SPHERICAL WAVES

When a wave travels far enough from its source, it starts spreading in all directions while its energy flow density $\mathbf{S}$ diminishes with distance as $1/r^2$,

$$\mathbf{S}(r,\theta,\phi) \xrightarrow[r\to\infty]{} F(\theta,\phi)\,\frac{\mathbf{n}}{r^2} \;+\; O\left(\frac{1}{r^3}\right) \tag{1}$$

for some general angular power distribution

$$\frac{dP}{d\Omega} \;=\; F(\theta,\phi). \tag{2}$$

The fields in such a wave — the $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ in an electromagnetic wave, or the over-density $\delta\rho(\mathbf{x},t)$ in a sound wave, or whatever — generally look like

$$\psi(r,\theta,\phi;t) \;=\; \frac{e^{ikr-i\omega t}}{r}\left(f(\theta,\phi) \,+\, O\left(\frac{1}{r}\right)\right). \tag{3}$$

Waves like these are called *divergent spherical waves* because their wave-fronts are spheres spreading out from the center as $r = v_{\text{phase}}t + \text{const}$.

In these notes, we shall learn about the the divergent spherical waves that are exact solutions of the wave equation(s). For simplicity, we shall start with the scalar waves before turning to the electromagnetic waves. Eventually, we shall see that the electric and the magnetic multipoles for $\ell = 1, 2, 3, \ldots$ emit spherical EM waves of specific types, and we shall spell out those waves in both far-, near-, and intermediate-distance zones.

Spherical Scalar Waves

Let's start with the waves of a complex scalar field $\phi(\mathbf{x},t)$, and focus on the harmonic waves of a fixed frequency ω, thus $\psi(\mathbf{x},t)=\psi(\mathbf{x})e^{-i\omega t}$ for $\psi(\mathbf{x})$ obeying the 3D wave equation

$$(\nabla^2+k^2)\psi(\mathbf{x}) \;=\; 0. \tag{4}$$

The scalar analog of the Poynting vector — the flow density of the wave's energy — is

$$\mathbf{S} \;=\; \operatorname{Im}\bigl(\psi^*\nabla\psi\bigr) \;=\; |\psi|^2\,\nabla\,\mathrm{phase}(\psi), \tag{5}$$

so a divergent scalar wave of a general form

$$\psi(r,\theta,\phi) \;=\; \frac{e^{ikr}}{r}\left(f(\theta,\phi)\,+\,O\left(\frac{1}{r}\right)\right) \tag{3}$$

indeed has a radially spreading energy flow

$$\begin{aligned}\mathbf{S}(r,\theta,\phi) \;&=\; \frac{|f(\theta,\phi)|^2}{r^2}\left(k\mathbf{n}\,+\,\nabla\,\mathrm{phase}(f)\,+\,O\left(\frac{1}{r}\right)\right)\\ &=\; \frac{k|f(\theta,\phi)|^2}{r^2}\left(\mathbf{n}\,+\,O\left(\frac{1}{kr}\right)\right)\end{aligned} \tag{6}$$

for

$$\frac{dP}{d\Omega} \;=\; k|f(\theta,\phi)|^2. \tag{7}$$

The real problem is finding the exact solutions of the wave equation (4) in the spherical wave form (3).

The simplest solution is the spherically symmetric wave

$$\psi(r\text{ only}) \;=\; f_0\,\frac{e^{ikr}}{r} \tag{8}$$

for $f(\theta,\phi)=f_0=\text{const}$. Indeed, for this wave

$$\nabla^2\psi(r\text{ only}) \;=\; \frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr} \;=\; \frac{1}{r}\frac{d^2}{dr^2}\bigl(r\psi(r)\bigr) \;=\; \frac{1}{r}\frac{d^2}{dr^2}\bigl(e^{ikr}\bigr) \;=\; \frac{1}{r}\left(-k^2e^{ikr}\right) \;=\; -k^2\psi(r). \tag{9}$$

But for all other $f(\theta,\phi)\neq\text{const}$, the exact solutions have not only the leading $O(1/r)$ term but also the subleading $O(1/r^2)$, $O(1/r^3)$, *etc.*, terms.

To find all such subleading terms, let's use the separation-of-variables method to solve the wave equation, *i.e.* look for the solutions in the form

$$\psi(r,\theta,\phi) \;=\; f(\theta,\phi)\times g(r). \tag{10}$$

In spherical coordinates, the Laplace operator has form

$$\nabla^2 \;=\; \frac{\partial^2}{\partial r^2} \;+\; \frac{2}{r}\frac{\partial}{\partial r} \;-\; \frac{1}{r^2}\hat{\mathbf{L}}^2 \tag{11}$$

where

$$\hat{\mathbf{L}} \;=\; -i\mathbf{x}\times\nabla \tag{12}$$

is the differential operator WRT the angular coordinates θ and ϕ. You can find its exact form in any quantum-mechanics textbook where $\hbar\hat{\mathbf{L}}$ is the orbital angular momentum operator in the coordinate basis. For the wave of the form (10),

$$(\nabla^2+k^2)(fg) \;=\; -(\hat{\mathbf{L}}^2 f)\times\frac{g}{r^2} \;+\; +\,f\times g'' \;+\; f\times\frac{2g'}{r} \;+\; f\times k^2 g, \tag{13}$$

hence

$$\frac{r^2}{fg}(\nabla^2+k^2)(fg) \;=\; -\frac{\hat{\mathbf{L}}^2 f}{f}(\theta,\phi) \;+\; \left(\frac{r^2 g''}{g} \;+\; \frac{2rg'}{g} \;+\; k^2r^2\right)(r), \tag{14}$$

which should vanish for a solution of the wave equation. And since the first term on the RHS depends only on the angular coordinates (θ,ϕ) while the second term depends only on the radial coordinate r, both terms must be constants, thus

$$\hat{\mathbf{L}}^2 f(\theta,\phi) \;=\; C\times f(\theta,\phi), \tag{15}$$

$$r^2 g''(r) \;+\; 2rg'(r) \;+\; k^2r^2 g(r) \;=\; C\times g(r), \tag{16}$$

$$\text{for the same constant}\quad C.$$

The spectrum of the $\hat{\mathbf{L}}^2$ operator should be familiar to all of your from the undergraduate quantum-mechanics class: eq. (15) has solutions $f(\theta,\phi)$ that are periodic in ϕ and non-singular at the poles $\theta = 0,\pi$ only for

$$C \;=\; \ell(\ell+1), \quad \text{integer } \ell = 0,1,2,3,\ldots. \tag{17}$$

And for each such ℓ, there are $(2\ell+1)$ independent solutions called *the spherical harmonics* $f(\theta,\phi) = Y_{\ell,m}(\theta,\phi)$, labeled by another integer m running from $-\ell$ to $+\ell$.

As to the radial equation (16), for $C = \ell(\ell+1)$ it becomes *spherical Bessel equation*

$$g''(r) \;+\; \frac{2}{r}\,g'(r) \;-\; \frac{\ell(\ell+1)}{r^2}\,g(r) \;+\; k^2 g(r) \;=\; 0 \tag{18}$$

whose 2 independent solutions are the *spherical Bessel functions* $j_\ell(kr)$ and $n_\ell(kr)$, thus

$$g(r) \;=\; Aj_\ell(kr) \;+\; Bn_\ell(kr). \tag{19}$$

The spherical Bessel functions are related to the ordinary (cylindrical) Bessel functions with half-integer indices

$$j_\ell(x) \;=\; \frac{J_{\ell+\frac{1}{2}}(x)}{\sqrt{2x}}\,, \qquad n_\ell(x) \;=\; \frac{N_{\ell+\frac{1}{2}}(x)}{\sqrt{2x}}\,. \tag{20}$$

More interestingly, the spherical Bessel functions are related to the elementary functions as

$$j_\ell(x),\; n_\ell(x) \;=\; \sin(x)\times\text{Polynomial}(1/x) \;+\; \cos(x)\times\text{Polynomial}(1/x). \tag{21}$$

Specifically,

$$j_0(x) \;=\; \frac{\sin x}{x}\,, \tag{22}$$

$$n_0(x) \;=\; -\frac{\cos\; x}{x}\,, \tag{23}$$

$$j_1(x) \;=\; \frac{\sin(x) \;-\; x\cos(x)}{x^2}\,, \tag{24}$$

$$
\begin{aligned}
n_1(x) &= -\frac{\cos(x) + x\sin(x)}{x^2}\,, &(25)\\
j_2(x) &= \frac{(3-x^2)\sin(x) - 3x\cos(x)}{x^3}\,, &(26)\\
n_2(x) &= -\frac{(3-x^2)\cos(x) - 3x\sin(x)}{x^3}\,, &(27)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
j_\ell(x) &= (-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\left(\frac{+\sin x}{x}\right), &(28)\\
n_\ell(x) &= (-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\left(\frac{-\cos x}{x}\right). &(29)
\end{aligned}
$$

Asymptotically,

$$
\begin{aligned}
\text{for } x\to 0:\quad j_\ell(x) &\approx \frac{x^\ell}{(2\ell+1)!!}\,, \quad n_\ell(x) \approx -\frac{(2\ell-1)!!}{x^{\ell+1}}\,, &(30)\\
\text{while for } x\to\infty:\quad j_\ell(x) &\approx \frac{\sin(x-\ell\frac{\pi}{2})}{x}\,, \quad n_\ell(x) \approx -\frac{\cos(x-\ell\frac{\pi}{2})}{x}\,. &(31)
\end{aligned}
$$

In particular, the *regular* spherical Bessel function $j_\ell(kr)$ is the unique solution of the radial equation that is regular at the coordinate origin. Therefore, the standing wave modes of the scalar field in a spherical cavity have general form

$$
\psi(r,\theta,\phi) = (\text{const}) \times j_\ell(kr) \times Y_{\ell,m}(\theta,\phi) \tag{32}
$$

for discrete values of k for which $j_\ell(kR) = 0$ or $j'_\ell(kR) = 0$, depending on the Dirichlet v. Neumann boundary conditions at the cavity's boundary.

But for a divergent spherical wave, a singularity at the origin is OK because the wave is must be generated by some compact oscillator at the origin. On the other hand, at long distances from the center the spherical wave should travel outward rather than inward, or stand in place. Consequently, its radial profile should be the complex combination of the two real Bessel functions (for each ℓ), namely the *Hankel function*

$$
h_\ell(x) = j_\ell(x) + in_\ell(x) = -i(-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\left(\frac{e^{+ix}}{x}\right).
$$

Or rather,

$$g_\ell(kr) \;=\; i^{\ell+1} h_\ell(kr), \tag{33}$$

which at long distances $kr \gg 1$ behave as

$$g_\ell(kr) \;\longrightarrow\; i^\ell\,\frac{\exp(+ikr - \ell\frac{\pi}{2})}{kr} \;=\; \frac{\exp(+ikr)}{kr}\,. \tag{34}$$

Consequently, the divergent spherical wave solutions of the scalar wave equation have form

$$\psi(r,\theta,\phi) \;=\; Ak \times g_\ell(kr) \times Y_{\ell,m}(\theta,\phi) \;\xrightarrow[r\to\infty]{}\; A \times \frac{e^{+ikr}}{r} \times Y_{\ell,m}(\theta,\phi). \tag{35}$$

Or rather, all the divergent spherical waves of a given wave number k are linear combinations of specific (ℓ, m) wave modes (35),

$$\psi(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} A_{\ell,m} k \times g_\ell(kr) \times Y_{\ell,m}(\theta,\phi) \tag{36}$$

for some coefficients $A_{\ell,m}$. At large distances, all such waves have form

$$\psi(r,\theta,\phi) \;\xrightarrow[r\to\infty]{}\; \frac{e^{+ikr}}{r} \times \left(f(\theta,\phi) \,+\, O\left(\frac{1}{kr}\right)\right) \tag{37}$$

for

$$f(\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} A_{\ell,m} \times Y_{\ell,m}(\theta,\phi). \tag{38}$$

Consequently, given the asymptotic angular function $f(\theta,\phi)$, we may reconstruct all the $A_{\ell,m}$ coefficients of the series (36) as

$$A_{\ell,m} \;=\; \oiint d^2\Omega(\theta,\phi)\, f(\theta,\phi) \times Y^*_{\ell,m}(\theta,\phi). \tag{39}$$

Finally, for future reference, let me spell out the radial profiles of the divergent spherical waves for the low $\ell = 0, 1, 2, 3$:

$$g_0(kr) = \frac{e^{+ikr}}{kr}\,, \tag{40}$$

$$g_1(kr) = \frac{e^{+ikr}}{kr}\left(1 + \frac{i}{kr}\right), \tag{41}$$
$$g_2(kr) = \frac{e^{+ikr}}{kr}\left(1 + \frac{3i}{kr} - \frac{3}{(kr)^2}\right), \tag{42}$$
$$g_3(kr) = \frac{e^{+ikr}}{kr}\left(1 + \frac{6i}{kr} - \frac{15}{(kr)^2} - \frac{15i}{(kr)^3}\right). \tag{43}$$

Note: these radial profiles are exact and valid for all kr: large, small, or intermediate.

Spherical Electromagnetic Waves

Each component of the EM fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ obeys the wave equation

$$\left(\nabla^2 + k^2\right)\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = 0, \tag{44}$$

so each component E_i or H_i can be expanded into divergent spherical waves along the lines of eq. (36). The difficulty here is coordinating the expansions of these 6 components to maintain the Maxwell equations

$$\nabla\cdot\mathbf{E} = 0, \tag{M1}$$
$$\nabla\cdot\mathbf{H} = 0, \tag{M2}$$
$$\nabla\times\mathbf{E} = +ikZ_0\mathbf{H}, \tag{M3}$$
$$\nabla\times\mathbf{H} = \frac{-ik}{Z_0}\mathbf{E}. \tag{M4}$$

Fortunately, some of these equations automatically follow from each other and the wave eqs. (44). Indeed, given any magnetic field $\mathbf{H}(\mathbf{x})$ which obeys

$$\nabla\cdot\mathbf{H} = 0 \quad\text{and}\quad (\nabla^2 + k^2)\mathbf{H} = 0, \tag{45}$$

then this magnetic field and the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{iZ_0}{k}\nabla\times\mathbf{H}(\mathbf{x}) \tag{46}$$

obey all the Maxwell equations: (M2) and (M4) by assumption; (M1) follows from $\mathbf{E}$ being a curl; and (M3) follows as

$$\nabla\times\mathbf{E} \;=\; \frac{iZ_0}{k}\nabla\times\nabla\mathbf{H} \;=\; \frac{iZ_0}{k}\Big(\nabla(\nabla\cdot\mathbf{H})\,-\,\nabla^2\mathbf{H}\Big)$$
$$\langle\!\langle\,\text{by assumptions}\,\rangle\!\rangle \tag{47}$$
$$=\; \frac{iZ_0}{k}\Big(0\,+\,k^2\mathbf{H}\Big) \;=\; ikZ_0\,\mathbf{H}.$$

Likewise, given any electric field $\mathbf{E}(\mathbf{x})$ which obeys

$$\nabla\cdot\mathbf{E} \;=\; 0 \quad\text{and}\quad (\nabla^2+k^2)\mathbf{E} \;=\; 0, \tag{48}$$

then this electric field and the magnetic field

$$\mathbf{H}(\mathbf{x}) \;=\; \frac{1}{ikZ_0}\nabla\times\mathbf{E}(\mathbf{x}) \tag{49}$$

obey all the Maxwell equations (M1) through (M4).

Thus, from the mathematical point of view, our problem reduces to finding divergent spherical waves of a single vector field $\mathbf{V}(\mathbf{x})$ — which can be either $\mathbf{H}(\mathbf{x})$ or $\mathbf{E}(\mathbf{x})$ — that obeys

$$\nabla\cdot\mathbf{V} \;=\; 0 \quad\text{and}\quad (\nabla^2+k^2)\mathbf{V} \;=\; 0. \tag{50}$$

Suppose we have such a vector field $\mathbf{V}(\mathbf{x})$, then the scalar field

$$\psi(\mathbf{x}) \;=\; \mathbf{x}\cdot\mathbf{V}(\mathbf{x}) \tag{51}$$

obeys the wave equation. Indeed,

$$\begin{aligned}\nabla^2\big(\psi\;=\;\mathbf{x}\cdot\mathbf{V}\;=\;x_iV_i\big) \;&=\; (\nabla^2x_i)V_i \;+\; 2(\nabla_jx_i)(\nabla_jV_i) \;+\; x_i(\nabla^2V_i)\\ &=\; 0 \;+\; 2\delta_{ij}\nabla_jV_i \;+\; x_i(\nabla^2V_i)\\ &=\; 2\nabla\cdot\mathbf{V} \;+\; \mathbf{x}\cdot\nabla^2\mathbf{V},\end{aligned} \tag{52}$$

hence

$$(\nabla^2+k^2)(\mathbf{x}\cdot\mathbf{V}) \;=\; 2\nabla\cdot\mathbf{V} \;+\; \mathbf{x}\cdot(\nabla^2+k^2)\mathbf{V} \;=\; 0\;+\;0. \tag{53}$$

Consequently, $\psi(\mathbf{x})$ can be expanded into divergent spherical waves along the lines of eq. (36), thus

$$\mathbf{x}\cdot\mathbf{V}(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} A_{\ell,m}\times g_\ell(kr)\times Y_{\ell,m}(\theta,\phi) \tag{54}$$

for some coefficients $A_{\ell,m}$, where $g_\ell(kr)=i^{\ell+1}h_\ell(kr)$, exactly as in eq. (34).

Since the electromagnetic waves have two vector fields obeying the conditions (50), we may apply eq. (54) to both of them. Thus, the most general divergent spherical EM wave should have

$$\begin{aligned} k\mathbf{x}\cdot\mathbf{E}(\mathbf{x}) \;&=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} E_{\ell,m}\times g_\ell(kr)\times Y_{\ell,m}(\theta,\phi),\\ k\mathbf{x}\cdot\mathbf{H}(\mathbf{x}) \;&=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} H_{\ell,m}\times g_\ell(kr)\times Y_{\ell,m}(\theta,\phi), \end{aligned} \tag{55}$$

for some coefficients $E_{\ell,m}$ and $H_{\ell,m}$. In a moment, we shall see that $E_{0,0}=0$ and $H_{0,0}$ — the EM waves have no $\ell=0$ modes, — while all the remaining coefficients $E_{\ell,m}$ and $H_{\ell,m}$ are independent. Consequently, each of these coefficients gives rise to a particular mode of a divergent spherical wave. Specifically:

- Transverse magnetic waves $\mathrm{TM}_{\ell,m}$ with

$$\begin{aligned} \mathbf{x}\cdot\mathbf{E}(\mathbf{x}) \;&=\; \frac{E_{\ell,m}}{k}\times g_\ell(kr)\times Y_{\ell,m}(\theta,\phi),\\ \mathbf{x}\cdot\mathbf{H}(\mathbf{x}) \;&\equiv\; 0. \end{aligned} \tag{56}$$

  These TM waves are generated by the oscillating electric multipole moments with the appropriate ℓ and m.

- Transverse electric waves $\mathrm{TE}_{\ell,m}$ with

$$\begin{aligned} \mathbf{x}\cdot\mathbf{E}(\mathbf{x}) \;&\equiv\; 0,\\ \mathbf{x}\cdot\mathbf{H}(\mathbf{x}) \;&=\; \frac{H_{\ell,m}}{k}\times g_\ell(kr)\times Y_{\ell,m}(\theta,\phi). \end{aligned} \tag{57}$$

  These TE waves are generated by the oscillating magnetic multipole moment with the appropriate ℓ and m.

No Monopole Waves

Before we study the TM and TE waves in detail, let's find why neither type of waves has the $\ell = 0$ 'monopole' mode. So let $\mathbf{V}(\mathbf{x})$ be the electric field $\mathbf{E}(\mathbf{x})$ or the magnetic field $\mathbf{H}(\mathbf{x})$; either way it must obey

$$(\nabla^2 + k^2)\mathbf{V} \;=\; 0 \quad \text{and} \quad \nabla\cdot\mathbf{V} \;=\; 0. \tag{58}$$

Let's Fourier transform this wave in all 3 space dimensions,

$$\widetilde{\mathbf{V}}(\mathbf{q}) \;=\; \iiint d^3\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\,\mathbf{V}(\mathbf{x}), \qquad \mathbf{V}(\mathbf{x}) \;=\; \iiint \frac{d^3\mathbf{q}}{(2\pi)^3}\, e^{+i\mathbf{q}\cdot\mathbf{x}}\,\widetilde{\mathbf{V}}(\mathbf{q}). \tag{59}$$

In terms of the $\widetilde{\mathbf{V}}(\mathbf{q})$, the wave equation (58) becomes

$$(k^2 - q^2)\widetilde{\mathbf{V}}(\mathbf{q}) \;=\; 0, \tag{60}$$

so $\widetilde{\mathbf{V}}(\mathbf{q})$ must have form

$$\widetilde{\mathbf{V}}(\mathbf{q}) \;=\; \mathbf{f}(\mathbf{n}_q) * \delta(|q| - k) \tag{61}$$

for some vector-valued function $\mathbf{f}$ of the direction $\mathbf{n}_q$ of $\mathbf{q}$. Furthermore, the zero-divergence equation (58) translates to

$$\mathbf{q}\cdot\widetilde{\mathbf{V}}(\mathbf{q}) \;\equiv\; 0 \quad\Longrightarrow\quad \mathbf{n}_q\cdot\mathbf{f}(\mathbf{n}_q) \;\equiv\; 0. \tag{62}$$

Mathematically, this makes $\mathbf{f}$ a vector field on the sphere spanned by the $\mathbf{n}_q$ that's everywhere tangent to the sphere. And there is a topological theorem that says that all such fields must have zeroes or singularities (or both) somewhere on the sphere. In particular, the spherically symmetric solutions of the $\mathbf{f}(\mathbf{n}_q)$ do not exist!

On the other hand, the would-be $\ell = 0$ $\mathrm{TM}_{0,0}$ or $\mathrm{TE}_{0,0}$ modes should involve the $Y_{0,0}(\theta, \phi)$ spherical harmonic that completely uniform in all directions, so these $\ell = 0$ modes should be spherically symmetric. But alas, such spherically symmetric EM waves are topologically impossible, so the $\ell = 0$ do not exist. *Quod erat demonstrandum.*

Transverse Magnetic Waves

Consider a TM wave with $\mathbf{x}\cdot\mathbf{H}\equiv 0$ while $\mathbf{x}\cdot\mathbf{E}$ is a partial wave with specific values of ℓ and m. Besides $\mathbf{x}\cdot\mathbf{H}=0$, the magnetic field of this wave also obeys

$$\begin{aligned}\hat{\mathbf{L}}\cdot\mathbf{H} &= -i(\mathbf{x}\times\nabla)\cdot\mathbf{H} \;=\; -i\mathbf{x}\cdot(\nabla\times\mathbf{H})\\ &= -\frac{k}{Z_0}\mathbf{x}\cdot\mathbf{E} \qquad \langle\!\langle \text{ by Maxwell eq. (M4) } \rangle\!\rangle\\ &= -\frac{1}{Z_0}E_{\ell,m}\times g_\ell(kr)\times Y_{\ell,m}(\theta,\phi).\end{aligned} \tag{63}$$

Note that the operator $\hat{\mathbf{L}}$ acts only on the angular dependence of (whatever it acts upon). Moreover, when acting on the spherical harmonics $Y_{\ell,m}(\theta,\phi)$, the $\hat{\mathbf{L}}$ preserves ℓ but may change m. Specifically,

$$\begin{aligned}\hat{L}_z Y_{\ell,m}(\theta,\phi) &= mY_{\ell,m}(\theta,\phi),\\ (\hat{L}_\pm=\hat{L}_x\pm i\hat{L}_y)Y_{\ell,m}(\theta,\phi) &= (\text{coeff})\times Y_{\ell,m\pm1}(\theta,\phi).\end{aligned} \tag{64}$$

Consequently, if we want

$$\hat{\mathbf{L}}\cdot\mathbf{H} \;=\; \hat{L}_zH_z \;+\; \tfrac{1}{2}\hat{L}_+H_- \;+\; \tfrac{1}{2}\hat{L}_-H_+ \tag{65}$$

(where $H_\pm=H_x\pm iH_y$) to be proportional to a specific $Y_{\ell,m}(\theta,\phi)$, we need all 3 terms in eq. (65) to be proportional to the same $Y_{\ell,m}(\theta,\phi)$, hence

$$\begin{aligned}H_z(r,\theta,\phi) &= \alpha\times g_\ell(kr)Y_{\ell,m}(\theta,\phi),\\ H_+(r,\theta,\phi) &= \beta\times g_\ell(kr)Y_{\ell,m+1}(\theta,\phi),\\ H_-(r,\theta,\phi) &= \gamma\times g_\ell(kr)Y_{\ell,m-1}(\theta,\phi),\end{aligned} \tag{66}$$

for some coefficients α,β,γ.

To determine these coefficients we use eq. (63) as well as the requirement $\mathbf{x}\cdot\mathbf{H}=0$ for all $\mathbf{x}$ and hence

$$\alpha\times Y_{\ell,m}(\theta,\phi)\times\cos\theta + \tfrac{1}{2}\beta\times Y_{\ell,m+1}(\theta,\phi)\times\sin\theta e^{-i\phi} + \tfrac{1}{2}\gamma\times Y_{\ell,m-1}(\theta,\phi)\times\sin\theta e^{+i\phi} \;=\; 0 \tag{67}$$

for all θ and ϕ. A simple solution for the this constraint is $\mathbf{H}(\mathbf{x})=\hat{\mathbf{L}}\psi(\mathbf{x})$ for some scalar field $\psi(\mathbf{x})$ because $\mathbf{x}\cdot\hat{\mathbf{L}}=0$ and hence $\mathbf{x}\cdot\mathbf{H}=\mathbf{x}\cdot\hat{\mathbf{L}}\psi=0$ for any scalar ψ. But to make sure the components of the $\mathbf{H}$ field depend on (r,θ,ϕ) as in eq. (66), we take

$$\mathbf{H}(r,\theta,\phi) \;=\; C * g_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi) \tag{68}$$

for some overall constant C. And the value of that constant follows from eq. (63):

$$\begin{aligned} \hat{\mathbf{L}}\cdot\mathbf{H} \;&=\; C * g_\ell(kr) * \hat{\mathbf{L}}^2 Y_{\ell,m}(\theta,\phi) \;=\; C * g_\ell(kr) * \ell(\ell+1) Y_{\ell,m}(\theta,\phi) \\ \text{and also} \;&=\; -\frac{1}{Z_0} E_{\ell,m} * g_\ell(kr) * Y_{\ell,m}(\theta,\phi), \end{aligned} \tag{69}$$

hence

$$C \;=\; -\frac{1}{\ell(\ell+1)Z_0} \times E_{\ell,m}\,. \tag{70}$$

Altogether, we may summarize the magnetic field of the $\mathrm{TM}_{\ell,m}$ wave as

$$\mathbf{H}(\mathbf{x}) \;=\; \hat{\mathbf{L}}\psi(\mathbf{x}) \quad \text{where} \quad \psi(r,\theta,\phi) \;=\; -\frac{E_{\ell,m}}{\ell(\ell+1)Z_0} * g_\ell(kr) * Y_{\ell,m}(\theta,\phi). \tag{71}$$

As to the electric field,

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{iZ_0}{k}\,\nabla\times\mathbf{H}(\mathbf{x}) \;=\; \frac{Z_0}{k}\,(\nabla\times i\hat{\mathbf{L}})\psi(\mathbf{x}), \tag{72}$$

where

$$\begin{aligned} \big(\nabla\times i\hat{\mathbf{L}}\big)_i \;&=\; \big(\nabla\times(\mathbf{x}\times\nabla)\big)_i \;=\; \nabla_j x_i \nabla_j \;-\; \nabla_j x_j \nabla_i \\ &=\; x_i\nabla^2 \;+\; \nabla_i \;-\; \nabla_i x_j \nabla_j \;-\; 2\nabla_i \\ &=\; x_i\nabla^2 \;-\; \nabla_i\left(1 \,+\, x_j\nabla_j \;=\; 1 \,+\, r\frac{\partial}{\partial r}\right), \end{aligned} \tag{73}$$

and hence

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{Z_0}{k}\left(\mathbf{x}\nabla^2 \;-\; \nabla\left(1+r\frac{\partial}{\partial r}\right)\right)\psi(\mathbf{x}). \tag{74}$$

$\star \quad \star \quad \star$

To understand the source emitting a TM wave, let's consider its EM fields — especially the electric field (74) — in the near zone of $kr \ll 1$. In this zone,

$$h_\ell(kr) \;\approx\; in_\ell(kr) \;\approx\; -i\frac{(2\ell-1)!!}{(kr)^{\ell+1}} \quad\Longrightarrow\quad g_\ell(kr) \;\approx\; \frac{i^\ell(2\ell-1)!!}{k^{\ell+1}} \times \frac{1}{r^{\ell+1}}\,, \tag{75}$$

hence

$$\psi \;\approx\; \frac{-i^\ell\,(2\ell-1)!!}{\ell(\ell+1)}\,\frac{E_{\ell,m}}{Z_0 k^{\ell+1}} \times \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\,, \tag{76}$$

and

$$\mathbf{H} \;=\; \hat{\mathbf{L}}\psi \;\approx\; \frac{-i^\ell\,(2\ell-1)!!}{\ell(\ell+1)}\,\frac{E_{\ell,m}}{Z_0 k^{\ell+1}} \times \frac{\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\,. \tag{77}$$

At the same time,

$$\nabla^2\frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}} \;=\; 0, \qquad \left(1+r\frac{\partial}{\partial r}\right)\frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}} \;=\; -\ell\frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\,, \tag{78}$$

hence eq. (74) becomes

$$\mathbf{E}(\mathbf{x}) \;\approx\; +\frac{Z_0}{k}\ell\,\nabla\psi(\mathbf{x}) = \;-\nabla\Phi(\mathbf{x}) \tag{79}$$

for

$$\Phi(\mathbf{x}) \;=\; -\ell\frac{Z_0}{k}\psi(\mathbf{x}) \;=\; \frac{i^\ell(2\ell-1)!!}{(\ell+1)}\,\frac{E_{\ell,m}}{k^{\ell+2}} \times \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\,. \tag{80}$$

Note: this near-zone electric field looks precisely like the field of an electric 2^ℓ–pole moment!

Back in January, I have defined the spherical components of the electric multipole tensors as

$$\mathcal{M}^{\rm el}_{\ell,m} \;=\; \sqrt{\frac{4\pi}{2\ell+1}}\iiint d^3\mathbf{y}\,\rho(\mathbf{y}) \times r_y^\ell Y^*_{\ell,m}(\theta_y,\phi_y), \tag{81}$$

hence the potential generated by the static multipole moments was

$$\Phi(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\sum_{\ell,m}\mathcal{M}^{\rm el}_{\ell,m}\sqrt{\frac{4\pi}{2\ell+1}} \times \frac{Y_{\ell,m}(\theta_x,\phi_x)}{r_x^{\ell+1}}\,. \tag{82}$$

For the harmonically oscillating multipole moments $\mathcal{M}^{\rm el}_{\ell,m} \times e^{-i\omega t}$, the *near-zone* electric field should oscillate with a similar amplitude $\mathbf{E} = -\nabla\Phi$ for exactly the same $\Phi(\mathbf{x})$ as in the expansion (82), although the medium-zone and the far-zone electric fields would be quite different.

Thus, the physical meaning of eq. (80) is that an oscillating electric multipole generates a spherically divergent TM wave with the same (ℓ, m) as the multipole. As to the wave's amplitude, eq. (80) gives us

$$\frac{i^\ell(2\ell-1)!!}{(\ell+1)}\,\frac{E_{\ell,m}}{k^{\ell+2}} \;=\; \frac{\mathcal{M}^{\rm el}_{\ell,m}}{4\pi\epsilon_0}\times\sqrt{\frac{4\pi}{2\ell+1}}\,, \tag{83}$$

hence

$$E_{\ell,m} \;=\; \frac{(\ell+1)(-i)^\ell}{(2\ell-1)!!\sqrt{4\pi(2\ell+1)}}\times\frac{k^{\ell+2}}{\epsilon_0}\times\mathcal{M}^{\rm el}_{\ell,m}\,. \tag{84}$$

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Now consider the far-zone fields of the same TM wave. For $kr \gg 1$, the magnetic field (71) becomes

$$\mathbf{H}(r,\theta,\phi) \;=\; -\frac{E_{\ell,m}}{\ell(\ell+1)Z_0}*g_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi) \;\longrightarrow\; -\frac{E_{\ell,m}}{\ell(\ell+1)Z_0k}*\frac{e^{ikr}}{r}*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi), \tag{85}$$

while the electric field obtains from the Maxwell equation (M4) as

$$\mathbf{E} \;=\; \frac{iZ_0}{k}\nabla\times\mathbf{H}. \tag{86}$$

Furthermore, for the far-zone magnetic field (85), $\nabla e^{ikr} = e^{ikr}ik\mathbf{n}$, while space derivative of all other factors carry an extra factor of $1/r$, thus

$$\nabla\times\left(\frac{e^{ikr}}{r}*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\right) \;=\; \frac{e^{ikr}}{r}\left(ik\mathbf{n}\times\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi) \;+\; O\left(\frac{1}{r}\right)\right), \tag{87}$$

and therefore

$$\mathbf{E}(r,\theta,\phi) \;\longrightarrow\; +\frac{E_{\ell,m}}{\ell(\ell+1)k}*\frac{e^{ikr}}{r}*\mathbf{n}(\theta,\phi)\times\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi). \tag{88}$$

In particular, in any local region of space far from the center, the spherical EM wave has

$$\mathbf{E} \;=\; -Z_0\mathbf{n}\times\mathbf{H}, \qquad \mathbf{H} \;=\; +\frac{1}{Z_0}\mathbf{n}\times\mathbf{E}, \tag{89}$$

exactly as for a plane wave traveling in the direction $\mathbf{n}$. Consequently, similar to a plane wave, the divergent spherical wave has Poynting vector

$$\mathbf{S} \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}\times\mathbf{H}^*\big) \;=\; \frac{Z_0}{2}\,|\mathbf{H}|^2\,\mathbf{n}, \tag{90}$$

which for the magnetic field (85) becomes

$$\mathbf{S} \;=\; \frac{|E_{\ell,m}|^2}{2Z_0\,\ell^2(\ell+1)^2k^2} * |\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)|^2 * \frac{\mathbf{n}}{r^2}\,. \tag{91}$$

Thus, the wave's energy indeed flows radially outwards while the flow density diminishes as $1/r^2$.

According to eq. (91), the wave power radiated in a particular direction (θ,ϕ) is

$$\frac{dP}{d\Omega} \;=\; \frac{|E_{\ell,m}|^2}{2Z_0\,\ell^2(\ell+1)^2k^2} \times |\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)|^2. \tag{92}$$

The angular dependence of this power follows from

$$\begin{aligned} |\hat{\mathbf{L}}\,Y_{\ell,m}(\theta,\phi)|^2 \;&=\; |\hat{L}_zY_{\ell,m}(\theta,\phi)|^2 \;+\; \tfrac{1}{2}|\hat{L}_+Y_{\ell,m}(\theta,\phi)|^2 \;+\; \tfrac{1}{2}|\hat{L}_-Y_{\ell,m}(\theta,\phi)|^2 \\ &=\; m^2|Y_{\ell,m}(\theta,\phi)|^2 \;+\; \tfrac{1}{2}(\ell-m)(\ell+1+m)|Y_{\ell,m+1}(\theta,\phi)|^2 \\ &\qquad +\; \tfrac{1}{2}(\ell+m)(\ell+1-m)|Y_{\ell,m-1}(\theta,\phi)|^2. \end{aligned} \tag{93}$$

For example, for $\ell=1$ and $m=0$ (linear dipole in z direction)

$$|\hat{\mathbf{L}}\,Y_{\ell,m}(\theta,\phi)|^2 \;=\; \frac{3}{4\pi}\times\sin^2\theta \tag{94}$$

while for $\ell=1$ and $m=\pm1$ (circular dipole in xy plane)

$$|\hat{\mathbf{L}}\,Y_{\ell,m}(\theta,\phi)|^2 \;=\; \frac{3}{4\pi}\times\frac{1+\cos^2\theta}{2}\,. \tag{95}$$

As to the net wave power radiated in all 4π direction, eq. (92) leads to

$$P_{\mathrm{net}} \;=\; \frac{|E_{\ell,m}|^2}{2Z_0\,\ell^2(\ell+1)^2k^2} \times \oiint d^2\Omega(\theta,\phi)\,|\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)|^2, \tag{96}$$

where

$$\begin{aligned}
\oiint d^2\Omega\,|\hat{\mathbf{L}}Y_{\ell,m}|^2 \;&= \oiint d^2\Omega\,\big(\hat{\mathbf{L}}Y_{\ell,m}\big)^*\cdot\big(\hat{\mathbf{L}}Y_{\ell,m}\big)\\
&\qquad\langle\!\langle \text{ by Hermiticity of the } \hat{\mathbf{L}} = -i\mathbf{x}\times\nabla \text{ operator } \rangle\!\rangle\\
&= \oiint d^2\Omega\,Y^*_{\ell,m} * \hat{\mathbf{L}}^2 Y_{\ell,m}\\
&= \oiint d^2\Omega\,Y^*_{\ell,m} * \ell(\ell+1)Y_{\ell,m}\\
&= \;\ell(\ell+1)\oiint d^2\Omega|Y_{\ell,m}|^2\\
&= \;\ell(\ell+1)\times 1,
\end{aligned}\tag{97}$$

hence

$$P_{\rm net} \;=\; \frac{|E_{\ell,m}|^2}{2Z_0\,\ell(\ell+1)k^2}\,. \tag{98}$$

Or in terms the multipole amplitude $\mathcal{M}^{\rm el}_{\ell,m}$ generating the wave — *cf.* eq. (84), — the net power becomes

$$\begin{aligned}
P_{\rm net} \;&=\; \frac{1}{2\ell(\ell+1)Z_0k^2}\times\left|\frac{(\ell+1)k^{\ell+2}\mathcal{M}^{\rm el}_{\ell,m}}{(2\ell-1)!!\,\sqrt{4\pi(2\ell+1)}\,\epsilon_0}\right|^2\\
&=\; C_\ell\times\frac{k^{2\ell+2}|\mathcal{M}^{\rm el}_{\ell,m}|^2}{Z_0\epsilon_0^2}
\end{aligned}\tag{99}$$

where

$$C_\ell \;=\; \frac{(\ell+1)}{8\pi\ell}\times\frac{1}{(2\ell-1)!!\,(2\ell+1)!!}\,. \tag{100}$$

In particular,

$$C_1 \;=\; \frac{1}{12\pi}\,,\quad C_2 \;=\; \frac{1}{240\pi}\,,\quad C_3 \;=\; \frac{1}{9450\pi}\,,\quad \ldots. \tag{101}$$

Also,

$$\frac{1}{Z_0\epsilon_0^2} \;=\; Z_0c^2, \tag{102}$$

so we may rewrite eq. (99) as

$$P_{\rm net} \;=\; C_\ell Z_0\omega^2k^{2\ell}\times|\mathcal{M}^{\rm el}_{\ell,m}|^2. \tag{103}$$

As a cross-check, let's compare this formula to what we have learned a few lectures ago for the electric dipole and the electric quadrupole radiation, *cf.* my notes. Relating the dipole moment vector and the quadrupole moment tensor to the spherical tensors $\mathcal{M}^{\rm el}_{\ell,m}$ for $\ell = 1,2$ according to

$$\begin{aligned}\sum_m |\mathcal{M}^{\rm el}_{1,m}|^2 &= |\mathbf{p}|^2,\\ \sum_m |\mathcal{M}^{\rm el}_{2,m}|^2 &= \tfrac{2}{3}\,{\rm tr}(\mathcal{Q}^\dagger\mathcal{Q}),\end{aligned} \tag{104}$$

we bring eq. (103) to the form

$$P^{\rm dipole}_{\rm net} = \frac{Z_0}{12\pi}\,\omega^2 k^2 \times |\mathbf{p}|^2, \tag{105}$$

$$P^{\rm quadrupole}_{\rm net} = \frac{Z_0}{240\pi}\,\omega^2 k^4 \times \frac{2}{3}\,{\rm tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big), \tag{106}$$

which is precisely what we had earlier in class.

Transverse Electric Waves

The TE waves work very similarly to the TM waves, except that the electric and the magnetic field swap their roles. Indeed, consider a TE spherical wave with $\mathbf{x}\cdot\mathbf{E}\equiv 0$ while $\mathbf{x}\cdot\mathbf{H}(\mathbf{x})$ is a partial wave with specific values ℓ and m. Besides the $\mathbf{x}\cdot\mathbf{E}$ condition, the electric field of this wave also obeys

$$\begin{aligned}\hat{\mathbf{L}}\cdot\mathbf{E} &= -i(\mathbf{x}\times\nabla)\cdot\mathbf{E} \;=\; -i\mathbf{x}\cdot(\nabla\times\mathbf{E})\\ &= kZ_0\,\mathbf{x}\cdot\mathbf{H} \qquad \langle\!\langle \text{ by the Maxwell eq. (M3) } \rangle\!\rangle\\ &= Z_0 H_{\ell,m} * g_\ell(kr) * Y_{\ell,m}(\theta,\phi).\end{aligned} \tag{107}$$

As in the TM case, the solution to this equation as well as $\mathbf{x}\cdot\mathbf{E}\equiv 0$ is

$$\mathbf{E}(r,\theta,\phi) = C * g_\ell(kr) * \hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi) \tag{108}$$

for some constant coefficient C, whose values obtains from

$$\begin{aligned}\hat{\mathbf{L}}\cdot\mathbf{E} &= C * g_\ell(kr) * \hat{\mathbf{L}}^2 Y_{\ell,m}(\theta,\phi) \;=\; C * g_\ell(kr) * \ell(\ell+1)Y_{\ell,m}(\theta,\phi)\\ \text{and also } &= Z_0 H_{\ell,m} * g_\ell(kr) * Y_{\ell,m}(\theta,\phi),\end{aligned} \tag{109}$$

hence

$$C \;=\; +\frac{Z_0}{\ell(\ell+1)} \times H_{\ell,m}\,. \tag{110}$$

Altogether, this gives us

$$\mathbf{E}(\mathbf{x}) \;=\; \hat{\mathbf{L}}\psi(\mathbf{x}) \quad \text{for} \quad \psi(r,\theta,\phi) \;=\; \frac{Z_0 H_{\ell,m}}{\ell(\ell+1)} * g_\ell(kr) * Y_{\ell,m}(\theta,\phi), \tag{111}$$

while

$$\begin{aligned} \mathbf{H}(\mathbf{x}) \;&=\; \frac{-i}{kZ_0}\,\nabla\times\mathbf{E}(\mathbf{x}) \;=\; \frac{-1}{kZ_0}(\nabla\times i\hat{\mathbf{L}})\psi(\mathbf{x}) \\ &=\; \frac{-1}{kZ_0}\left(\mathbf{x}\nabla^2 \;-\; \nabla\left(1+r\frac{\partial}{\partial r}\right)\right)\psi(\mathbf{x}). \end{aligned} \tag{112}$$

To understand the source of a TE wave, let's look at the EM fields — especially the magnetic field (112) — in the near zone $kr \ll 1$. In this zone

$$g_\ell(kr) \;\longrightarrow\; \frac{i^\ell(2\ell-1)!!}{k^{\ell+1}} \times \frac{1}{r^{\ell+1}}\,, \tag{113}$$

hence

$$\psi(r,\theta,\phi) \;\longrightarrow\; \frac{i^\ell(2\ell-1)!!}{\ell(\ell+1)} * \frac{Z_0 H_{\ell,m}}{k^{\ell+1}} * \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\,, \tag{114}$$

where

$$\nabla^2\left(\frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\right) \;=\; 0 \tag{115}$$

and

$$\nabla\left(1+r\frac{\partial}{\partial r}\right)\left(\frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\right) \;=\; -\ell\nabla\left(\frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\right). \tag{116}$$

Consequently, in the near zone $kr \ll 1$ the magnetic field (112) of the TE wave becomes a gradient field, specifically

$$\mathbf{H}(r,\theta,\phi) \;\longrightarrow\; -\nabla\left(\frac{i^\ell(2\ell-1)!!}{(\ell+1)} * \frac{H_{\ell,m}}{k^{\ell+2}} * \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\right). \tag{117}$$

Apart from the (implicit) time dependence $e^{-i\omega t}$, this gradient field looks exactly like the magnetostatic field of a magnetic 2^ℓ-pole. Indeed, in terms of the spherical components

$\mathcal{M}^{\rm mag}_{\ell,m}$ of such a 2^ℓ-pole tensor, its magnetic field is

$$\mathbf{H}(r,\theta,\phi) \;=\; -\nabla\left(\frac{\mathcal{M}^{\rm mag}_{\ell,m}}{\sqrt{4\pi(\ell+1)}} * \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}}\right). \tag{118}$$

For an oscillating rather than stationary magnetic multipole $\mathcal{M}^{\rm mag}_{\ell,m} * e^{-i\omega t}$, the magnetic field would oscillate with an amplitude that looks just like (118) in the near zone, although in the intermediate and the far zones it would look quite different. Thus, comparing eqs. (117) and (118), we may identify the near-zone $\mathrm{TE}_{\ell,m}$ wave with the near-zone radiation emitted by an oscillating magnetic multipole $\mathcal{M}^{\rm mag}_{\ell,m} * e^{-i\omega t}$. Consequently, we may go beyond the near zone and identify the $\mathrm{TE}_{\ell,m}$ divergent spherical wave — at all distances from the origin, short, long, or intermediate — with the wave emitted by the oscillating magnetic 2^ℓ-pole.

As to the amplitude of the TE wave emitted by a specific $\mathcal{M}^{\rm mag}_{\ell,m}$ multipole, it also follows from the comparison of eqs. (117) and (118):

$$\frac{i^\ell(2\ell-1)!!}{(\ell+1)} \times \frac{H_{\ell,m}}{k^{\ell+2}} \;=\; \frac{\mathcal{M}^{\rm mag}_{\ell,m}}{\sqrt{4\pi(2\ell+1)}}\,, \tag{119}$$

hence

$$H_{\ell,m} \;=\; \frac{(-i)^\ell(\ell+1)}{(2\ell-1)!!} \times \frac{1}{\sqrt{4\pi(2\ell+1)}} \times k^{\ell+2}\mathcal{M}^{\rm mag}_{\ell,m}\,. \tag{120}$$

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Similar to the TM waves, in the far zone $kr \gg 1$ of a TE wave the EM fields locally look like the fields or a plane wave that happens to travel in the radial direction $\mathbf{n}$,

$$\mathbf{E} \;=\; -Z_0\mathbf{n}\times\mathbf{H}, \qquad \mathbf{H} \;=\; +\frac{1}{Z_0}\,\mathbf{n}\times\mathbf{E}, \tag{121}$$

To see how this works, let's go back to eq. (111) for the electric field and take the far-zone limit

$$g_\ell(kr) \;\longrightarrow\; \frac{e^{+ikr}}{kr}\,, \tag{122}$$

hence

$$\mathbf{E}(r,\theta,\phi) \;\longrightarrow\; +\frac{Z_0 H_{\ell,m}}{\ell(\ell+1)k} * \frac{e^{ikr}}{r} * \hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi). \tag{123}$$

As to the magnetic field $\mathbf{H}$, it follows from the Maxwell eq. (M3),

$$\mathbf{H} \;=\; \frac{-i}{kZ_0}\,\nabla\times\mathbf{E} \tag{124}$$

where in the far zone $\nabla \;=\; ik\mathbf{n} + O(1/r)$. Consequently,

$$\mathbf{H} \;\approx\; +\frac{1}{Z_0}\,\mathbf{n}\times\mathbf{E} \tag{125}$$

and hence the other eq. (121).

The Poynting vector of a locally-plane-like wave (121) is

$$\mathbf{S} \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}^*\times\mathbf{H}\big) \;=\; \frac{|\mathbf{E}|^2}{2Z_0}\,\mathbf{n}, \tag{126}$$

which for a far-zone TE wave (123) becomes

$$\mathbf{S} \;=\; \frac{Z_0}{2k^2} * \frac{|H_{\ell,m}|^2}{\ell^2(\ell+1)^2} * |\hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi)|^2 * \frac{\mathbf{n}}{r^2}\,. \tag{127}$$

Similar for the TM wave, the energy of the TE wave spreads out radially so the flow density diminishes as $1/r^2$, so the relevant feature of this energy is the power emitted into a particular direction,

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0}{2k^2} * \frac{|H_{\ell,m}|^2}{\ell^2(\ell+1)^2} * |\hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi)|^2. \tag{128}$$

Note that the direction dependence of this power is exactly the same as for the TM wave with similar ℓ and m, namely

$$\frac{dP}{d\Omega} = (\text{const}) \times \begin{pmatrix} m^2 \times |Y_{\ell,m}(\theta,\phi)|^2 \\ +\frac{1}{2}(\ell-m)(\ell+1+m) \times |Y_{\ell,m+1}(\theta,\phi)|^2 \\ +\frac{1}{2}(\ell-m)(\ell+1-m) \times |Y_{\ell,m-1}(\theta,\phi)|^2 \end{pmatrix}. \tag{129}$$

For example, a linear magnetic dipole pointing in z direction corresponds to $\ell = 1$, $m = 0$, hence

$$\frac{dP}{d\Omega} \propto \sin^2\theta. \tag{130}$$

Finally, the net power of a divergent spherical TE wave is

$$P_{\text{net}} = \frac{Z_0}{2k^2} \times \frac{|H_{\ell,m}|^2}{\ell^2(\ell+1)^2} \times \oiint d^2\Omega |\hat{\mathbf{L}} Y_{\ell,m}|^2 \tag{131}$$

where the integral over the directions evaluates to $\ell(\ell+1)$, *cf.* eq. (97), hence

$$P_{\text{net}} = \frac{Z_0}{2k^2} \times \frac{|H_{\ell,m}|^2}{\ell(\ell+1)}. \tag{132}$$

Or in terms of the magnetic multipole amplitude that generates the TE wave — *cf.* eq. (120), — the net power is

$$P_{\text{net}} = C_\ell \times Z_0 k^{2\ell+2} \times \left|\mathcal{M}_{\ell,m}^{\text{mag}}\right|^2, \tag{133}$$

where

$$C_\ell \stackrel{\text{def}}{=} \frac{(\ell+1)}{8\pi\ell} \times \frac{1}{(2\ell-1)!!\,(2\ell+1)!!}, \tag{134}$$

exactly as in the similar eq. (103) for the radiation of the electric multipole. For example, a magnetic dipole oscillator — for which

$$\sum_m \left|\mathcal{M}_{1,m}^{\text{mag}}\right|^2 = |\mathbf{m}|^2, \tag{135}$$

emits net power

$$P_{\text{net}}^{\text{mag.dipole}} = \frac{Z_0 k^4 |\mathbf{m}|^2}{12\pi}, \tag{136}$$

exactly as in my my notes on multipole radiation.

Intermediate zone

In the intermediate zone of $kr \sim 1$ — also known as the induction zone — we may no longer approximate the radial profile $g_\ell(kr)$ as either $e^{ikr}/(kr)$ or $(\text{coeff})/(kr)^{\ell+1}$. Instead, we heed we need to know

$$g_\ell(kr) \;=\; i^{\ell+1} h_\ell(kr) \;=\; \frac{e^{+ikr}}{kr} \times \text{Polynomial of degree } \ell \text{ in } \frac{i}{kr} \tag{137}$$

in all its details. For low $\ell = 1, 2, 3$,

$$g_1(kr) \;=\; \frac{e^{+ikr}}{kr}\left(1 + \frac{i}{kr}\right), \tag{41}$$

$$g_2(kr) \;=\; \frac{e^{+ikr}}{kr}\left(1 + \frac{3i}{kr} - \frac{3}{(kr)^2}\right), \tag{42}$$

$$g_3(kr) \;=\; \frac{e^{+ikr}}{kr}\left(1 + \frac{6i}{kr} - \frac{15}{(kr)^2} - \frac{15i}{(kr)^3}\right), \tag{43}$$

while for higher ℓ, they can be obtained as

$$g_\ell(x) \;=\; x^\ell \left(\frac{-i}{x}\frac{d}{dx}\right)^\ell \frac{e^{+ix}}{x}\,. \tag{138}$$

Given such radial profiles, the $\text{TM}_{\ell,m}$ wave has fields

$$\begin{aligned} \mathbf{H}(r,\theta,\phi) \;&=\; -\frac{E_{\ell,m}}{\ell(\ell+1)Z_0} * g_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi), \\ \mathbf{E}(r,\theta,\phi) \;&=\; \frac{iZ_0}{k}\,\nabla\times\mathbf{H}(r,\theta,\phi), \end{aligned} \tag{139}$$

while the $\text{TE}_{\ell,m}$ wave has fields

$$\begin{aligned} \mathbf{E}(r,\theta,\phi) \;&=\; \frac{Z_0 H_{\ell,m}}{\ell(\ell+1)} * g_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi), \\ \mathbf{H}(r,\theta,\phi) \;&=\; \frac{-i}{Z_0 k}\,\nabla\times\mathbf{E}(r,\theta,\phi). \end{aligned} \tag{140}$$

Or if you know the multipole source of the wave in a tensor form, you may replace the spherical harmonics $Y_{\ell,m}(\theta,\phi)$ — or rather than their combinations with the amplitudes

$E_{\ell,m}$ or $H_{\ell,m}$ with tensor products

$$\begin{aligned}
\sum_m E_{\ell,m} Y_{\ell,m}(\mathbf{n}) &\longrightarrow E^{(\ell)}_{i_1,\ldots,i_\ell} n_{i_1}\cdots n_{i_\ell}\,,\\
\sum_m H_{\ell,m} Y_{\ell,m}(\mathbf{n}) &\longrightarrow H^{(\ell)}_{i_1,\ldots,i_\ell} n_{i_1}\cdots n_{i_\ell}\,,
\end{aligned} \tag{141}$$

for the appropriate it symmetric traceless tensors $E^{(\ell)}_{i_1,\ldots,i_\ell}$ or $H^{(\ell)}_{i_1,\ldots,i_\ell}$.

Sometimes it's convenient to separate the intermediate-zone EM fields into their longitudinal (radial) and transverse (angular) components. For a TM wave, the magnetic field (139) is purely transverse while the electric field has both longitudinal and transverse components. Specifically,

$$\begin{aligned}
E_r &= \mathbf{n}\cdot\mathbf{E} = \frac{iZ_0}{k}\,\mathbf{n}\cdot(\nabla\times\mathbf{H}) = -\frac{Z_0}{rk}\,\hat{\mathbf{L}}\cdot\mathbf{H}\\
&= +\frac{E_{\ell,m}}{\ell(\ell+1)kr} * g_\ell(kr) * \hat{\mathbf{L}}^2 Y_{\ell,m}(\mathbf{n})\\
&= E_{\ell,m} * \frac{g_\ell(kr)}{kr} * Y_{\ell,m}(\mathbf{n}),
\end{aligned} \tag{142}$$

while

$$\mathbf{E}_t = -\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = -i\frac{Z_0}{k}\,\mathbf{n}\times(\mathbf{n}\times(\nabla\times\mathbf{H})) = \frac{i}{k}\,\mathbf{n}\times(\mathbf{n}\times(\nabla\times\hat{\mathbf{L}}))\psi \tag{143}$$

for

$$\psi(r,\theta,\phi) = \frac{E_{\ell,m}}{\ell(\ell+1)}\,g_\ell(kr)\,Y_{\ell,m}(\theta,\phi). \tag{144}$$

The differential operator in eq. (143) includes

$$\big[\mathbf{n}\times(\nabla\times\hat{\mathbf{L}})\big]_i = n_j\nabla_i L_j - n_j\nabla_j L_i \tag{145}$$

where

$$\begin{aligned}
n_j\nabla_i L_j &= \nabla_i(n_j L_j) - (\nabla_i n_j)L_j = \nabla_i(\mathbf{n}\cdot\hat{\mathbf{L}}=0) - \frac{\delta_{ij}-n_i n_j}{r}\,L_j\\
&= -\frac{L_i}{r} + \frac{n_i}{r}\,(\mathbf{n}\cdot\hat{\mathbf{L}}=0) = -\frac{L_i}{r}
\end{aligned} \tag{146}$$

while

$$n_j \nabla_j L_i \;=\; \frac{\partial}{\partial r} L_i\,, \tag{147}$$

so altogether

$$\mathbf{n} \times (\nabla \times \hat{\mathbf{L}}) \;=\; -\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \hat{\mathbf{L}}. \tag{148}$$

In the context of eq. (143), this means

$$\begin{aligned} \mathbf{E}_t \;&=\; \frac{-i}{k}\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \mathbf{n} \times \hat{\mathbf{L}}\psi \\ &=\; -i\frac{E_{\ell,m}}{\ell(\ell+1)k} * \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) g_\ell(kr) * \mathbf{n} \times \hat{\mathbf{L}} Y_{\ell,m}(\mathbf{n}). \end{aligned} \tag{149}$$

Altogether, the $\mathrm{TM}_{\ell,m}$ wave has fields

$$H_r \;=\; 0, \tag{150}$$

$$\mathbf{H}_t \;=\; -\frac{E_{\ell,m}}{\ell(\ell+1)Z_0} * g_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,m}(\mathbf{n}), \tag{151}$$

$$E_r \;=\; +E_{\ell,m} * \frac{g_\ell(kr)}{kr} * Y_{\ell,m}(\mathbf{n}), \tag{152}$$

$$\mathbf{E}_t \;=\; -i\frac{E_{\ell,m}}{\ell(\ell+1)} * \left(\frac{1}{kr} \,+\, \frac{\partial}{\partial(kr)}\right) g_\ell(kr) * \mathbf{n} \times \hat{\mathbf{L}} Y_{\ell,m}(\mathbf{n}). \tag{153}$$

As to the TE waves, we get similar formulae after swapping the electric and the magnetic fields with each other, or rather

$$\mathbf{E}^{\mathrm{TE}} \;=\; Z_0 \mathbf{H}^{\mathrm{TM}}, \qquad \mathbf{H}^{\mathrm{TE}} \;=\; \frac{-1}{Z_0}\,\mathbf{E}^{\mathrm{TM}}, \tag{154}$$

thus the $\mathrm{TE}_{\ell,m}$ wave has fields

$$E_r \;=\; 0, \tag{155}$$

$$\mathbf{E}_t \;=\; +\frac{Z_0 H_{\ell,m}}{\ell(\ell+1)} * g_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,m}(\mathbf{n}), \tag{156}$$

$$H_r \;=\; +H_{\ell,m} * \frac{g_\ell(kr)}{kr} * Y_{\ell,m}(\mathbf{n}), \tag{157}$$

$$\mathbf{H}_t \;=\; -i\frac{H_{\ell,m}}{\ell(\ell+1)} * \left(\frac{1}{kr} \,+\, \frac{\partial}{\partial(kr)}\right) g_\ell(kr) * \mathbf{n} \times \hat{\mathbf{L}} Y_{\ell,m}(\mathbf{n}). \tag{158}$$

PARTIAL WAVE ANALYSIS OF SCATTERING

Partial Scalar Waves

Consider scattering of a scalar wave $\psi(\mathbf{x})$ off some spherically symmetric obstacle. In quantum mechanics, this obstacle is usually a short-ranged central potential $V(r)$, although it can also be a reflective — or partially reflective — sphere with non-trivial boundary conditions. In any case, far away from the obstacle the scalar field $\psi(\mathbf{x})$ obeys the free wave equation, thus

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{x}) \xrightarrow[r\to\infty]{} 0, \tag{159}$$

and we are looking for solutions of the form

$$\psi(\mathbf{x}) = \psi_{\text{incident}}(\mathbf{x}) + \psi_{\text{scattered}}(\mathbf{x}) \xrightarrow[r\to\infty]{} \exp(ikz) + f(\theta)\,\frac{\exp(ikr)}{r}\,. \tag{160}$$

Note: by the spherical symmetry of the scattering object, the direction of the incident plane wave does not matter, so without loss of generality we make that direction the z axis. Likewise, the scattering amplitude $f(\mathbf{n})$ depends only on the angle between the incident wave and the direction $\mathbf{n}$ of the scattering, thus in the spherical coordinates $f(\theta)$ rather than $f(\theta,\phi)$.

To understand the physical meaning of the scattering solution (160), consider a Gaussian wave packet

$$\Psi(\mathbf{x},t) = \int\frac{dk}{\sqrt{2\pi}\delta k}\,e^{-(k-k_0)^2/2\delta k^2}\times\psi_k(\mathbf{x})e^{-i\omega(k)t} \tag{161}$$

where $\psi_k(\mathbf{x})$ is as in eq. (160) and δk is very small. Consequently, at large r we get

$$\psi_{\text{inc}}(\mathbf{x},t) \xrightarrow[r\to\infty]{} \exp(ik_0z - i\omega_0t)\times\exp\bigl(-(z-vt)^2/a^2\bigr), \tag{162}$$

$$\psi_{\text{sc}}(\mathbf{x},t) \xrightarrow[r\to\infty]{} \frac{\exp(ik_0r - i\omega_0t)}{r}\times f(\theta)\times\exp\bigl(-(r-vt)^2/a^2\bigr), \tag{163}$$

$$\text{where } a = \frac{1}{\delta k} \tag{164}$$

$$\text{and } v = \frac{d\omega}{dk}\,. \tag{165}$$

Thus, the incident wave packet moves steadily forward at the group velocity v, the scattered wave does not exist at early times $t \ll -a/v$, while at late times $t \gg +v/a$ it spreads out in all directions. At late times, the incident wave packet and the scattered wave packet move at the same speed but in different directions, so they stop overlapping for

$$t \gg \frac{a/v}{1-\cos\theta} \implies z_{\rm inc} - z_{\rm sc} = vt - vt\cos\theta \gg a. \tag{166}$$

Consequently, at large distances — and hence late times — we may ignore the interference between the incident and the scattered waves but calculate their energy flow densities $\mathbf{S}_{\rm inc}$ and $\mathbf{S}_{\rm sc}$ as if they were independent waves. Thus,

$$\mathbf{S}_{\rm inc} = k\hat{\mathbf{z}}, \qquad \mathbf{S}_{\rm sc} = \frac{k|f(\theta)|^2}{r^2}\,\mathbf{n} \tag{167}$$

and hence the scattering cross-section

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \tag{168}$$

Going back to the scalar wave equation and its scattering solutions, let's use the spherical symmetry of the equation to separate the variables in spherical coordinates,

$$\psi(r,\theta,\phi) = \sum_{\ell,m} C_{\ell,m}\sqrt{4\pi(2\ell+1)}Y_{\ell,m}(\theta,\phi)\times\psi_\ell(r). \tag{169}$$

Moreover, thanks to the axial symmetry of the scattering solutions (160), — ψ should depend on r and θ but not ϕ, all the $C_{\ell,m}$ with $m\neq 0$ must vanish. As to the $m=0$ modes, $Y_{\ell,0}(\theta,\phi)=\sqrt{(2\ell+1)/4\pi}\,P_\ell(\cos\theta)$, hence

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} C_\ell(2\ell+1)P_\ell(\cos\theta)\times\psi_\ell(r) \tag{170}$$

where $P_\ell(x)$ are the Legendre polynomials. The radial functions $\psi_\ell(r)$ in the sum (170) obey the radial wave equations

$$\psi_\ell''(r) + \frac{2}{r}\psi_\ell'(r) - \frac{\ell(\ell+1)}{r^2}\psi_\ell(r) + k^2\psi_\ell(r) = \begin{pmatrix}\text{perturbation by}\\ \text{the scatterer}\end{pmatrix} \xrightarrow[r\to\infty]{} 0. \tag{171}$$

Consequently, outside the scatterer the radial waves become linear combinations of the spherical Bessel functions $j_\ell(kr)$ and $n_\ell(kr)$, and if the perturbation potential or boundary condition (on the surface of some reflecting sphere) are real, then for each ℓ we should have a real linear combination

$$\psi_\ell(r) \;=\; \cos\delta_\ell \times j_\ell(kr) \;-\; \sin\delta_\ell \times n_\ell(kr) \tag{172}$$

for some angle δ_ℓ called *the phase shift*. The reason for this name is the asymptotic behavior of the radial solution at large r, — meaning both $r \gg R_{\text{scatterer}}$ and $kr \gg 1$. For $kr \gg 1$, the spherical Bessel functions asymptote to

$$j_\ell(kr) \;\xrightarrow[kr\gg 1]{}\; \frac{\sin\bigl(kr - \ell\frac{\pi}{2}\bigr)}{kr}\,, \qquad n_\ell(kr) \;\xrightarrow[kr\gg 1]{}\; -\frac{\cos\bigl(kr - \ell\frac{\pi}{2}\bigr)}{kr}\,, \tag{173}$$

hence for large radii

$$\psi_\ell(r) \;\xrightarrow[r\to\infty]{}\; \cos\delta\frac{\sin\bigl(kr - \ell\frac{\pi}{2}\bigr)}{kr} \;+\; \sin\delta\frac{\cos\bigl(kr - \ell\frac{\pi}{2}\bigr)}{kr} \;=\; \frac{\sin\bigl(kr - \ell\frac{\pi}{2} + \delta_\ell\bigr)}{kr}\,. \tag{174}$$

In this formula, δ_ℓ shifts the phase of the asymptotic sine wave from the no-scattering asymptotic behavior

$$\begin{aligned} \psi_\ell^{\text{free}}(r) \;&=\; j_\ell(kr) \text{ @ all } r \qquad \langle\!\langle \text{ because } \psi_\ell^{\text{free}}(r) \text{ should stay finite for } r\to 0 \,\rangle\!\rangle \\ &\xrightarrow[kr\gg 1]{}\; \frac{\sin\bigl(kr - \ell\frac{\pi}{2}\bigr)}{kr}\,. \end{aligned} \tag{175}$$

Next, let's assemble the partial waves for different ℓ's into the sum

$$\begin{aligned} \psi(r,\theta) \;&=\; \sum_{\ell=0}^{\infty} C_\ell(2\ell+1)P_\ell(\cos\theta) \times \psi_\ell(r) \\ &=\; \sum_{\ell=0}^{\infty} C_\ell(2\ell+1)P_\ell(\cos\theta) \times \Bigl(\cos\delta_\ell \times j_\ell(kr) \;-\; \sin\delta_\ell \times \; n_\ell(kr)\Bigr) \end{aligned} \tag{176}$$

and choose the coefficients C_ℓ such that the net wave has asymptotic behavior (160) at large distances. The key to this choice is the following **Lemma:**

$$\int\limits_{-1}^{+1} e^{ikrc}\,P_\ell(c)\,dc \;=\; 2i^\ell j_\ell(kr). \tag{177}$$

Since the Legendre polynomial form a complete orthogonal basis of functions of $c \in [-1,+1]$

normalized to

$$\int_{-1}^{+1} dc\, P_\ell(c) P_{\ell'}(c) \;=\; \frac{2}{2\ell+1}\,\delta_{\ell,\ell'}\,, \tag{178}$$

the Lemma (177) leads to

$$e^{ikrc} \;=\; \sum_\ell \frac{2\ell+1}{2} P_\ell(c) \times \int_{-1}^{+1} dc'\, P_\ell(c') \times e^{ikrc'} \;=\; \sum_\ell (2\ell+1) i^\ell j_\ell(kr) \times P_\ell(c). \tag{179}$$

Identifying c in this formula with $\cos\theta$, we see that the incident wave decomposes into the spherical waves as

$$\psi_{\rm inc} \;=\; \exp(ikz) \;=\; \exp(ikr\cos\theta) \;=\; \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(kr) \times P_\ell(\cos\theta). \tag{180}$$

At the same time, the scattered wave is purely divergent: its asymptotic behavior is

$$\psi_{\rm sc}(r,\theta) \;=\; \frac{f(\theta)}{r} \times e^{+ikr} \qquad \text{without an } e^{-ikr} \text{ term}, \tag{181}$$

so for each partial wave we should have

$$\psi_\ell^{\rm sc}(r) \; \xrightarrow[r\to\infty]{} \; A_\ell \times \frac{e^{+ikr}}{r} \tag{182}$$

for some overall complex coefficient A_ℓ, or in terms of the spherical Bessel functions

$$\psi_\ell^{\rm sc}(r) \;=\; A_\ell k \times i^{\ell+1} h_\ell(kr) \;=\; A_\ell k i^{\ell+1} \times \bigl(j_\ell(kr) \,+\, i n_\ell(kr)\bigr) \; \xrightarrow[kr\gg 1]{} \; A_\ell \times \frac{e^{+ikr}}{r}\,. \tag{183}$$

Altogether, the scattered wave should have form

$$\psi_{\rm sc}(r,\theta) \;=\; \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell+1} A_\ell P_\ell(\cos\theta) \times \bigl(h_\ell(kr) \;=\; j_\ell(kr) \,+\, i n_\ell(kr)\bigr), \tag{184}$$

hence adding the incident wave (180) we build

$$\psi^{\rm net}(r,\theta) \;=\; \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell P_\ell(\cos\theta)\times\Big((1+iA_\ell)\times j_\ell(kr)\;-\;A_\ell\times n_\ell(kr)\Big). \tag{185}$$

Comparing this formula to eq. (176), we find the same general behavior provided

$$C_\ell\times\cos\delta_\ell \;=\; iA_\ell\;+\;1 \quad\text{and}\quad C_\ell\times(-\sin\delta_\ell)\;=\;-A_\ell\,. \tag{186}$$

Solving these equations gives us

$$C_\ell \;=\; \exp(i\delta_\ell), \qquad A_\ell \;=\; \sin\delta_\ell\times\exp(i\delta_\ell)\;=\;\frac{e^{2i\delta_\ell}-1}{2i}\,. \tag{187}$$

Coming back to the scattered wave, eq. (184) leads to

$$\begin{aligned}
\psi_{\rm sc}(r,\theta) \;&=\; \sum_{\ell=0}^{\infty}(2\ell+1)A_\ell P_\ell(\cos\theta)\times i^\ell h_\ell(kr)\\
&\xrightarrow[kr\gg1]{}\; \sum_{\ell=0}^{\infty}(2\ell+1)A_\ell P_\ell(\cos\theta)\times\frac{e^{+ikr}}{kr}\\
&=\;\frac{e^{+ikr}}{kr}\times\sum_{\ell=0}^{\infty}(2\ell+1)A_\ell P_\ell(\cos\theta)\\
&=\;f(\theta)\times\frac{e^{+ikr}}{r}
\end{aligned} \tag{188}$$

for the *scattering amplitude*

$$f(\theta) \;=\; \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)A_\ell P_\ell(\cos\theta)\,. \tag{189}$$

The coefficients A_ℓ here should be as in eq. (187), thus

$$f(\theta) \;=\; \sum_{\ell=0}^{\infty}\frac{e^{2i\delta_\ell}-1}{2ik}\times(2\ell+1)P_\ell(\cos\theta). \tag{190}$$

The partial scattering cross-section follows from the amplitude (190) as

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \tag{191}$$

where

$$|f(\theta)|^2 = \sum_{\ell,\ell'} \frac{(\exp(+2i\delta_\ell)-1)(\exp(-2i\delta_{\ell'})-1)}{4k^2} \times (2\ell+1)(2\ell'+1)P_\ell(\cos\theta)P_{\ell'}(\cos\theta). \tag{192}$$

Consequently, integrating this partial cross-section over the 4π directions to obtain the total cross-section, we obtain

$$\begin{aligned}
\sigma_{\text{tot}} &= \oiint d^2\Omega\ |f|^2 \\
&= \int_0^\pi |f|^2 \times 2\pi\sin\theta\,d\theta \\
&= \sum_{\ell,\ell'} \frac{(\exp(+2i\delta_\ell)-1)(\exp(-2i\delta_{\ell'})-1)}{4k^2} \times \\
&\qquad \times (2\ell+1)(2\ell'+1)\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\,2\pi\sin\theta\,d\theta
\end{aligned} \tag{193}$$

On the last line here

$$\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\,2\pi\sin\theta\,d\theta = 2\pi\int_{-1}^{+1} P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\,d\cos\theta = \frac{4\pi}{2\ell+1}\times\delta_{\ell,\ell'}, \tag{194}$$

hence

$$\begin{aligned}
\sigma_{\text{tot}} &= \sum_\ell \left|\frac{\exp(2i\delta_\ell)-1}{2k}\right|^2 \times 4\pi(2\ell+1) \\
&= \frac{4\pi}{k^2}\sum_{\ell=0}^\infty (2\ell+1)\sin^2(\delta_\ell).
\end{aligned} \tag{195}$$

Curiously, the same combination of phase shifts also govern the imaginary part of the forward scattering amplitude $f(\theta = 0)$. Indeed, for $\theta = 0$ all $P_\ell(\cos 0) = P_\ell(1) = 1$, hence

$$f(\theta = 0) \;=\; \sum_{\ell=0}^{\infty}(2\ell+1)\times\frac{e^{2i\delta_\ell}-1}{2ik} \;=\; \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\times e^{i\delta_\ell}\sin\delta_\ell\,, \tag{196}$$

and therefore the imaginary part of this forward amplitude is

$$\operatorname{Im} f(\theta = 0) \;=\; \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\times\sin^2\delta_\ell\,. \tag{197}$$

Comparing this formula to eq. (195) for the total cross-section, we immediately see that

$$\sigma_{\text{tot}} \;=\; \frac{4\pi}{k}\operatorname{Im} f(\theta = 0). \tag{198}$$

This relation is knows as the Optical Theorem.

Scattering off a Small Hard Sphere

A hard sphere is a spherical surface which cannot be penetrated by a particle or a wave. In quantum mechanics, its implemented by the infinite-wall potential

$$V(r) \;=\; \begin{cases} 0 & \text{for } r > R, \\ +\infty & \text{for } r < R. \end{cases} \tag{199}$$

Consequently, the wave-function $\psi(r,\theta,\phi)$ obeys the un-perturbed wave equation outside the sphere,

$$(\nabla^2 + k^2)\psi(r,\theta,\phi) \;=\; 0 \quad \text{for } r \;>\; R, \tag{200}$$

but also the Dirichlet boundary conditions on the sphere's surface

$$\psi(r,\theta,\phi) \;=\; 0 \quad \text{for } r \;=\; R \text{ and any } \theta,\phi. \tag{201}$$

Separating the variables in the spherical coordinates, we see that outside the sphere we have the usual

$$\psi(r,\theta) \;=\; \sum_\ell C_\ell (2\ell+1) P_\ell(\cos\theta) \times \psi_\ell(r) \tag{202}$$

where the radial ψ_ℓ are solutions of the free radial wave equations and hence linear combinations of the spherical Bessel functions. Specifically,

$$\psi_\ell(r) \;=\; \cos\delta_\ell \times j_\ell(kr) \;-\; \sin\delta_\ell \times n_\ell(kr) \tag{203}$$

for some phase shift δ_ℓ, which obtains from the Dirichlet boundary condition

$$\psi_\ell(r=R) \;=\; 0, \tag{204}$$

hence

$$\tan\delta_\ell \;=\; \frac{j_\ell(kR)}{n_\ell(kR)}\,. \tag{205}$$

Alas, this formula is not particularly transparent, so let us explore the particularly simple limit of a small hard sphere, $R \ll (1/k)$.

In this limit,

$$j_\ell(kR) \;\approx\; \frac{(kR)^\ell}{(2\ell+1)!!}\,, \qquad n_\ell(kR) \;\approx\; -\frac{(2\ell-1)!!}{(kR)^{\ell+1}}\,, \tag{206}$$

so eq. (205) for the phase shifts yields

$$\tan\delta_\ell \;=\; -\frac{(kR)^{2\ell+1}}{(2\ell-1)!!\,(2\ell+1)!!}\,. \tag{207}$$

In particular,

$$\tan\delta_0 \;\approx\; -(kR), \qquad \tan\delta_1 \;\approx\; -\frac{(kR)^3}{3} \qquad \tan\delta_2 \;\approx\; -\frac{(kR)^5}{45}\,,\;\ldots. \tag{208}$$

Note that for $kR \ll 1$ all the phase shifts are negative and small, and their magnitudes rapidly decrease with ℓ. Thus, to the leading order in (kR) we may approximate

$$\delta_0 \approx -kR, \qquad \text{other } \delta_\ell \approx 0. \tag{209}$$

In this approximation, the scattering amplitude becomes

$$f(\theta) \approx \frac{e^{2i\delta_0}-1}{2ik} \times P_0(\cos\theta) + 0 \approx \frac{2i\delta_0}{2ik} \times 1 \approx -R, \tag{210}$$

hence isotropic scattering cross-section

$$\frac{d\sigma}{d\Omega} = |f|^2 \approx R^2 \quad \text{in all directions}, \tag{211}$$

and the total scattering cross-section is

$$\sigma_{\text{tot}} = 4\pi R^2. \tag{212}$$

Note: this total scattering cross-sections is 4 times larger than the geometric cross-section $\sigma_{\text{geom}} = \pi R^2$ of the sphere in question. However, this discrepancy does not raise a paradox since one should not expect the geometric optics to work around objects of size $R \ll \lambda$.

Partial Electromagnetic Waves

Now consider scattering of the electromagnetic waves from a spherically symmetric target. Again, at large distances from the target the EM field obey the free Maxwell equations, and we are looking for solutions of the form

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}(\mathbf{x}) \xrightarrow[r\to\infty]{} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_{\text{inc}}(\mathbf{x}) + \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_{\text{sc}}(\mathbf{x}) \tag{213}$$

where the incident wave is a plane wave traveling in z direction, while the scattered wave is a divergent spherical wave. For a spherically symmetric problem we separate the variables in spherical coordinates, hence a most general solution of the wave equation becomes a superposition of the spherical TE and TM waves with all possible ℓ and m,

$$\begin{aligned}
\mathbf{E}(r,\theta,\phi) &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left[C^{\rm TE}_{\ell,m}F_\ell(r)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\,+\,\frac{i}{k}C^{\rm TM}_{\ell,m}\,\nabla\times\Big(F_\ell(r)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\Big)\right],\\
Z_0\mathbf{H}(r,\theta,\phi) &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left[C^{\rm TM}_{\ell,m}F_\ell(r)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\,-\,\frac{i}{k}C^{\rm TE}_{\ell,m}\nabla\times\Big(F_\ell(r)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\Big)\right],
\end{aligned} \tag{214}$$

where the radial profiles $F_\ell(r)$ obey the spherical Bessel equation,

$$\left(\frac{d^2}{dr^2}\,+\,\frac{2}{r}\,\frac{d}{dr}\,-\,\frac{\ell(\ell+1)}{r^2}\,+\,k^2\right)F_\ell(r)\;=\;0, \tag{215}$$

Or rather, they obey it outside of the scattering target. Consequently, outside the target, the $F_\ell(r)$ are a linear combination of spherical Bessel functions $j_\ell(kr)$ and $n_\ell(kr)$, or equivalently of the spherical Hankel functions $h_\ell = j_\ell + in_\ell$ and its conjugate $h^*_\ell = j_\ell - in_\ell$. Physically, the $h_\ell(kr)$ accounts for the divergent part of the radial wave (energy moves from the center outward) while the $h^*_\ell(kr)$ accounts for the convergent part (energy moves from the infinity to the center). For a perfectly reflecting target, these two components must have equal magnitudes, thus we should have

$$F_\ell(r)\;=\;e^{2i\delta_\ell}h_\ell(kr)\;+\;e^{-i\delta_\ell}h^*(kr)\;=\;2\cos\delta_\ell\times j_\ell(kr)\;-\;2\sin\delta_\ell\times n_\ell(kr) \tag{216}$$

for some phase shift δ_ℓ. But if the target both absorbs and scatters the incident EM power, then the convergent component should have a larger magnitude than the divergent component. Thus, in terms of the spherical Hankel functions

$$F_\ell(r)\;\propto\;h^*_\ell(kr)\,+\,\alpha_\ell\times h_\ell(kr)\;\xrightarrow[kr\gg1]{}\;i^{\ell+1}\times\frac{e^{-ikr}}{kr}\;+\;\alpha_\ell i^{-\ell-1}\times\frac{e^{+ikr}}{kr}, \tag{217}$$

where α_ℓ is a complex number of magnitude $|\alpha_\ell|\leq 1$.

The values of the reflection coefficients α_ℓ depend on the details of the scattering target and its surface, for example the radius and the surface impedance of an opaque sphere. By spherical symmetry, the α_ℓ depend on ℓ but not on m of a spherical wave. On the other hand, the TE wave and the TM wave with the same ℓ may have different reflection coefficients

$$\alpha_\ell^{\rm TE} \;\neq\; \alpha_\ell^{\rm TM}. \tag{218}$$

I wish I had enough class-time to give you an example of calculating the reflection coefficients $\alpha_\ell^{\rm TE}$ and $\alpha_\ell^{\rm TM}$ for some spherically symmetric reflector. Instead, let me simply quote Jackson's example (from §10.4) of an opaque sphere of radius a with surface wave impedance Z_s:

$$\begin{aligned} \alpha_\ell^{\rm TM} \;&=\; \frac{\left[\frac{d}{dx}\,+\,\frac{1}{x}\,+\,i\frac{Z_s}{Z_0}\right]\big(n_\ell(x)+ij_\ell(x)\big)}{\left[\frac{d}{dx}\,+\,\frac{1}{x}\,+\,i\frac{Z_s}{Z_0}\right]\big(n_\ell(x)-ij_\ell(x)\big)} \quad @\ x\,=\,ra,\\ \alpha_\ell^{\rm TE} \;&=\; \frac{\left[\frac{d}{dx}\,+\,\frac{1}{x}\,+\,i\frac{Z_0}{Z_s}\right]\big(n_\ell(x)+ij_\ell(x)\big)}{\left[\frac{d}{dx}\,+\,\frac{1}{x}\,+\,i\frac{Z_0}{Z_s}\right]\big(n_\ell(x)-ij_\ell(x)\big)} \quad @\ x\,=\,ra. \end{aligned} \tag{219}$$

Note: for $Z_s = 0$ (a perfectly conducting sphere), or for $Z_s = \infty$ (a perfect insulator), or for any other purely imaginary Z_s, there is no absorption of the EM waves but only reflection, and indeed, for all these cases eqs. (219) yield

$$|\alpha_\ell^{\rm TM,TE}| \;=\; 1 \quad\Longrightarrow\quad \alpha_\ell^{\rm TM,TE} \;=\; \exp\!\big(2i\delta_\ell^{\rm TM,TE}\big) \tag{220}$$

for some phase shifts $\delta_\ell^{\rm TM,TE}$. But for all other values of the sphere's wave impedance, there is both reflection and absorption, and eqs. (219) yield

$$|\alpha_\ell^{\rm TM,TE}| \;<\; 1. \tag{221}$$

In particular, for a small opaque sphere with $kR \ll 1$, eqs. (219) yield

$$\alpha_\ell^{\rm TM} \;=\; 1\,+\,\frac{2(kR)^{2\ell+1}}{(2\ell-1)!!\,(2\ell+1)!!}\times\left[\frac{\ell+1}{\ell}i\,-\,\frac{(2\ell+1)}{\ell^2}(kR)\times\frac{Z_s}{Z_0}\,+\,O\big((kR)^2\big)\right], \tag{222}$$

$$\alpha_\ell^{\rm TE} \;=\; 1\,+\,\frac{2(kR)^{2\ell+1}}{(2\ell-1)!!\,(2\ell+1)!!}\times\left[\frac{\ell+1}{\ell}i\,-\,\frac{(2\ell+1)}{\ell^2}(kR)\times\frac{Z_0}{Z_s}\,+\,O\big((kR)^2\big)\right], \tag{223}$$

and hence

$$
\begin{aligned}
1 - \left|\alpha_\ell^{\rm TM}\right|^2 &\approx \frac{4(kr)^{2\ell+2}}{[(2\ell-1)!!\,\ell]^2} \times \mathrm{Re}\left(\frac{Z_s}{Z_0}\right), && (224)\\
1 - \left|\alpha_\ell^{\rm TM}\right|^2 &\approx \frac{4(kr)^{2\ell+2}}{[(2\ell-1)!!\,\ell]^2} \times \mathrm{Re}\left(\frac{Z_0}{Z_s}\right). && (225)
\end{aligned}
$$

⋆ ⋆ ⋆

Anyhow, given the complex reflection coefficients $\alpha_\ell^{\rm TM}$ and $\alpha_\ell^{\rm TE}$, we may decompose the most general harmonic EM wave outside the scattering target into a superposition of spherical TE and TM waves,

$$
\begin{aligned}
\mathbf{E}(r,\theta,\phi) &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left[\begin{array}{c} C_{\ell,m}^{\rm TE}\Big(h_\ell^*(kr)+\alpha_\ell^{\rm TE}h_\ell(kr)\Big)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi) \\ +\,\frac{i}{k}C_{\ell,m}^{\rm TM}\nabla\times\Big(\Big(h_\ell^*(kr)+\alpha_\ell^{\rm TM}h_\ell(kr)\Big)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\Big)\end{array}\right], \\
Z_0\mathbf{H}(r,\theta,\phi) &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell}\left[\begin{array}{c} C_{\ell,m}^{\rm TM}\Big(h_\ell^*(kr)+\alpha_\ell^{\rm TM}h_\ell(kr)\Big)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi) \\ -\,\frac{i}{k}C_{\ell,m}^{\rm TE}\nabla\times\Big(\Big(h_\ell^*(kr)+\alpha_\ell^{\rm TE}h_\ell(kr)\Big)*\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)\Big)\end{array}\right],
\end{aligned}
\qquad (226)
$$

for some general coefficients $C_{\ell,m}^{\rm TE}$ and $C_{\ell,m}^{\rm TM}$. To find these coefficients for the scattering solution (213), we start by decomposing the incident plane wave into spherical TM and TE waves. Since the incident wave is regular at the origin, in its decomposition all the radial profiles are also regular at the origin, which means they must be proportional to the $j_\ell(kr)$. As to their angular dependence, for a scalar wave the axial symmetry of the incident wave excluded all the modes with $m \neq 0$. For the EM wave, the analysis is more involved due to the polarization vector $\mathbf{E}_0$ of the incident wave and its interplay with the $\hat{\mathbf{L}}$ operator acting on the spherical harmonics $Y_{\ell,m}(\theta,\phi)$. Consequently, instead of restricting m to zero we end up with a restriction to $m = \pm 1$ only. Furthermore, for the circularly polarized incident wave we have only one allowed value of m:

$$
\begin{aligned}
&\text{for}\quad \mathbf{E}_0 = \frac{E_0}{\sqrt{2}}(1,+i,0), \quad \text{only} \quad m = +1, \\
&\text{for}\quad \mathbf{E}_0 = \frac{E_0}{\sqrt{2}}(1,-i,0), \quad \text{only} \quad m = -1.
\end{aligned}
\qquad (227)
$$

Thus,

$$\begin{aligned}
\mathbf{E}^{\rm inc}_{L,R}(\mathbf{x}) &= \frac{E_0}{\sqrt{2}}(1,\pm i,0)e^{ikr\cos\theta}\\
&= \sum_{\ell=1}^{\infty}\left[A^{\rm TE}_\ell j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\,+\,\frac{i}{k}A^{\rm TM}_\ell\,\nabla\times\Big(j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Big)\right],\\
Z_0\mathbf{H}^{\rm inc}_{L,R}(\mathbf{x}) &= \mp i\mathbf{E}^{\rm inc}_{L,R}(\mathbf{x})\\
&= \sum_{\ell=1}^{\infty}\left[A^{\rm TM}_\ell j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\,-\,\frac{i}{k}A^{\rm TE}_\ell\,\nabla\times\Big(j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Big)\right],
\end{aligned}\tag{228}$$

for some coefficients $A^{\rm TE}_\ell$ and $A^{\rm TM}_\ell$. Let me skip the calculation of these coefficients; it's spelled out in §10.3 of Jackson's textbook. Translating Jackson's formulae into the notations of these notes, we have

$$A^{\rm TE}_\ell = E_0N_\ell i^\ell,\quad A^{\rm TM}_\ell = E_0N_\ell i^{\ell\mp1},\quad \text{for}\quad N_\ell = \sqrt{\frac{2\pi(2\ell+1)}{\ell(\ell+1)}},\tag{229}$$

hence

$$\begin{aligned}
\mathbf{E}^{\rm inc}(r,\theta,\phi) &= E_0\sum_{\ell=1}^{\infty}N_\ell i^\ell\left[j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\pm\frac{1}{k}\nabla\times\Big(j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Big)\right],\\
\mathbf{H}^{\rm inc}(r,\theta,\phi) &= \frac{E_0}{Z_0}\sum_{\ell=1}^{\infty}N_\ell i^{\ell\mp1}\left[j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\pm\frac{1}{k}\nabla\times\Big(j_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Big)\right],
\end{aligned}\tag{230}$$

As to the scattered EM wave, it has the same (ℓ,m) modes as the incident wave: all $\ell=1,2,3,\ldots$, but only $m=+1$ or only $m=-1$, depending on the incident wave's helicity. Also, the scattered wave is purely divergent, so the radial profiles of all the modes are proportional to the $h_\ell(kr)$ without any contribution from the $h^*_\ell(kr)$. Altogether, this means

$$\begin{aligned}
\mathbf{E}^{\rm sc}(r,\theta,\phi) &= E_0\sum_{\ell=1}^{\infty}N_\ell i^\ell\left[\begin{array}{c} iB^{TE}_\ell*h_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\\ \pm\frac{i}{k}B^{TM}_\ell*\nabla\times\Big(h_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Big)\end{array}\right],\\
\mathbf{H}^{\rm sc}(r,\theta,\phi) &= \frac{E_0}{Z_0}\sum_{\ell=1}^{\infty}N_\ell i^{\ell\mp1}\left[\begin{array}{c} iB^{TM}_\ell*h_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\\ \pm\frac{i}{k}B^{TE}_\ell*\nabla\times\Big(h_\ell(kr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Big)\end{array}\right],
\end{aligned}\tag{231}$$

for some complex coefficients B_ℓ^{TE} and B_ℓ^{TM}. Altogether, the net incident + scattered wave is

$$\begin{aligned}
\mathbf{E}^{\rm net}(r,\theta,\phi) &= E_0\sum_{\ell=1}^\infty N_\ell i^\ell \begin{bmatrix} \bigl(j_\ell(kr)\,+\,iB_\ell^{\rm TE}\times h_\ell(kr)\bigr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi) \\ \pm\frac{1}{k}\nabla\times\Bigl(\bigl(j_\ell(kr)\,+\,iB_\ell^{\rm TM}\times h_\ell(kr)\bigr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Bigr)\end{bmatrix},\\
\mathbf{H}^{\rm net}(r,\theta,\phi) &= \frac{E_0}{Z_0}\sum_{\ell=1}^\infty N_\ell i^{\ell\mp1} \begin{bmatrix} \bigl(j_\ell(kr)\,+\,iB_\ell^{\rm TM}\times h_\ell(kr)\bigr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi) \\ \pm\frac{1}{k}\nabla\times\Bigl(\bigl(j_\ell(kr)\,+\,iB_\ell^{\rm TE}\times h_\ell(kr)\bigr)*\hat{\mathbf{L}}Y_{\ell,\pm1}(\theta,\phi)\Bigr)\end{bmatrix}.
\end{aligned}\tag{232}$$

At the same time, this scattering solution should have form (226) where the radial profile of each TM or TE mode has the right ratio $\alpha_\ell^{\rm TE\,or\,TM}$ between the incoming and the outgoing waves. Thus, for each mode we should have

$$\begin{aligned}
E_0N_\ell i^\ell\Bigl(j_\ell(kr)\,+\,iB_\ell^{\rm TE}\times h_\ell(kr)\Bigr) &= C_{\ell,\pm1}^{\rm TE}\times\Bigl(h_\ell^*(kr)\,+\,\alpha_\ell^{\rm TE}\times h_\ell(kr)\Bigr),\\
E_0N_\ell i^{\ell\mp1}\Bigl(j_\ell(kr)\,+\,iB_\ell^{\rm TM}\times h_\ell(kr)\Bigr) &= C_{\ell,\pm1}^{\rm TM}\times\Bigl(h_\ell^*(kr)\,+\,\alpha_\ell^{\rm TM}\times h_\ell(kr)\Bigr).
\end{aligned}\tag{233}$$

Using

$$j_\ell(kl) \;=\; \tfrac{1}{2}h_\ell(kr)\;+\;\tfrac{1}{2}h_\ell^*(kr)\tag{234}$$

and matching the coefficients of $h_\ell^*(kr)$ and $h_\ell(kr)$ on both sides of eqs. (233), we find all the C_ℓ and B_ℓ coefficients in terms of the α_ℓ. Specifically,

$$C_{\ell,\pm1}^{\rm TE} \;=\; \tfrac{1}{2}E_0N_\ell i^\ell,\qquad C_{\ell,\pm1}^{\rm TM} \;=\; \tfrac{1}{2}E_0N_\ell i^{\ell\mp1},\tag{235}$$

while

$$B_\ell^{\rm TE} \;=\; \frac{\alpha_\ell^{\rm TE}-1}{2i}\,,\qquad B_\ell^{\rm TM} \;=\; \frac{\alpha_\ell^{\rm TM}-1}{2i}\,.\tag{236}$$

Plugging these coefficients back into eq. (231) for the scattered wave, we arrive at

$$\begin{aligned}
\mathbf{E}_{\rm sc}(\mathbf{x}) &= E_0\sum_{\ell=1}^{\infty} N_\ell i^{\ell+1}\left[\begin{array}{l} \dfrac{\alpha_\ell^{TE}-1}{2i} * h_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,\pm1}(\theta,\phi) \\ \pm\, \dfrac{\alpha_\ell^{TM}-1}{2ik} * \nabla\times\Big(h_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,\pm1}(\theta,\phi)\Big)\end{array}\right] \\
\mathbf{H}_{\rm sc}(\mathbf{x}) &= \frac{E_0}{Z_0}\sum_{\ell=1}^{\infty} N_\ell i^{\ell}\left[\begin{array}{l} \pm\dfrac{\alpha_\ell^{TM}-1}{2i} * h_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,\pm1}(\theta,\phi) \\ +\, \dfrac{\alpha_\ell^{TE}-1}{2ik} * \nabla\times\Big(h_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,\pm1}(\theta,\phi)\Big)\end{array}\right]
\end{aligned} \tag{237}$$

where N_ℓ is as in eq. (229).

In the far zone of $kr \gg 1$, we may approximate

$$\begin{aligned}
i^{\ell+1} h_\ell(kr) &\approx \frac{e^{ikr}}{kr}, \quad \text{same for all } \ell, \\
\nabla\times\Big(i^{\ell+1} h_\ell(kr) * \hat{\mathbf{L}} Y_{\ell,\pm1}\Big) &\approx \frac{e^{ikr}}{kr} * ik\mathbf{n}\times\hat{\mathbf{L}} Y_{\ell,\pm1}\,.
\end{aligned} \tag{238}$$

Consequently, the far-zone scattered fields (237) become

$$\begin{aligned}
\mathbf{E}_{\rm sc} &= E_0\frac{e^{ikr}}{r}\sum_{\ell=1}^{\infty} N_\ell\left[\frac{\alpha_\ell^{TE}-1}{2ik} * \hat{\mathbf{L}} Y_{\ell,\pm1}(\mathbf{n}) \pm \frac{\alpha_\ell^{TM}-1}{2ik} * i\mathbf{n}\times\hat{\mathbf{L}} Y_{\ell,\pm1}(\mathbf{n})\right], \\
\mathbf{H}_{\rm sc} &= \frac{E_0}{Z_0}\frac{e^{ikr}}{r}\sum_{\ell=1}^{\infty} N_\ell\left[\mp i\frac{\alpha_\ell^{TM}-1}{2ik} * \hat{\mathbf{L}} Y_{\ell,\pm1}(\mathbf{n}) + \frac{\alpha_\ell^{TE}-1}{2ik} * \mathbf{n}\times Y_{\ell,\pm1}(\mathbf{n})\right],
\end{aligned} \tag{239}$$

or in other words,

$$\mathbf{E}_{\rm sc} = E_0\frac{e^{ikr}}{r} * \mathbf{f}(\mathbf{n}), \qquad \mathbf{H}_{\rm sc} = -\frac{E_0}{Z_0}\frac{e^{ikr}}{r} * \mathbf{n}\times\mathbf{f}(\mathbf{n}), \tag{240}$$

for the *scattering amplitude*

$$\mathbf{f}(\mathbf{n}) = \sum_{\ell=1}^{\infty} N_\ell\left[\frac{\alpha_\ell^{TE}-1}{2ik} * \hat{\mathbf{L}} Y_{\ell,\pm1}(\mathbf{n}) + \frac{\alpha_\ell^{TM}-1}{2ik} * (\pm i\mathbf{n})\times\hat{\mathbf{L}} Y_{\ell,\pm1}(\mathbf{n})\right]. \tag{241}$$

In terms of this scattering amplitude,

$$\mathbf{S}_{\mathrm{sc}} = \frac{|E_0|^2}{2Z_0}\,|\mathbf{f}|^2\,\frac{\mathbf{n}}{r^2} \tag{242}$$

while $S_{\mathrm{inc}} = (|E_0|^2/2Z_0)$, hence partial scattering cross-section

$$\frac{d\sigma}{d\Omega} = \frac{r^2\mathbf{S}_{\mathrm{sc}}\cdot\mathbf{n}}{S_{\mathrm{inc}}} = |\mathbf{f}(\mathbf{n})|^2. \tag{243}$$

To integrate this partial cross-section over the directions, we use

$$\begin{aligned}
\oiint d^2\Omega\,\left(\hat{\mathbf{L}}Y_{\ell,\pm1}(\mathbf{n})\right)^*\cdot\left(\hat{\mathbf{L}}Y_{\ell',\pm1}(\mathbf{n})\right) &= \delta_{\ell,\ell'}\times\ell(\ell+1),\\
\oiint d^2\Omega\,\left(\hat{\mathbf{L}}Y_{\ell,\pm1}(\mathbf{n})\right)^*\cdot\left(i\mathbf{n}\times\hat{\mathbf{L}}Y_{\ell',\pm1}(\mathbf{n})\right) &= 0,\\
\oiint d^2\Omega\,\left(i\mathbf{n}\times\hat{\mathbf{L}}Y_{\ell,\pm1}(\mathbf{n})\right)^*\cdot\left(i\mathbf{n}\times\hat{\mathbf{L}}Y_{\ell',\pm1}(\mathbf{n})\right) &= \delta_{\ell,\ell'}\times\ell(\ell+1).
\end{aligned} \tag{244}$$

Consequently, in the integral

$$\oiint d^2\Omega\left(\mathbf{f}(\mathbf{n}) = \sum \text{terms}\right)^*\cdot\left(\mathbf{f}(\mathbf{n}) = \sum{}' \text{terms}'\right) = \sum\sum{}'\oiint d^2\Omega(\text{term})^*\cdot(\text{term}') \tag{245}$$

all integrals with term$'\neq$ term vanish, while the remaining integrals add up to

$$\sum_{\ell=1}^{\infty} N_\ell^2\times\ell(\ell+1)\times\left(\left|\frac{\alpha_\ell^{\mathrm{TE}}-1}{2ik}\right|^2 + \left|\frac{\alpha_\ell^{\mathrm{TM}}-1}{2ik}\right|^2\right) = \frac{2\pi}{4k^2}\sum_\ell(2\ell+1)\left(|\alpha_\ell^{\mathrm{TE}}-1|^2+|\alpha_\ell^{\mathrm{TM}}-1|^2\right). \tag{246}$$

Altogether, the net scattering cross-section of EM waves is

$$\sigma_{\mathrm{scattering}}^{\mathrm{net}} = \frac{2\pi}{4k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(|\alpha_\ell^{\mathrm{TE}}-1|^2 + |\alpha_\ell^{\mathrm{TM}}-1|^2\right). \tag{247}$$

In particular, if there is no absorption but only scattering, then

$$\alpha_\ell^{\rm TE} \;=\; \exp\bigl(2i\delta_\ell^{\rm TE}\bigr), \quad \alpha_\ell^{\rm TM} \;=\; \exp\bigl(2i\delta_\ell^{\rm TM}\bigr), \tag{248}$$

for some phase shifts $\delta_\ell^{\rm TE}$ and $\delta_\ell^{\rm TM}$, and consequently

$$\sigma_{\rm scattering}^{\rm net} = \frac{2\pi}{k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\Bigl(\sin^2\bigl(\delta_\ell^{\rm TE}\bigr) \,+\, \sin^2\bigl(\delta_\ell^{\rm TM}\bigr)\Bigr). \tag{249}$$

On the other hand, suppose some (or all) $|\alpha_\ell| < 1$, and let's calculate the net absorption cross-section

$$\sigma_{\rm absorption}^{\rm net} \stackrel{\rm def}{=} \frac{P_{\rm absorbed}}{S_{\rm incident}}\,. \tag{250}$$

Let's go back a single partial wave $\mathrm{TE}_{\ell,m}$ or $\mathrm{TM}_{\ell,m}$ with

$$\begin{aligned}\begin{pmatrix}\mathbf{E}\text{ or}\\ Z_0\mathbf{H}\end{pmatrix}(\mathbf{x}) \;&=\; C_{\ell,m}\Bigl(h_\ell^*(kr) \,+\, \alpha_\ell\times h_\ell(kr)\Bigr) * \hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n})\\ &\xrightarrow[kr\gg1]{}\; \frac{C_{\ell,m}}{kr}\Bigl(i^{\ell+1}e^{-ikr} \,+\, \alpha_\ell\times i^{-\ell-1}e^{+ikr}\Bigr) * \hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n}).\end{aligned} \tag{251}$$

In the wave-packet analysis, the convergent wave e^{-ikr} appears at early times $t \approx -r/c$ while the divergent wave appears at later times $t \approx +r/c$, so we may ignore the interference between the two waves. Instead, we may treat their respective wave powers separate from each other. Thus, for $kr \gg 1$,

$$\begin{aligned}\mathbf{S}_{\rm conv} \;&\approx\; \frac{|C_{\ell,m}|^2}{2Z_0k^2}\,\frac{-\mathbf{n}}{r^2} * |\hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n})|^2,\\ \mathbf{S}_{\rm div} \;&\approx\; \frac{|C_{\ell,m}|^2}{2Z_0k^2}\,\frac{+\mathbf{n}}{r^2} * |\hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n})|^2 * |\alpha_\ell|^2,\end{aligned} \tag{252}$$

hence after integrating over the directions

$$\begin{aligned}P_{\rm conv} \;&=\; \frac{\ell(\ell+1)|C_{\ell,m}|^2}{2Z_0k^2}\,,\\ P_{\rm div} \;&=\; \frac{\ell(\ell+1)|C_{\ell,m}|^2}{2Z_0k^2} * |\alpha_\ell|^2.\end{aligned} \tag{253}$$

The difference between the convergent and the divergent powers is absorbed by the scattering target, thus

$$P_{\rm abs} \;=\; \frac{\ell(\ell+1)|C_{\ell,m}|^2}{2Z_0k^2} * (1-|\alpha_\ell|^2). \tag{254}$$

Next, consider a superposition of several (or infinitely many) spherical TM or TE waves with different (ℓ, m). The interference between these waves causes complicated angular dependence of the convergent and divergent wave powers. But thanks to eqs. (15), one we integrate over the directions, all the interference terms cancel out, and the net convergent and divergent powers (253) simply add up,

$$\begin{aligned} P^{\rm net}_{\rm conv} &= \sum_{\ell,m}\sum_{TE,TM} \frac{\ell(\ell+1)|C_{\ell,m}|^2}{2Z_0k^2}\,, \\ P^{\rm net}_{\rm div} &= \sum_{\ell,m}\sum_{TE,TM} \frac{\ell(\ell+1)|C_{\ell,m}|^2}{2Z_0k^2} * |\alpha_\ell|^2, \end{aligned} \tag{255}$$

and likewise, the net absorbed power is

$$P^{\rm net}_{\rm abs} = \sum_{\ell,m}\sum_{TE,TM} \frac{\ell(\ell+1)|C_{\ell,m}|^2}{2Z_0k^2} * (1-|\alpha_\ell|^2). \tag{256}$$

In the context of the scattering solution — the plane incident wave plus the scattered wave — the amplitudes $C_{\ell,m}$ are

$$C^{\rm TE}_{\ell,\pm1} \;=\; \tfrac12 E_0 N_\ell i^\ell, \qquad C^{\rm TM}_{\ell,\pm1} \;=\; \tfrac12 E_0 N_\ell i^{\ell\mp1}, \tag{235}$$

hence the net convergent power is

$$P^{\rm net}_{\rm conv} \;=\; \frac{|E_0|^2}{8Z_0k^2}\sum_{\ell=1}^\infty \ell(\ell+1)N_\ell^2\times 2, \tag{257}$$

while the net divergent power is

$$P^{\rm net}_{\rm div} \;=\; \frac{|E_0|^2}{8Z_0k^2}\sum_{\ell=1}^\infty \ell(\ell+1)N_\ell^2\times \left(\left|\alpha^{\rm TE}_\ell\right|^2 + \left|\alpha^{\rm TM}_\ell\right|^2\right). \tag{258}$$

Depending on both magnitudes and phases of the reflection coefficients $\alpha_\ell^{\rm TE,TM}$, some of this divergent power rejoins the incident wave while the rest becomes the scattered wave.

Finally, the difference between the net converging power and the net diverging power is the net absorbed power, thus

$$P_{\rm abs}^{\rm net} = \frac{|E_0|^2}{8Z_0k^2}\sum_{\ell=1}^{\infty}\ell(\ell+1)N_\ell^2\times\left(2-\left|\alpha_\ell^{\rm TE}\right|^2-\left|\alpha_\ell^{\rm TM}\right|^2\right). \tag{259}$$

In this formula

$$\ell(\ell+1)N_\ell^2 = 2\pi(2\ell+1) \tag{260}$$

while

$$\frac{|E_0|^2}{2Z_0} = S_{\rm inc}\,, \tag{261}$$

hence

$$P_{\rm abs}^{\rm net} = \frac{2\pi S_{\rm inc}}{4k^2}\times\sum_{\ell=1}^{\infty}(2\ell+1)\times\left(2-\left|\alpha_\ell^{\rm TE}\right|^2-\left|\alpha_\ell^{\rm TM}\right|^2\right) \tag{262}$$

and therefore the net absorption cross-section

$$\sigma_{\rm abs}^{\rm net} = \frac{2\pi}{4k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\times\left(2-\left|\alpha_\ell^{\rm TE}\right|^2-\left|\alpha_\ell^{\rm TM}\right|^2\right). \tag{263}$$

For a continuous medium made of many scattering bodies, both scattering and absorption contribute to the attenuation of the incident wave,

$$S_{\rm inc} = S_0\exp(-n\sigma_1 z) \tag{264}$$

where n is the density of the scattering bodies, and σ_1 is the net cross-section of intercepting the incident wave by a single body,

$$\begin{aligned}\sigma_{\rm interception} &= \sigma_{\rm scattering}^{\rm net} + \sigma_{\rm absorption}^{\rm net}\\ &= \frac{2\pi}{4k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(\left|\alpha_\ell^{\rm TE}-1\right|^2+\left|\alpha_\ell^{\rm TM}-1\right|^2\right)\\ &\quad+\frac{2\pi}{4k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(2-\left|\alpha_\ell^{\rm TE}\right|^2-\left|\alpha_\ell^{\rm TM}\right|^2\right)\\ &= \frac{\pi}{k^2}\sum_{\ell=1}^{\infty}(2\ell+1)\left(2-{\rm Re}\,\alpha_\ell^{\rm TE}-{\rm Re}\,\alpha_\ell^{\rm TM}\right).\end{aligned} \tag{265}$$

Note: it is this interception cross-section that is related to the forward scattering amplitude by the optical theorem:

$$\sigma_{\text{interception}} \;=\; \frac{4\pi}{k}\,\mathrm{Im}\big(\mathbf{e}_0^*\cdot\mathbf{f}_E(\theta=0)\big). \tag{266}$$

Indeed, for a circularly polarized incident wave with $\mathbf{e}_0 = (1,\pm i,0)/\sqrt{2}$ and $\mathbf{n}=\mathbf{n}_0=(0,0,1)$, we have

$$\mathbf{e}_0^*\cdot\hat{\mathbf{L}} \;=\; \mathbf{e}_0^*\cdot\big(\pm i\mathbf{n}\times\hat{\mathbf{L}}\big) \;=\; \frac{1}{\sqrt{2}}\hat{L}_{\mp}\,, \tag{267}$$

hence

$$\mathbf{e}_0^*\cdot\mathbf{f}_E(\theta=0) \;=\; \sum_{\ell=1}^{\infty}\frac{N_\ell}{\sqrt{2}}\left(\frac{\alpha_\ell^{\mathrm{TE}}-1}{2ik}+\frac{\alpha_\ell^{\mathrm{TM}}-1}{2ik}\right)\hat{L}_{\mp}Y_{\ell,\pm1}(\theta=0). \tag{268}$$

In this formula,

$$\hat{L}_{\mp}Y_{\ell,\pm1}(\theta,\phi) = \sqrt{\ell(\ell+1)}Y_{\ell,0}(\theta,\phi) \;=\; \sqrt{\ell(\ell+1)}\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta) \tag{269}$$

where $P_\ell(\cos\theta)$ becomes 1 for $\theta=0$. Consequently,

$$\frac{N_\ell}{\sqrt{2}}\,\hat{L}_{\mp}Y_{\ell,\pm1}(\theta=0) \;=\; \sqrt{\frac{2\pi(2\ell+1)}{2\ell(\ell+1)}} * \sqrt{\ell(\ell+1)}\sqrt{\frac{2\ell+1}{4\pi}} \;=\; \frac{2\ell+1}{2}, \tag{270}$$

and therefore

$$\mathbf{e}_0^*\cdot\mathbf{f}_E(\theta=0) \;=\; \frac{1}{4ik}\sum_{\ell=1}^{\infty}(2\ell+1)\Big(\alpha_\ell^{\mathrm{TE}}+\alpha_\ell^{\mathrm{TM}}-2\Big). \tag{271}$$

Taking the imaginary part of this forward scattering amplitude, we get

$$\mathrm{Im}\big[\mathbf{e}_0^*\cdot\mathbf{f}_E(\theta=0)\big] \;=\; \frac{1}{4k}\sum_{\ell=1}^{\infty}(2\ell+1)\Big(2-\mathrm{Re}\,\alpha_\ell^{\mathrm{TE}}-\mathrm{Re}\,\alpha_\ell^{\mathrm{TM}}\Big), \tag{272}$$

and comparing this formula to eq. (265) for the total interception cross-section we immediately see that

$$\sigma_{\text{interception}} \;=\; \frac{4\pi}{k}\,\mathrm{Im}\big(\mathbf{e}_0^*\cdot\mathbf{f}_E(\theta=0)\big). \tag{266}$$

Quod erat demonstrandum.