frac{2^8}{3^8}\,\alpha^4\,\frac{c}{a}\,. \tag{S.13}$$

Numerically,

$$\frac{c}{a} \approx 5.66\cdot 10^{18}\ \mathrm{s}^{-1}, \qquad \alpha\approx\frac{1}{137}\,, \tag{S.14}$$

hence

$$\Gamma \approx 6.3\cdot 10^8\ \mathrm{s}^{-1}. \tag{S.15}$$

In other words, the average lifetime of the excited 2p state is $\Gamma^{-1}\approx 1.6$ ns.

Problem **2**:

In the electric dipole approximation, the allowed radiative transitions are the transitions between states with a non-zero matrix element of the electric dipole moment between them, $\langle 2|\,\hat{\mathbf{p}}\,|1\rangle \neq 0$. By the rotational and parity symmetries of the atomic states, such non-zero matrix elements are allowed only for

$$|j_1 - j_2| \;\leq\; 1 \;\leq\; j_1 + j_2\,, \quad |m_1^j - m_2^j| \;\leq\; 1, \quad \text{and} \quad \text{parity}_2 \;=\; -\text{parity}_1\,. \tag{S.16}$$

Moreover, for transitions involving a single electron — especially in a hydrogen-like atom or ion — between states with definite m^ℓ and m^s, we also need

$$|\ell_1 - \ell_2| \;\leq\; 1, \quad |m_1^\ell - m_2^\ell| \;\leq\; 1, \quad m_2^s \;=\; m_1^s, \tag{S.17}$$

and

$$(-1)^{\ell_2} \;=\; \text{parity}_2 \;=\; -\text{parity}_1 \;=\; -(-1)^{\ell_1}, \tag{S.18}$$

hence

$$\ell_2 \;=\; \ell_1 \;\pm\; 1 \quad (\text{but not } \ell_2 = \ell_1). \tag{S.19}$$

In this problem, we are going to ignore the electron's spin state — it's going to stay the same through the whole cascade of transitions, — and focus on the remaining quantum numbers n, ℓ and $m = m^\ell$.

Let's start with the state $|1\rangle = |n_1, \ell_1, m_1\rangle$ with $m_1 = \ell_1 = n_1 - 1$. The next state $|2\rangle = |n_2, \ell_2, m_2\rangle$ in the photon-emission cascade must have a lower energy than the initial state, which requires

$$n_2 \;<\; n_1 \tag{S.20}$$

and hence

$$m_2 \;\leq\; \ell_2 \;\leq\; n_2 \,-\, 1 \;\leq\; n_1 \,-\, 2. \tag{S.21}$$

On the other hand, by the selection rules (S.17), this state must have

$$m_2 \;\geq\; m_1 \,-\, 1 \quad \text{and} \quad \ell_2 \;\geq\; \ell_1 \,-\, 1, \tag{S.22}$$

which for the initial state with $m_1 = \ell_1 = n_1 - 1$ means

$$m_2, \ell_2 \ \geq \ n_1 \ - \ 2. \tag{S.23}$$

Finally, the only way to reconcile the inequalities (S.21) and (S.23) is to have

$$m_2 \ = \ \ell_2 \ = \ n_2 - 1 \ = \ n_1 \ - \ 2. \tag{S.24}$$

Thus, in the electric dipole approximation to the photon emission, if the initial state has $m_1 = \ell_1 = n_1 - 1$ then the final state (of the first transition) is a similar state with $m_2 = \ell_2 = n_2 - 1$ for $n_2 = n_1 - 1$. *Quod erat demonstrandum.*

Problem **3**:

First, the generalia. In the multipole moment expansion for the EM radiation by small objects — classical or quantum, — the m^{th} order in the expansion leads to

$$\mathbf{f}_m \ \sim \ q\omega(kR)^{m+1} \tag{S.25}$$

hence

$$\text{Power} \ \sim \ Z_0 q^2 \omega^2 (kR)^{2m+2} \tag{S.26}$$

for the classical radiation or

$$\Gamma \ \sim \ \alpha\omega(kR)^{2m+2} \tag{S.27}$$

for a quantum transition. In terms of the specific electric or magnetic multipole moments, the m^{th} order of the multipole expansion includes the electric 2^{m+1}–pole moment and the magnetic 2^m–pole moment. Thus,

$$\begin{aligned} &\text{for an electric } 2^\ell \text{ pole transition,} \quad \Gamma \ \sim \ \alpha\omega(kR)^{2\ell}, \\ &\text{for a magnetic } 2^\ell \text{ pole transition,} \quad \Gamma \ \sim \ \alpha\omega(kR)^{2\ell+2}. \end{aligned} \tag{S.28}$$

Now let's deal with the specific metastable nuclei, starting with the cobalt $\mathrm{Co}^{58\mathrm{m}}$ — or rather $\mathrm{Co}^{58\mathrm{m}1}$ isomer. According to the Wikipedia page on isotopes of cobalt, the longer-lived cobalt-58m1 isomer has angular momentum and parity $J^P = 5+$, and it decays (by internal conversion of the γ ray) to the ground state of cobalt-58 which has $J^P = 2+$. By the selection rules of angular momentum and parity, the transition matrix element must have

$$\ell \;\geq\; |J_1 - J_2| \;=\; 3 \quad \text{but} \quad \ell \;\leq\; J_1 \,+\, J_2 \;=\; 7 \tag{S.29}$$

and positive parity. Positive parity means electric multipole for even ℓ and magnetic multipole for odd ℓ, thus the complete list of the allowed multipole moments is:

- the magnetic octupole, order $m = 3$;
- the electric 16–pole, order $m = 3$;
- the magnetic 32–pole, order $m = 5$;
- the electric 64–pole, order $m = 5$;
- the magnetic 128–pole, order $m = 7$.

In particular, the lowest-order multipoles are the magnetic octupole and the electric 16-pole, hence

$$\Gamma \;\sim\; \alpha\omega(kR)^8. \tag{S.30}$$

Next, consider the hafnium-178m2 isomer. According to the Wikipedia page on isotopes of hafnium, the hafnium-178 isotope has 3 long-lived metastable excited states AKA isomers — denoted $\mathrm{Hf}^{178\mathrm{m}1}$, $\mathrm{Hf}^{178\mathrm{m}2}$, and $\mathrm{Hf}^{178\mathrm{m}3}$, — which decay by γ-emission or internal conversion along the cascade

$$\mathrm{Hf}^{178\mathrm{m}3} \;\to\; \mathrm{Hf}^{178\mathrm{m}2} \;\to\; \mathrm{Hf}^{178\mathrm{m}1} \;\to\; \mathrm{Hf}^{178}_{\text{ground state}}\,. \tag{S.31}$$

Let's focus on the middle transition of this cascade with

$$\hbar\omega \;=\; U(\mathrm{m}2) \;-\; U(\mathrm{m}1) \;=\; 1.298\ \mathrm{MeV} \tag{S.32}$$

and a particularly long half-life of 31 years. According to the isotope table on the Wikipedia page, the m2 isomer has angular momentum and parity $J^P = 16^+$ while the m1 isomer has $J^P = 8^-$. Consequently, by the selection rules, the multipole moment involved in this transition must have

$$\ell \;\geq\; |16-8| \;=\; 8 \quad \text{but} \quad \ell \;\leq\; 16+8 \;=\; 24, \tag{S.33}$$

and also negative parity — which means magnetic multipole for even ℓ and electric multipole for odd ℓ. The list of multipole moments agreeing with these criteria starts with the magnetic $\ell = 8$ moment and the electric $\ell = 9$ moment, and continues all the way to the magnetic $\ell = 24$ moment. In terms of eq. (S.28), both the magnetic 2^8–pole and the electric 2^9–pole moments contribute at the order $m = 8$, hence

$$\Gamma \;\sim\; \alpha\omega(kR)^{18}, \tag{S.34}$$

while the remaining allowed multipoles contribute at higher orders to much smaller effect.

Finally, consider the extraordinarily long lived tantalum-180m isomer. According to the Wikipedia page on isotopes of tantalum, this isomer state has angular momentum and parity $J^P = 9^-$ while the ground state has $J^P = 1^+$. Consequently, the multipole matrix elements allowed between these states must have

$$\ell \;\geq\; |J_1 - J_2| \;=\; 8 \quad \text{but} \quad \ell \;\leq\; J_1 \,+\, J_2 \;=\; 10 \quad \Longrightarrow \quad \ell \;=\; 8, 9, 10, \tag{S.35}$$

and negative parity, which limits the list of the allowed multipole moments to

- the magnetic 2^8–pole, order $m = 8$;
- the electric 2^9–pole, order $m = 8$;
- the magnetic 2^{10}–pole, order $m = 10$.

In particular, the lowest-order multipoles are the magnetic 2^8–pole and the electric 2^9–pole, which lead to

$$\Gamma \;\sim\; \alpha\omega(kR)^{18}. \qquad \text{(S.36).}$$

PS: Although eq. (S.34) for the hafnium-178m2 and eq. (S.36) for the tantalum-180m have similar powers of the (kR) factors suppressing their decays, the tantalum isomer decays much slower than the hafnium isomer because the kR factor itself is much smaller for the tantalum transition. Indeed, while the two nuclei have similar radii $R \approx 6.9$ fm, the hafnium transition has

$$\hbar\omega \;\approx\; 1.3\text{ MeV} \quad\Longrightarrow\quad k \;\approx\; 6.6\text{ pm}^{-1} \quad\Longrightarrow\quad kR \;\approx\; 45\cdot 10^{-3}, \tag{S.37}$$

while the tantalum transition has much smaller energy

$$\hbar\omega \;\approx\; 77\text{ keV} \quad\Longrightarrow\quad k \;\approx\; 0.39\text{ pm}^{-1} \quad\Longrightarrow\quad kR \;\approx\; 2.7\cdot 10^{-3}. \tag{S.38}$$

When you take these (kR) factors to the 18^{th} power, you get a 10^{-24} suppression factor for the hafnium and 10^{-46} suppression factor for the tantalum. And that's why the hafnium-178m2 isomer has a half-life of 'only' 31 years while the tantalum-180m isomer lives for at least 10^{15} years (experimental lower limit) and maybe a lot longer.

Problem **4**, preamble:

Far away from the antenna,

$$\mathbf{A}(r,\theta,\phi) \;\approx\; \mu_0\,\frac{e^{ikr}}{r}\,\mathbf{f}(\theta,\phi) \tag{S.39}$$

and hence the power (per solid angle) radiated by the antenna in the direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega} \;=\; \frac{\omega^2 Z_0}{2c^2}\,|\mathbf{n}\times\mathbf{f}(\mathbf{n})|^2 \tag{S.40}$$

for

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{1}{4\pi}\iiint\limits_{\text{antenna}} d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{S.41}$$

Eq. (S.40) for the power is exact — as long as the power is measured far away from the antenna — but the integral in eq. (S.41) often takes various approximations to calculate, for example, the multipole expansion for short antennas. However, in this problem we are going to calculate the exact $\mathbf{f}(\theta)$ for the antenna in question in part (a), and then compare it to the multipole expansion in later parts.

Problem **4**(a):

For a thin antenna we approximate

$$\mathbf{J}(x,y,z) \;=\; \delta(x)\delta(y)I(z)\,\hat{\mathbf{z}} \tag{S.42}$$

where $\hat{\mathbf{z}} = (0,0,1)$ is the unit vector in the z direction, hence

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{\hat{\mathbf{z}}}{4\pi}\int dz\,I(z)\,\exp(-ik(\mathbf{n}\cdot\hat{\mathbf{z}})z), \tag{S.43}$$

or in terms of the angle θ between the direction $\mathbf{n}$ towards the observer and the z axis,

$$\mathbf{f}(\theta) \;=\; \frac{\hat{\mathbf{z}}}{4\pi}\int dz\,I(z)\,\exp(-ik(\cos\theta)z). \tag{S.44}$$

For the antenna current as in eq. (12), the integral here evaluates to

$$\begin{aligned}
\mathbf{f}(\theta) \;&=\; \frac{\hat{\mathbf{z}}}{4\pi}\int\limits_{-L/2}^{+L/2} I_0\,\sin(kz)\,\exp(-i(k\cos\theta)z) \qquad \langle\!\langle\,\text{for } k=(2\pi/\lambda) \text{ and } L=\lambda\,\rangle\!\rangle\\
&=\; \frac{I_0\hat{\mathbf{z}}}{4\pi k}\int\limits_{-\pi}^{+\pi} dx\,\sin(x)\,\exp(-ix\cos\theta)\\
&=\; \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\int\limits_{-\pi}^{+\pi} dx\,\Big(\exp(i(1-\cos\theta)x)\;-\;\exp(i(-1-\cos\theta))\Big)\\
&=\; \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\left[\frac{\exp(i(1-\cos\theta)x)}{i(1-\cos\theta)}\;-\;\frac{\exp(i(-1-\cos\theta)x)}{i(-1-\cos\theta)}\right]\Bigg|_{x=-\pi}^{x=+\pi}
\end{aligned} \tag{S.45}$$

where

$$\begin{aligned}
\exp(i(\pm1-\cos\theta)x)\Big|_{x=-\pi}^{x=+\pi} \;&=\; e^{\pm i\pi}e^{-i\pi\cos\theta}\;-\;e^{\mp i\pi}e^{+i\pi\cos\theta}\\
&=\; -e^{-i\pi\cos\theta}\;+\;e^{+i\pi\cos\theta}\;=\;2i\sin(\pi\cos\theta),
\end{aligned} \tag{S.46}$$

hence

$$\mathbf{f}(\theta) = \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\left[\frac{2i\sin(\pi\cos\theta)}{i(1-\cos\theta)} - \frac{2i\sin(\pi\cos\theta)}{i(-1-\cos\theta)}\right]$$
$$= \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\sin(\pi\cos\theta)\left[\frac{2}{1-\cos\theta} - \frac{2}{-1-\cos\theta} = \frac{4}{1-\cos^2\theta}\right] \qquad (S.47)$$
$$= \frac{I_0\hat{\mathbf{z}}}{2\pi ik}\,\frac{\sin(\pi\cos\theta)}{\sin^2\theta}.$$

Problem **4**(b):

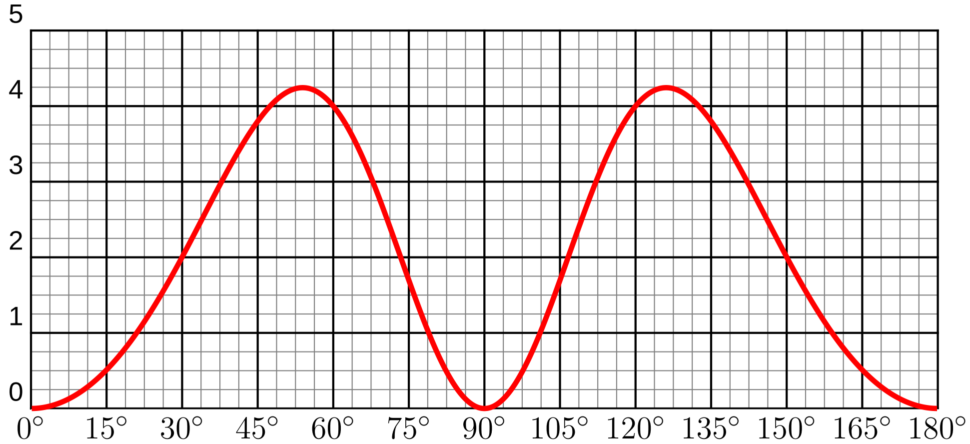
As a vector, $\mathbf{f}(\theta)$ always points in the z direction, hence

$$|\mathbf{n}\times\mathbf{f}|^2 = |\mathbf{f}|^2\times\sin^2\theta. \qquad (S.48)$$

Consequently, the EM power (per solid angle) emitted in the direction $\mathbf{n}$ at angle θ from the z axis is

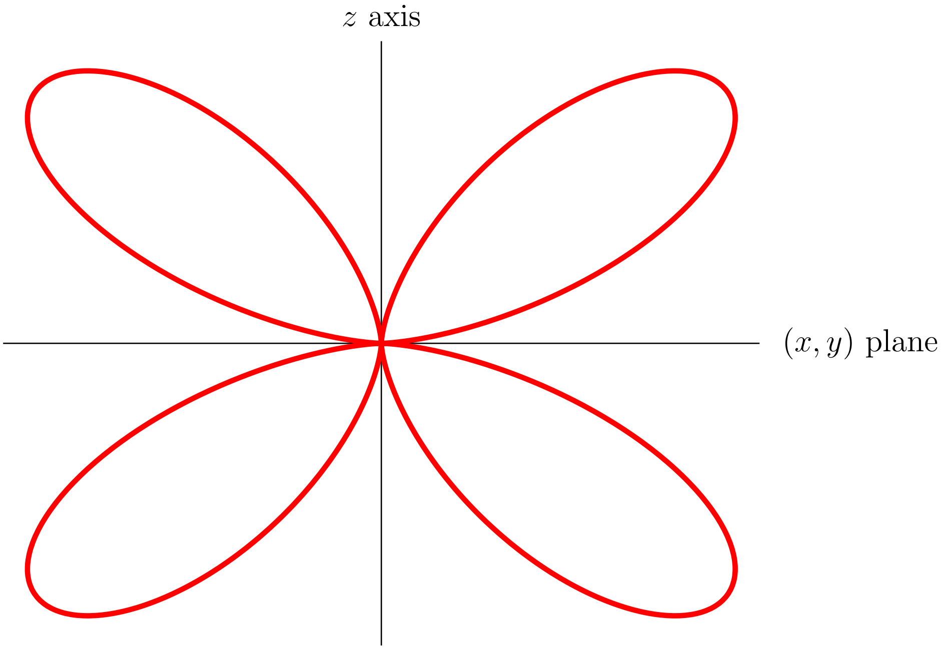
$$\frac{dP}{d\Omega} = \frac{Z_0\omega^2}{2c^2}\,|\mathbf{n}\times\mathbf{f}|^2$$
$$= \frac{Z_0\omega^2}{2c^2}\,|\mathbf{f}|^2\,\sin^2\theta$$
$$= \frac{Z_0\omega^2}{2c^2}\,\frac{|I_0|^2}{4\pi^2k^2}\times\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \qquad (S.49)$$
$$= \frac{Z_0|I_0|^2}{8\pi^2}\times\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}.$$

Let's plot the angular distribution of this power. The regular plot



shows no radiation at all in the direction of the antenna itself or ⊥ to the antenna. Instead, the radiation is peaked at some intermediate angles; taking the numeric derivative of eq. (S.49), we find the power maxima at $\theta \approx 53.9^\circ$ and $\theta \approx 126.1^\circ$, both being 36.1° away from the (x, y) plane normal to the antenna.

For completeness sake, here is the radiative power diagram representing the same plot:



(S.50)

But please note that this 2D diagram is but a vertical cross-section of the 3D power diagram, which has a rotational symmetry around the z axis. So while the 2D diagram (S.50) has 4 distinct 'leaves', the 3D diagram has only 2 lobes: the upper lobe including the upper-left and the upper-right 'leaves', and the lower lobe including the lower-left and the lower-right 'leaves'.

Problem **4**(c):

The net power emitted by the antenna in question obtains as an integral

$$\begin{aligned} P_{\rm net} &= \iint d^2\Omega \, \frac{dP}{d\Omega} = \frac{Z_0|I_0|^2}{8\pi^2} \times 2\pi \int_0^\pi d\theta \, \sin\theta \, \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \\ &= \frac{Z_0|I_0|^2}{4\pi} \times \int_{-1}^{+1} dc \, \frac{\sin^2(\pi c)}{1-c^2} \end{aligned} \tag{S.51}$$

where on the last line I have changed the integration variable from θ to $c = \cos\theta$. Unfortunately, this integral does not evaluate in terms of elementary functions but only in terms of the cosine integral function

$$\mathrm{Cin}(x) \stackrel{\text{def}}{=} \int\limits_0^x \frac{1-\cos(t)}{t}\,dt. \tag{S.52}$$

Indeed,

$$\frac{\sin^2(2\pi c)}{1-c^2} = \frac{1}{4}\bigl(1-\cos(2\pi c)\bigr)\left(\frac{1}{1-c}+\frac{1}{1+c}\right) \tag{S.53}$$

hence

$$\begin{aligned}
\int\limits_{-1}^{+1} dc\,\frac{\sin^2(\pi c)}{1-c^2} &= \frac{1}{4}\int\limits_{-1}^{+1} dc\left(\frac{1-\cos(2\pi c)}{1-c}+\frac{1-\cos(2\pi c)}{1+c}\right)\\
&\quad\langle\!\langle\,\text{using } c\to -c \text{ symmetry}\,\rangle\!\rangle\\
&= \frac{1}{2}\int\limits_{-1}^{+1} dc\,\frac{1-\cos(2\pi c)}{1+c}\\
&\quad\langle\!\langle\,\text{changing variable from } c \text{ to } t=2\pi(1+c)\,\rangle\!\rangle\\
&= \frac{1}{2}\int\limits_{0}^{4\pi} dt\,\frac{1-\cos(t)}{t}\\
&= \tfrac{1}{2}\,\mathrm{Cin}(4\pi),
\end{aligned} \tag{S.54}$$

and therefore

$$P_{\text{net}} = \frac{\mathrm{Cin}(4\pi)}{4\pi}\times\frac{Z_0|I_0|^2}{2}\,. \tag{S.55}$$

In terms of the antenna's radiative resistance, this means

$$R_{\text{rad}} = \mathrm{Re}(Z_{\text{rad}}) = \frac{\mathrm{Cin}(4\pi)}{4\pi}\times Z_0\,. \tag{S.56}$$

Numerically,

$$\frac{\mathrm{Cin}(4\pi)}{4\pi} \approx 0.247833, \tag{S.57}$$

thus

$$R_{\rm rad} \;\approx\; 0.247833\,Z_0 \;\approx\; 93.366\ \Omega. \qquad \text{(S.58)}$$

Problem **4**(d):

All the magnetic multipole moments involve integrals of the form

$$\iiint d^3\mathbf{y}\,(\mathbf{J}(\mathbf{y})\times\mathbf{y})\,(\mathbf{y}\cdot\mathbf{n})^{\text{some power}}. \qquad \text{(S.59)}$$

For the linear antennas, all such integrals vanish since $\mathbf{y}$ is confined to a single axis and the current $\mathbf{y}$ is parallel to that axis, hence $\mathbf{J}\times\mathbf{y}=0$.

As to the electric multipole moments, they follow from the oscillating electric charge density

$$\rho(\mathbf{x}) \;=\; \frac{1}{i\omega}\,\nabla\cdot\mathbf{J}(\mathbf{x}), \qquad \text{(S.60)}$$

which for the linear antenna in question becomes

$$\rho(x,y,z) \;=\; \frac{1}{i\omega}\,\delta(x)\delta(y)\frac{dI(z)}{dz} \;=\; \frac{I_0}{ic}\,\delta(x)\delta(y)\cos(kz). \qquad \text{(S.61)}$$

This distribution is symmetric WRT $z\to -z$ and hence WRT to the space reflection $\mathbf{x}\to-\mathbf{x}$. Consequently, this distribution has no odd-parity multipole moments but only-even-parity moments. For the electric multipole moments

$$\mathcal{M}^E_{\ell,m} \;=\; \sqrt{\frac{4\pi}{2\ell+1}}\iiint d^3\mathbf{x}\,\rho(\mathbf{x})\,r^\ell Y^*_{\ell,m}(\theta,\phi), \qquad \text{(S.62)}$$

the parity is $(-1)^\ell$, so all the odd-ℓ moments have negative parities and hence must vanish for the positive-parity $\rho(-\mathbf{x})=\rho(+\mathbf{x})$.

Thus, the only non-vanishing multipole moments of the antenna in question are the electric multipole with even $\ell=2,4,6,\ldots$. *Quod erat demonstrandum.*

Problem $\mathbf{4}$(e):

In terms of the electric and magnetic multipole moments of the antenna, the $\mathbf{f}_m$ is related to the electric 2^{m+1}–poles and the magnetic 2^m-poles. For even m, both of these multipole moments vanish by symmetry (*cf.* part (c)), so we should have $\mathbf{f}_m = 0$.

Another way to obtain this result is by looking at the explicit integrals (7). For the linear antenna in question, the 3D integral (7) becomes a 1D integral over z,

$$\mathbf{f}_m(\mathbf{n}) \;=\; \frac{(-ik)^m}{4\pi m!}\int\limits_{-L/2}^{+L/2} dz\, I(z)\hat{\mathbf{z}}\,(z\cos\theta)^m \;=\; \frac{(-ik)^m}{4\pi m!}\,I_0\hat{\mathbf{z}}\int\limits_{-L/2}^{+L/2} dz\,\sin(kz)\,(z\cos\theta)^m. \quad \text{(S.63)}$$

The integrand here is symmetric WRT to $z \to -z$ for odd m and odd for even m, so for the symmetric integration limits $z = \mp(L/2)$, the integral vanishes for all even m.

Now let's calculate the integral (S.63) for the odd m. Changing the integration variable from z to $x = kz$ and using $kL = 2\pi$, we get

$$\mathbf{f}_m(\mathbf{n}) \;=\; \frac{(-i)^m}{4\pi m!}\,\frac{I_0\hat{\mathbf{z}}}{k}\,(\cos\theta)^m\int\limits_{-\pi}^{+\pi} dx\,\sin(x)\,x^m \qquad \text{(S.64)}$$

where the remaining integrals obtain from the recursive relation

$$\mathcal{I}_m \;\stackrel{\text{def}}{=}\; \int\limits_{-\pi}^{+\pi} dx\,x^m\sin(x) \;=\; 2\pi^m \;-\; m(m-1)\mathcal{I}_{m-2} \qquad \langle\!\langle\,\text{for odd } m \text{ only}\,\rangle\!\rangle. \qquad \text{(S.65)}$$

Indeed,

$$\begin{aligned} x^m\sin(x) \;+\; m(m-1)x^{m-2}\sin(x) \;&=\\ &= \Big(x^m\sin(x) \;-\; mx^{m-1}\cos(x)\Big) \;+\; m\Big(x^{m-1}\cos(x) \;+\; (m-1)x^{m-2}\sin(x)\Big)\\ &= \frac{d}{dx}\Big(-x^m\cos(x)\Big) \;+\; \frac{d}{dx}\Big(mx^{m-1}\sin(x)\Big), \end{aligned} \qquad \text{(S.66)}$$

hence

$$
\begin{aligned}
\mathcal{I}_m \;+\; \mathcal{I}_{m-2} \;&=\; \int\limits_{-\pi}^{+\pi} dx \left(x^m \sin(x) \;+\; m(m-1)x^{m-2}\sin(x)\right)\\
&=\; \int\limits_{-\pi}^{+\pi} dx\,\frac{d}{dx}\left(-x^m\cos(x) \;+\; mx^{m-1}\sin(x)\right)\\
&=\; \left(-x^m\cos(x) \;+\; mx^{m-1}\sin(x)\right)\Big|_{-\pi}^{+\pi}\\
&=\; \left(-(+\pi)^m\cos(\pi) \;+\; m(+\pi)^{m-1}\sin(+\pi)\right)\\
&\qquad-\;\left(-(-\pi)^m\cos(-\pi) \;+\; m(-\pi)^{m-1}\sin(-\pi)\right)\\
&=\; \left(+(+\pi)^m \;+\; 0\right) \;-\; \left(+(-\pi)^m \;+\; 0\right)\\
&=\; (+\pi)^m \;-\; (-\pi)^m \;=\; 2\pi^m \quad \langle\!\langle\,\text{for an odd } m\,\rangle\!\rangle
\end{aligned}
\tag{S.67}
$$

and therefore the recursive relation (S.65). Working it out, we get

$$
\begin{aligned}
\mathcal{I}_1 \;&=\; 2\pi,\\
\mathcal{I}_3 \;&=\; 2\pi^3 \;-\; 6\mathcal{I}_1 \;=\; 2\pi^3 \;-\; 12\pi,\\
\mathcal{I}_5 \;&=\; 2\pi^5 \;-\; 20\mathcal{I}_3 \;=\; 2\pi^5 \;-\; 40\pi^3 \;+\; 240\pi,\\
\mathcal{I}_7 \;&=\; 2\pi^7 \;-\; 42\mathcal{I}_5 \;=\; 2\pi^7 \;-\; 84\pi^5 \;+\; 1680\pi^3 \;-\; 10080\pi,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{S.68}
$$

Finally, plugging the first 3 of these relations into eq. (S.64) for the $\mathbf{f}_{\mathrm{odd}\,m}$, we get

$$\mathbf{f}_1(\theta) \;=\; \frac{1}{2}\,(-i\cos\theta)\,\frac{I_0\hat{\mathbf{z}}}{k}\,, \tag{S.69}$$

$$\mathbf{f}_3(\theta) \;=\; \frac{\pi^2-6}{12}\,(-i\cos\theta)^3\,\frac{I_0\hat{\mathbf{z}}}{k}\,, \tag{S.70}$$

$$\mathbf{f}_5(\theta) \;=\; \frac{\pi^4-20\pi^2+120}{240}\,(-i\cos\theta)^5\,\frac{I_0\hat{\mathbf{z}}}{k}\,. \tag{S.71}$$

Problem **4**(f):

In light of eqs. (S.69) through (S.71),

$$\begin{aligned} \mathbf{f}_a(\theta) &= \frac{I_0\hat{\mathbf{z}}}{2k}\,(-i\cos\theta), \\ \mathbf{f}_b(\theta) &= \frac{I_0\hat{\mathbf{z}}}{2k}\,(-i\cos\theta)\left(1-\frac{\pi^2-6}{6}\,\cos^2\theta\right), \\ \mathbf{f}_c(\theta) &= \frac{I_0\hat{\mathbf{z}}}{2k}\,(-i\cos\theta)\left(1-\frac{\pi^2-6}{6}\,\cos^2\theta+\frac{\pi^4-20\pi^2+120}{120}\,\cos^4\theta\right). \end{aligned} \tag{S.72}$$

For each of these approximations,

$$\frac{dP}{d\Omega} = \frac{Z_0k^2}{2}\,|\mathbf{f}(\theta)|^2\sin^2\theta, \tag{S.73}$$

thus

$$\frac{dP_a}{d\Omega} = \frac{Z_0|I_0|^2}{8}\times\sin^2\theta\cos^2\theta, \tag{S.74}$$

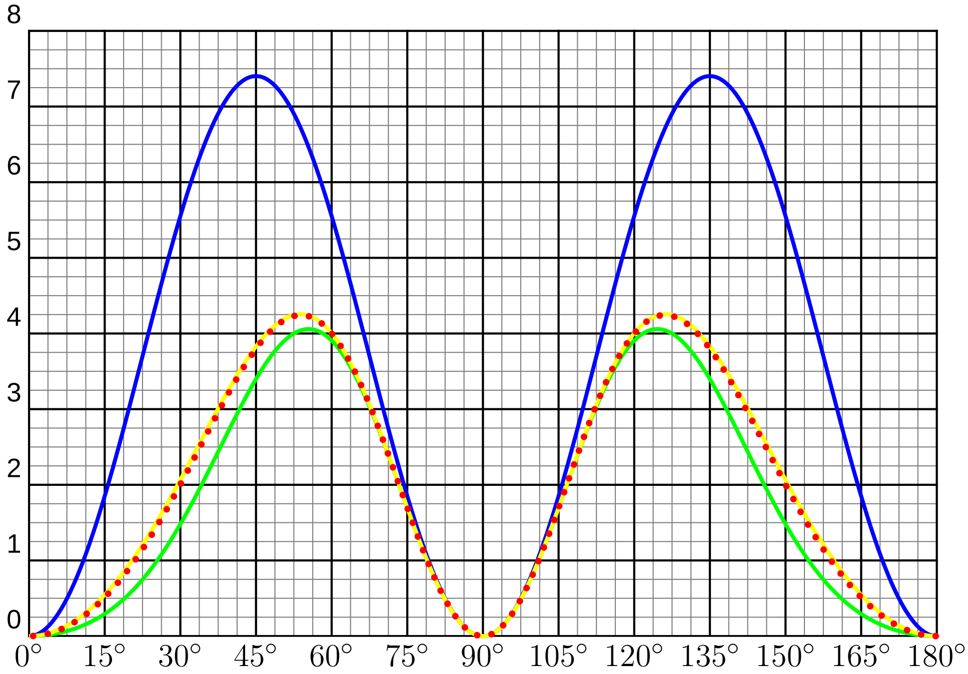
$$\frac{dP_b}{d\Omega} = \frac{Z_0|I_0|^2}{8}\times\sin^2\theta\cos^2\theta\times\left(1-\frac{\pi^2-6}{6}\,\cos^2\theta\right)^2, \tag{S.75}$$

$$\frac{dP_c}{d\Omega} = \frac{Z_0|I_0|^2}{8}\times\sin^2\theta\cos^2\theta\times\left(1-\frac{\pi^2-6}{6}\,\cos^2\theta+\frac{\pi^4-20\pi^2+120}{120}\,\cos^4\theta\right). \tag{S.76}$$

Note that in all these approximation, no power is radiated either along the antenna's axis z nor perpendicular to that axis, and in part (b) we saw that the exact angular distribution of the radiated power also has the same zero-power directions.

To get a more detailed comparison, let's plot the approximate angular power distributions

(S.74), (S.75), (S.76) as well as the exact power distribution (S.49) on the same graph:



(S.77)

On this plot, the solid blue line is the (a) approximation, the solid green line is the (b) approximation, the solid yellow line is the (c) approximation, and the dotted red line is the exact result from part (a).

As to the net power radiated by the antenna, integrating eqs. (S.74), (S.75), and (S.76) over the solid angle yields

$$\begin{aligned}
P_{\rm net}(a) &= \frac{Z_0|I_0|^2}{2}\times\left(\frac{2\pi}{15}\approx 0.418879\right),\\
P_{\rm net}(b) &= \frac{Z_0|I_0|^2}{2}\times\left(\frac{88\pi}{315}-\frac{4\pi^3}{135}+\frac{\pi^5}{1134}\approx 0.228805\right),\\
P_{\rm net}(c) &= \frac{Z_0|I_0|^2}{2}\times\left(\frac{17894\pi}{45045}-\frac{292\pi^3}{5005}+\frac{2831\pi^5}{810810}-\frac{\pi^7}{10530}+\frac{\pi^9}{1029600}\approx 0.249645\right),
\end{aligned}\tag{S.78}$$

as compared to the 'exact' result from part (c)

$$P_{\rm net}\approx\frac{Z_0|I_0|^2}{2}\times\left(\frac{\mathrm{Cin}(4\pi)}{4\pi}\approx 0.247833\right).\tag{S.79}$$

Comparing all these results, we see that the pure electric quadrupole approximation (a) looks qualitatively similar to the exact result, but quantitatively overestimates the net power by about 70%. Also, in the pure quadrupole approximation, the directional power is maximal at $\theta = 45°$ or $\theta = 135°$, while the exact calculation from part (b) shows maxima at $\theta \approx 53.9°$ and $\theta \approx 126.1°$, almost 9° closer to the (x, y) plane. However, the next approximation (b) — which adds the electric 16-pole term to the quadrupole — is much more accurate, both WRT the net power (underestimates by only 8%) and the direction of the maximal power (off by only 1.5°). And our last approximation (c) — which includes the electric quadrupole, the 16-pole, and the 64-pole terms — is so accurate that the yellow and the red lines on the graph (S.77) completely overlap each other.

Thus, although the leading multipole approximation to the full-wavelength antenna is rather crude, adding a few subleading multipoles makes for a *much* better approximation.