Magnetization, Bound Currents, and the H Field

In magnetic materials, atoms and molecules may have non-zero average magnetic moments $\langle\mathbf{m}\rangle$. Such magnetic moments are usually induced by an external magnetic fields, but in permanent magnets they can remain long after the external field is switched off. One way or the other, the atomic/molecular magnetic moments lead to macroscopic *magnetization*

$$\mathbf{M} \;=\; \frac{\text{net magnetic dipole moment}}{\text{volume}}\,. \tag{1}$$

In these notes, we shall explore the macroscopic magnetic field $\mathbf{B}(x,y,z)$ due to such magnetization.

Let's start with the vector potential $\mathbf{A}(x,y,z)$ due to a single magnetic dipole $\mathbf{m}$,

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\,\mathbf{m}\times\frac{\hat{\mathbf{r}}}{r^2} \;=\; \frac{\mu_0}{4\pi}\,\mathbf{m}\times\nabla\left(\frac{-1}{r}\right). \tag{2}$$

By the superposition principle, the vector potential of a bunch of magnetic dipoles is a sum of potentials like (2), namely

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\sum_i^{\text{dipoles}}\mathbf{m}_i\times\nabla_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}_i|}\right). \tag{3}$$

Likewise, for a continuous distribution of dipoles with macroscopic magnetization $\mathbf{M}(x,y,z)$, we have

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\mathbf{M}(\mathbf{r}')\times\nabla_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right). \tag{4}$$

To simplify this expression, we note that

$$\nabla_{\mathbf{r}}\left(\frac{-1}{|\mathbf{r}-\mathbf{r}'|}\right) \;=\; \nabla_{\mathbf{r}'}\left(\frac{+1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{5}$$

and then integrate by parts:

$$\mathbf{M}(\mathbf{r}')\times\nabla_{\mathbf{r}'}\left(\frac{+1}{|\mathbf{r}-\mathbf{r}'|}\right) \;=\; -\nabla_{\mathbf{r}'}\left(\frac{+1}{|\mathbf{r}-\mathbf{r}'|}\right)\times\mathbf{M}(\mathbf{r}') \;=\; -\nabla_{\mathbf{r}'}\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) \;+\; \frac{\nabla\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{6}$$

Applying these formulae to the integrand of eq. (4), we arrive at

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\nabla\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;-\;\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\nabla_{\mathbf{r}'}\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right). \tag{7}$$

Moreover, the second term on the RHS can be rewritten as a surface integral using the Lemma I shall prove in the Appendix to these notes, namely *for any vector field* $\mathbf{T}(\mathbf{r})$ *and any volume* $\mathcal{V}$ *with surface* $\mathcal{S}$*,*

$$\iiint\limits_{\mathcal{V}}(\nabla\times\mathbf{T})d^3\mathrm{Vol} \;=\; -\iint\limits_{\mathcal{S}}\mathbf{T}\times\mathbf{d^2A} \;=\; -\iint\limits_{\mathcal{S}}(\mathbf{T}(\mathbf{r})\times\mathbf{n}(\mathbf{r}))\,d^2A \tag{8}$$

where $\mathbf{n}(\mathbf{r})$ *is a unit vector normal to to the surface* $\mathcal{S}$ *at point* $\mathbf{r}$. In the present context, $\mathbf{M}(r')/|\mathbf{r}-\mathbf{r}'|$ plays the role of $\mathbf{T}(\mathbf{r}')$, $\mathcal{V}$ is the volume occupied by the magnetic material and $\mathcal{S}$ is its outer surface. Thus altogether, the vector potential (7) becomes

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint\limits_{\mathcal{V}} d^3\mathrm{Vol}'\,\frac{\nabla\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;+\;\frac{\mu_0}{4\pi}\iint\limits_{\mathcal{S}} d^2A\,\frac{\mathbf{M}(\mathbf{r}')\times\mathbf{n}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{9}$$

which has the form of a vector potential created by the volume current

$$\mathbf{J}_b(\mathbf{r}') \;=\; \nabla\times\mathbf{M}(\mathbf{r}') \tag{10}$$

and the surface current

$$\mathbf{K}_b(\mathbf{r}') \;=\; \mathbf{M}(\mathbf{r}')\times\mathbf{n}(\mathbf{r}'); \tag{11}$$

indeed, in terms of these current, the potential (9) becomes

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint\limits_{\mathcal{V}} d^3\mathrm{Vol}'\,\frac{\mathbf{J}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;+\;\frac{\mu_0}{4\pi}\iint\limits_{\mathcal{S}} d^2A\,\frac{\mathbf{K}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{12}$$

The currents (10) and (11) are called the *bound currents*, by analogy with the bound charges in a dielectric.

- EXAMPLE: A uniformly magnetized cylinder, magnetization $\mathbf{M}$ in the direction of the cylinder's axis.

$$\begin{aligned} \text{inside the cylinder} \quad \mathbf{J}_b &= \nabla\times\mathbf{M} = \mathbf{0}, \\ \text{on the top and the bottom disks} \quad \mathbf{K}_b &= \mathbf{M}\times\mathbf{n} = \mathbf{0}, \\ \text{on the outer wall} \quad \mathbf{K}_b &= \mathbf{M}\times\mathbf{n} = M\hat{\mathbf{z}}\times\hat{\mathbf{s}} = M\hat{\phi}. \end{aligned} \tag{13}$$

Thus, the magnetic field due to magnetization is the same as the field due to a surface current $\mathbf{K} = M\hat{\phi}$ flowing in circles around the cylinder's outer surface, — which is similar to the free current flowing around a solenoid. Consequently, the magnetic field of the magnetized cylinder is similar to the solenoid's field; in particular, for a cylinder whose length L is much longer than its diameter $2R$, the magnetic field is

$$\begin{aligned} \text{inside the cylinder} \quad \mathbf{B} &\approx \mu_0 K\hat{\mathbf{z}} = \mu_0\mathbf{M}, \\ \text{outside the cylinder} \quad \mathbf{B} &\approx \mathbf{0}. \end{aligned} \tag{14}$$

- EXAMPLE: A uniformly magnetized ball. Let's use spherical coordinates where $\mathbf{M}$ points towards the 'North pole' $\theta = 0$. Then

$$\begin{aligned} \text{inside the ball} \quad \mathbf{J}_b &= \nabla\times\mathbf{M} = \mathbf{0}, \\ \text{on the spherical surface} \quad \mathbf{K} &= \mathbf{M}\times\mathbf{n} = M\hat{\mathbf{z}}\times\hat{\mathbf{r}} = M\sin\theta\,\hat{\phi}. \end{aligned} \tag{15}$$

Thus, the magnetic field due to uniform magnetization of the ball is the same as the field of a surface current $\mathbf{K} = M\sin\theta\hat{\phi}$ flowing in circles along the latitudes of the sphere. The specific form of this surface current is similar to the rotating charged sphere example from my notes on the vector potential, with M playing the role of $\omega R\sigma$. Consequently, proceeding exactly as in that example, we find the magnetic field inside the magnetized ball to be uniform

$$\mathbf{B}_{\text{inside}} = \frac{2}{3}\,\mu_0\mathbf{M}, \tag{16}$$

while outside the ball we have pure dipole field

$$\mathbf{B}_{\text{outside}}(\mathbf{r}) = \frac{\mu_0 R^3}{3}\,\frac{3(\mathbf{M}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{M}}{r^3} \tag{17}$$

corresponding to the net magnetic moment

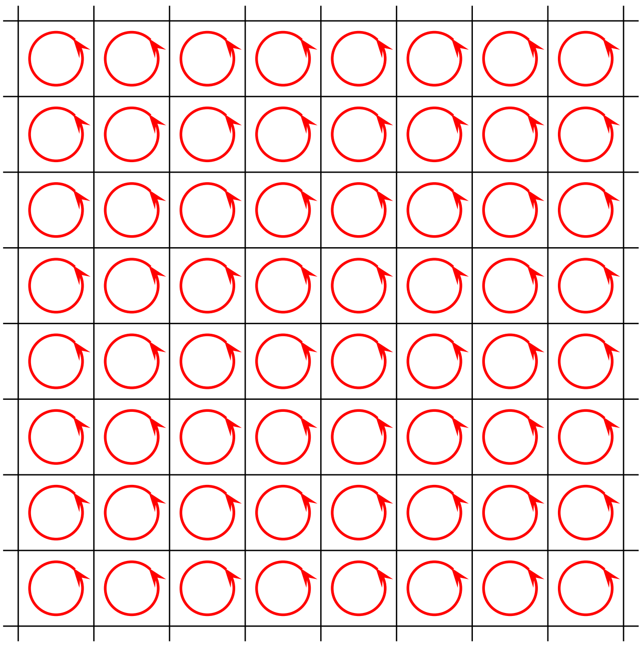
$$\mathbf{m}_{\text{net}} = \frac{4\pi R^3}{3}\mathbf{M} = \text{Volume(ball)}\,\mathbf{M}. \tag{18}$$

Origin of the Bound Currents

Physically, the bound currents on the surface of a magnetized material (and also in its volume for a non-uniform $\mathbf{M}$) originate from the mis-cancellation of the microscopic currents inside the atoms which give rise to their magnetic moments. As a model of how this works, consider a large $L \times L \times L$ a uniformly magnetized material with $\mathbf{M}$ pointing in the z direction of the cube. For simplicity, let's assume the material in question is a crystal with a simple cubic lattice — each tiny $a \times a \times a$ cube occupied by a single atom, — and further more, assume each atom has the same magnetic moment

$$\mathbf{m} = a^3\mathbf{M}. \tag{19}$$

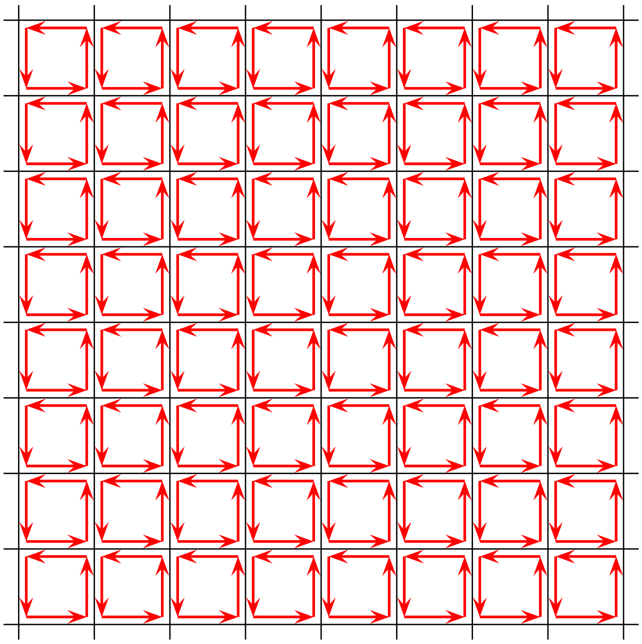
The picture below shows a single slice of this cubic lattice along the (x, y) plane, or rather a small part of that slice:



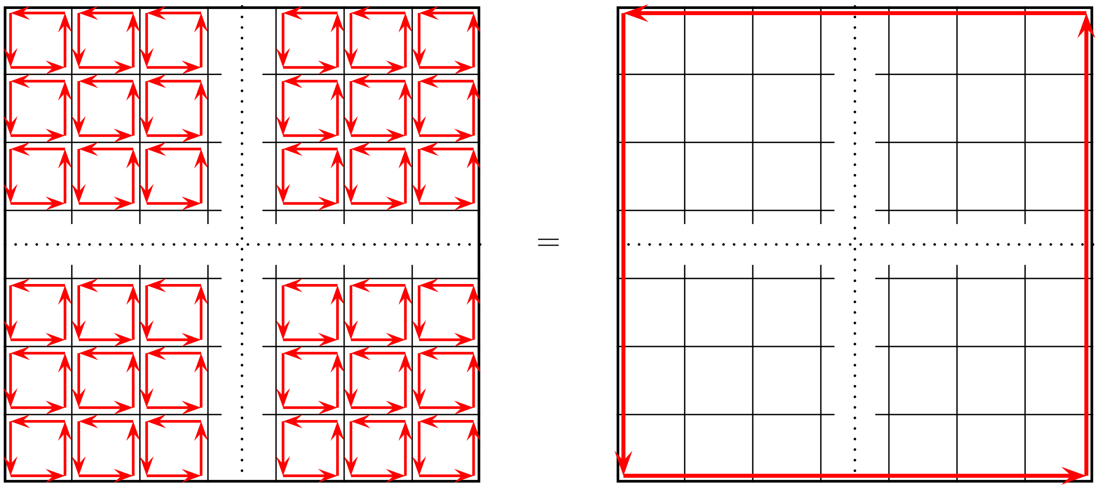
The red loops here stand for the current loops creating the atomic magnetic moments $\mathbf{m}$. We do not know the radii or even the shapes of these loops or the currents which flow through them; we do not even know if the currents are line currents or volume currents. But for our purposes all such details do not matter, all we care is the net magnetic moment $\mathbf{m}$ of each atom. As far as the *macroscopic* magnetic field $\mathbf{B}(\mathbf{r})$ is concerned, we may replace each atom with an $a \times a \times a$ cube with the surface current

$$I = \frac{m}{a^2} \quad \Longrightarrow \quad K = \frac{I}{a} = \frac{m}{a^3} \tag{20}$$

flowing around the 4 vertical sides of the cube. Here is the picture of such a cubic model, or rather, of a small part of a single slice of the cubic lattice:



Inside each atom, the current flows counterclockwise. But when we look at the boundary between two neighboring atoms, we immediately see that over that boundary, the currents of the two atoms flow in opposite directions. And since they have the same magnitude $I = m/a^2$, they cancel each other! Thus, *in the middle of the crystal* all the atomic currents cancel each other and there is no net current. However, *at the outer boundary of the crystal,* *there is no cancellation:*



Instead, there is un-canceled current $I = m/a^2$ flowing counterclockwise around the entire outer boundary of the crystal, or rather of single atomic layer of the crystal. The next layer on top of the layer shown on the above picture has a similar current, and so is every other layer, which makes for a surface current density

$$K = \frac{I}{a} = \frac{m}{a^3} = M \tag{21}$$

over the 4 vertical sides of the whole magnetized cube. In vector notations, the surface current on each side of the cube is

$$\mathbf{K} = \mathbf{M} \times \mathbf{n} \tag{22}$$

where $\mathbf{n}$ is the unit vector normal to the side in question.

The above toy model explains the physical origin of the surface bound current. It does not have a volume bound current since we assumed a uniform magnetization inside the magnetic material. To model a non-uniform magnetization we should give different atoms different magnetic moments $\mathbf{m}$ and hence different atomic currents. Consequently, at the boundary of two neighboring atoms we would no longer have exact cancellation of their currents, and that would give rise to bound volume currents inside the bulk of the magnetized material. Assuming for simplicity that all atomic magnetic moments point in z directions but their magnitudes *slowly* depend on x and y coordinated of the atom, the net current on the boundary between two atoms neighboring in x direction is

$$\mathbf{K} = (-\hat{\mathbf{y}}) \times a \frac{\partial}{\partial x}\left(\frac{m_z}{a^3}\right) \tag{23}$$

while on the boundary between two atoms neighboring in y direction

$$\mathbf{K} = (+\hat{\mathbf{x}}) \times a \frac{\partial}{\partial y}\left(\frac{m_z}{a^3}\right). \tag{24}$$

Averaging these currents over the volume of the magnetized crystal, we get the volume bound current

$$\mathbf{J}_b = \frac{1}{a}\mathbf{K} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial y} - \hat{\mathbf{y}}\frac{\partial}{\partial x}\right)\frac{m_z}{a^3} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial y} - \hat{\mathbf{y}}\frac{\partial}{\partial x}\right) M_z = \nabla \times \mathbf{M}. \tag{25}$$

The Magnetic Intensity Field H

In dielectric materials, it's convenient to use two kinds of electric field: the electric tension field $\mathbf{E}$ and the electric displacement field $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$. Likewise, in magnetic materials it's convenient to use two kinds of magnetic field: the *magnetic intensity field* **H** and the *magnetic induction field* **B**. Here is how this works.

⋆ *Microscopically,* there is only one kind of magnetic field, namely the $\mathbf{B}(\mathbf{r})$.

- *Macroscopically,* the magnetic induction field $\mathbf{B}_{\text{macro}}(\mathbf{r})$ is the microscopic magnetic field averaged over some distance scale $a \gg$ typical distances between the atoms,

$$\mathbf{B}_{\text{macro}}(\mathbf{r}) = \iiint \frac{d^3\mathbf{r}'}{a^3} f\left(\frac{\mathbf{r}' - \mathbf{r}}{a}\right) \mathbf{B}_{\text{micro}}(\mathbf{r}') \tag{26}$$

for some non-negative averaging function f which integrates to 1, for example

$$f(\Delta\mathbf{r}/a) = \frac{3}{4\pi}\Theta(a - |\mathbf{r}|) \quad \text{or} \quad f(\Delta\mathbf{r}/a) = \pi^{-3/2}\exp\bigl(-(\Delta\mathbf{r})^2/a^2\bigr). \tag{27}$$

- The macroscopic magnetic induction field obeys similar equations to the microscopic magnetic field, namely the zero divergence equation and the Ampere law,

$$\nabla\cdot\mathbf{B}_{\text{macro}} = 0 \quad\text{and}\quad \nabla\times\mathbf{B}_{\text{macro}} = \mu_0\mathbf{J}^{\text{net}}_{\text{macro}}\,. \tag{28}$$

In the macroscopic Ampere law here, the net electric current includes both the bound current $\mathbf{J}_b(\mathbf{r})$ due to magnetization as well as the free macroscopic current $\mathbf{J}_f(\mathbf{r})$ due to electric conductance. Altogether, this gives us

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}_b + \mu_0\mathbf{J}_f = \mu_0\nabla\times\mathbf{M} + \mu_0\mathbf{J}_f\,, \tag{29}$$

which we may rewrite as

$$\nabla\times(\mathbf{B}-\mu_0\mathbf{M}) = \mu_0\mathbf{J}_f\,. \tag{30}$$

⋆ The macroscopic field $\mathbf{H}(\mathbf{r})$ — called the *magnetic intensity field* or the *magnetic tension field* — is defined as

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}). \qquad \langle\!\langle\,\text{note the minus sign!}\,\rangle\!\rangle \tag{31}$$

In light of eq. (30), the $\mathbf{H}$ field obeys the Ampere Law which involves only the free current,

$$\nabla\times\mathbf{H} = \mathbf{J}_f\,. \tag{32}$$

In the integral form, this Ampere Law becomes

$$\oint_{\mathcal{L}}\mathbf{H}\cdot d\vec{\ell} = I_{\text{free}}[\text{through loop }\mathcal{L}]. \tag{33}$$

- On the other hand, the $\mathbf{B}$ field is divergence-less, $\nabla\cdot\mathbf{B}=0$, come hell or high water, while

$$\nabla\cdot\mathbf{H} = \frac{1}{\mu_0}\nabla\cdot\mathbf{B} - \nabla\cdot\mathbf{M} = -\nabla\cdot\mathbf{M}, \quad\text{which does not have to vanish.} \tag{34}$$

∘ In MKSA units, the H field is measured in Amperes-per-meter, 1 A/m, while the B field is measured in Teslas.

- In Gauss units, H is defined as $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, so in vacuum $\mathbf{H} = \mathbf{B}$. Nevertheless, the Gauss units for the two kinds of magnetic fields have different names: the unit of H field is called the Oersted, 1 Oe, after Hans Cristian Ørsted, while the unit of B field is called the Gauss, 1 G, after Carl Friedrich Gauss. In vacuum, $H = 1$ Oe means $B = 1$ G and vice verse.

Let me conclude this section with a simple example of the Ampere Law for the H field. Consider a long straight copper wire with a round cross-section carrying uniform current density $\mathbf{J}_f = (I/\pi R^2)\hat{\mathbf{z}}$. Copper is a linear diamagnetic, so in magnetic field it develops magnetization $\mathbf{M}$ in the opposite direction from $\mathbf{B}$ and $\mathbf{H}$, specifically

$$\mathbf{M} = \chi_m \mathbf{H} \quad \text{for a negative magnetic susceptibility } \chi_m[\text{Cu}] \approx -9.6 \cdot 10^{-6}. \tag{35}$$

The $\mathbf{H}$ field of the long straight wire follows directly from the Ampere Law (33) and the symmetries of the wire

$$\begin{aligned} \text{outside the wire, for } s > R, \quad \mathbf{H} &= \frac{I}{2\pi s}\,\hat{\phi}, \\ \text{inside the wire, for } s < R, \quad \mathbf{H} &= \frac{Is}{2\pi R^2}\,\hat{\phi}, \end{aligned} \tag{36}$$

cf. my earlier notes on the Ampere Law. Consequently, outside the wire

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi} \tag{37}$$

while inside the wire

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu_0(1 + \chi_m)\mathbf{H} = \frac{\mu_0(1 + \chi_m) I s}{2\pi R^2}\,\hat{\phi}. \tag{38}$$

More interestingly, inside the wire the magnetization $\mathbf{M} = \chi_m \mathbf{H}$ gives rise to the bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{\chi_m I}{2\pi R^2}\,\nabla \times (s\hat{\phi}) \tag{39}$$

where

$$\nabla \times (s\hat{\phi}) \;=\; \nabla \times (x\hat{\mathbf{y}} - y\hat{\mathbf{x}}) \;=\; 2\hat{\mathbf{z}}, \tag{40}$$

hence

$$\mathbf{J}_b \;=\; \frac{\chi_m I}{\pi R^2}\,\hat{\mathbf{z}} \;=\; \chi_m \mathbf{J}_f\,. \tag{41}$$

But please note that the copper is diamagnetic, $\chi_m < 0$, so the bound current in the bulk of the wire flows in the opposite direction from the free current!

Besides the volume bound current, there is also a surface bound current

$$\mathbf{K} \;=\; \mathbf{M}(s=R)\times\mathbf{n} \;=\; \frac{\chi_m I}{2\pi R}\,\hat{\phi}\times\hat{\mathbf{r}} \;=\; \frac{\chi_m I}{2\pi R}\,(-\hat{\mathbf{z}}), \tag{42}$$

note the minus sign! However, there is another negative sign due to $\chi_m < 0$, so the bound current on the surface flows in the same direction as the free current.

Finally, the net bound current through the wire is zero:

$$I_b^{\rm net} \;=\; \pi R^2 \times J_{b,z} \;+\; 2\pi R \times K_{b,z} \;=\; +\chi_m I \;-\; \chi_m I \;=\; 0. \tag{43}$$

The above example was particularly simple due to symmetries of the wire. In less symmetric situations, we cannot find the magnetic fields from just the Ampere Law. Instead, we must solve the field equations

$$\nabla\cdot\mathbf{B} \;=\; 0, \quad \nabla\times\mathbf{H} \;=\; \mathbf{J}_f\,, \tag{44}$$

together with the *boundary conditions* at the surface of the magnetic material.

Boundary Conditions

Consider the outer surface of some magnetic material. Pick any point on the surface and let ζ be a coordinate normal to the surface at that point. Then, any discontinuities of the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ at the surface give rise to δ-function terms in the derivatives $\nabla\cdot\mathbf{B}$ and $\nabla\times\mathbf{H}$, specifically

$$\nabla\cdot\mathbf{B} \;=\; \delta(\zeta)\,\mathbf{n}\cdot\mathrm{disc}(\mathbf{B}) \;+\; \text{finite}, \tag{45}$$

$$\nabla\times\mathbf{H} \;=\; \delta(\zeta)\,\mathbf{n}\times\mathrm{disc}(\mathbf{H}) \;+\; \text{finite}. \tag{46}$$

Combining these formulae with eqs. (44) for the fields, we immediately see that

$$\mathbf{n}\cdot\mathrm{disc}(\mathbf{B}) \;=\; 0 \quad\Longrightarrow\quad \mathrm{disc}(B_\perp) \;=\; 0, \tag{47}$$

$$\mathbf{n}\times\mathrm{disc}(\mathbf{H}) \;=\; \mathbf{K}_f \quad\Longrightarrow\quad \mathrm{disc}(\mathbf{H}_\|) \;=\; \mathbf{K}_f\times\mathbf{n}, \tag{48}$$

regardless of the bound surface current. In particular, when there is no *free* current flowing along the surface,

$$\mathrm{disc}(B_\perp) \;=\; 0, \quad \mathrm{disc}(\mathbf{H}_\|) \;=\; \mathbf{0}, \tag{49}$$

which means that *the normal component of the* **B** *field and the tangential components of the* **H** *field must be continuous across the surface*; on the other hand, *the tangential components of the* **B** *field and the normal component of the* **H** *field are discontinuous*.

These rules are very similar to the rules for the electric tension and displacement fields **E** and **D** at the surface of a dielectric material:

- $\mathbf{E}_\|$ and $D_\perp$ are continuous across the surface, but $E_\perp$ and $\mathbf{D}_\|$ may jump, just like
- $\mathbf{H}_\|$ and $B_\perp$ are continuous across the surface, but $H_\perp$ and $\mathbf{B}_\|$ may jump.

The electric-magnetic analogy extends to the equations obeyed by the respective fields in the absence of *free* charges or currents:

$$\text{when } \rho_f \;=\; 0, \quad \nabla\times\mathbf{E} \;=\; 0, \quad \nabla\cdot\mathbf{D} \;=\; 0, \quad \mathbf{D} \;=\; \epsilon_0\mathbf{E} \;+\; \mathbf{P}; \tag{50}$$

$$\text{when } J_f \;=\; 0, \quad \nabla\times\mathbf{H} \;=\; 0, \quad \nabla\cdot\mathbf{B} \;=\; 0, \quad \mathbf{B} \;=\; \mu_0\mathbf{H} \;+\; \mu_0\mathbf{M}. \tag{51}$$

Consequently, the solutions for the dielectric boundary problems apply equally well to similar boundary problems for the magnetic materials. For example, in homework set#8 you (should) have calculated the electric field inside a uniformly polarized dielectric ball; the solution gives a pure dipole field outside the ball (with dipole moment $\mathbf{p} = \mathbf{P} \times \frac{4\pi}{3} R^3$) while inside the ball there is uniform field

$$\mathbf{D} = \frac{2}{3}\mathbf{P}, \quad \mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}. \tag{52}$$

Now we may recycle this solution to find the magnetic field of a uniformly magnetized ball without doing any new calculations at all — all we need is to translate the dielectric ball solution to the magnetic language. Thus, outside the magnetized ball the magnetic field is a pure magnetic dipole field (with magnetic moment $\mathbf{m} = \mathbf{M} \times \frac{4\pi}{3} R^3$), while inside the ball there is uniform field

$$\mathbf{B} = \frac{2\mu_0}{3}\mathbf{M}, \quad \mathbf{H} = -\frac{1}{3}\mathbf{M}. \tag{53}$$

Note that despite complete absence of any free currents anywhere, the $\mathbf{H}$ field does not vanish either inside or outside the ball.

Linear Magnetic Materials

Many magnetic materials are linear: the magnetization respond linearly to the magnetic field which causes it,

$$\mathbf{M} = (\text{const})\mathbf{B}, \tag{54}$$

although by analogy with the linear dielectrics, this formula is usually written as

$$\mathbf{M} = \chi_m \mathbf{H}. \tag{55}$$

The dimensionless constant coefficient χ_m here is called the *magnetic susceptibility* of the material in question.

- *Diamagnetic* materials have small negative susceptibilities, $\chi_m < 0$, $|\chi_m| \ll 1$.
- *Paramagnetic* materials have small positive susceptibilities, $\chi_m > 0$, $|\chi_m| \ll 1$.

- *Ferromagnetic* materials have large positive susceptibilities, $\chi_m > 0$, $|\chi_m| \gg 1$.

In light of eq. (55),

$$\mathbf{B} \;=\; \mu_0(\mathbf{H}+\mathbf{M}) \;=\; \mu_0(1+\chi_m)\mathbf{H}, \tag{56}$$

which is usually written as

$$\mathbf{B} \;=\; \mu_{\text{abs}}\mathbf{H} \;=\; \mu_{\text{rel}}\mu_0\mathbf{H}, \quad \mu_{\text{rel}} \;=\; \chi_m \;+\; 1. \tag{57}$$

The coefficient μ_{rel} here (often written as simply μ) is called the *relative magnetic permeability* of the magnetic material in question, while $\mu_{\text{abs}} = \mu_{\text{rel}}\mu_0$ is called the *absolute magnetic permeability.* In this terminology, the μ_0 constant of the MKSA system of units is the magnetic permeability of the vacuum.

In the bulk of a *uniform* magnetic material,

$$\mu_0\mathbf{J}_{\text{net}} \;=\; \nabla\times\mathbf{B} \;=\; \mu_{\text{rel}}\mu_0\nabla\times\mathbf{H} \;=\; \mu_{\text{rel}}\mu_0\mathbf{J}_{\text{free}}\,, \tag{58}$$

hence the bound currents shadows the free currents,

$$\mathbf{J}_{\text{net}} \;=\; \mu_{\text{rel}}\mathbf{J}_{\text{free}} \quad\Longrightarrow\quad \mathbf{J}_{\text{bound}} \;=\; \chi_m\mathbf{J}_{\text{free}}\,. \tag{59}$$

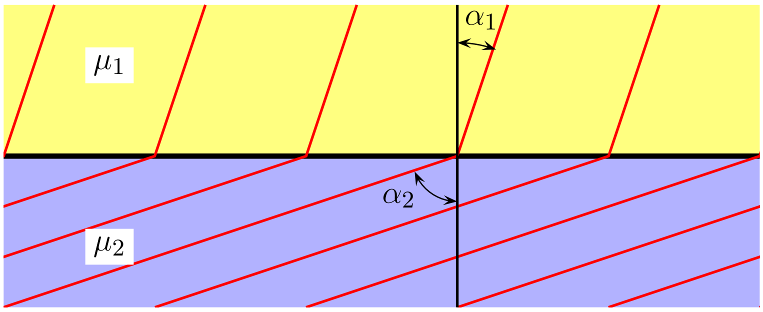
This, in the middle of a ferromagnetic conductor like iron or nickel, there is a bound current flowing in the same direction as the free current, but much stronger in magnitude. In the middle of a paramagnetic conductor like aluminum, the bound current also flows in the same direction as the free current, but its magnitude is much weaker. On the other hand, in the middle of a diamagnetic conductor like copper, the bound current flows in the opposite direction from the free current, as we already saw a few pages back.

But eq. (59) apply only to the volume currents in the bulk of a uniform material. At the outer surface of the magnetic material — or at the interface between two different magnetic materials — there are surface bound currents not related to any free currents. Instead, we have to calculate them from the boundary conditions for the **H** and **B** fields. Specifically, at the interface of two uniform magnetic materials,

$$\mathbf{H}_1^{||} = \mathbf{H}_2^{||} \implies \frac{1}{\mu_1}\mathbf{B}_1^{||} = \frac{1}{\mu_2}\mathbf{B}_2^{||}, \tag{60}$$

$$B_1^{\perp} = B_2^{\perp} \implies \mu_1 H_1^{\perp} = \mu_2 H_2^{\perp}. \tag{61}$$

For example, consider bending of the magnetic field at the interface:



From the angles α_1 and α_2 on this picture we find that

$$\frac{B_1^{||}}{B_1^{\perp}} = \tan\alpha_1\,, \quad \frac{B_2^{||}}{B_2^{\perp}} = \tan\alpha_2\,, \tag{62}$$

and therefore

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{B_1^{||}}{B_1^{\perp}} \Big/ \frac{B_2^{||}}{B_2^{\perp}} = \frac{B_1^{||}}{B_2^{||}} \Big/ \frac{B_1^{\perp}}{B_2^{\perp}}. \tag{63}$$

But according to eqs. (60) and (61),

$$\frac{B_2^{\perp}}{B_1^{\perp}} = 1 \quad \text{while} \quad \frac{B_1^{||}}{B_2^{||}} = \frac{\mu_1}{\mu_2}\,, \tag{64}$$

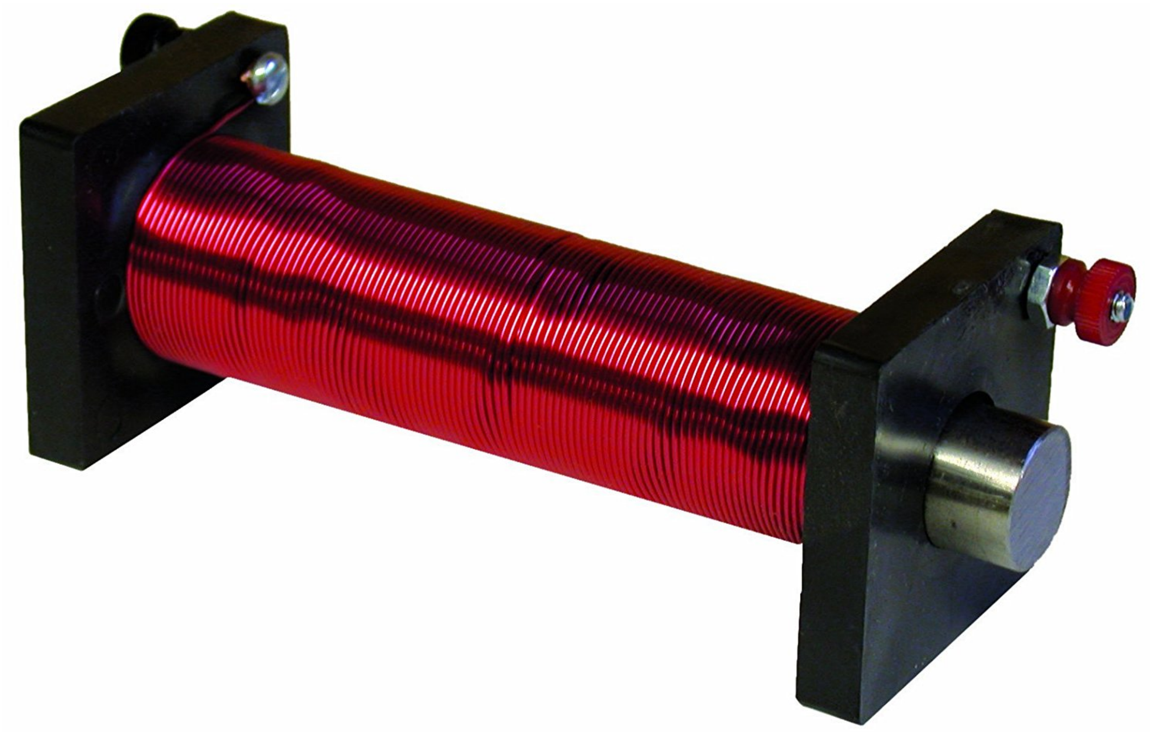
hence

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_1}{\mu_2} \Big/ 1 = \frac{\mu_1}{\mu_2}. \tag{65}$$

This refraction law is exactly similar to the bending of the electric field line at the boundary of two uniform dielectrics.

Electromagnets

To make a strong electromagnet powered by a moderate electric current one needs to fill the electromagnet with a ferromagnetic core, usually iron or steel. As an example of how this works, consider a solenoid with an iron core, such as shown on this picture:



To simplify the calculation of the magnetic field created by this solenoid, let's assume its length is much longer than its diameter, so to a first approximation we may treat it as infinitely long, hence the cylindrical symmetry. By this symmetry (and also by the reflection $z \to -z$ symmetry), the magnetic field everywhere — both inside and outside the solenoid — points in z direction while its magnitude depends only on s. Furthermore, by the Ampere Law applied to the **H** field,

$$H(s_1) \;-\; H(s_2) \;=\; \frac{IN}{L} \quad \text{for any } s_1 < R,\; s_2 > R, \tag{66}$$

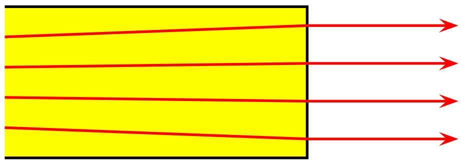
hence uniform **H** inside the solenoid, uniform **H** outside the solenoid, and therefore

$$\mathbf{H}_{\text{inside}} = \frac{IN}{L}\,\hat{\mathbf{z}}, \quad \mathbf{H}_{\text{outside}} = 0. \tag{67}$$

But the inside of the solenoid is filled with the iron core of high relative permeability $\mu \gg 1$, so the **B** field inside that core is μ times larger than it would be in the air-filled solenoid,

$$\mathbf{B}_{\text{inside}} = \mu\mu_0\mathbf{H} = \mu\,\frac{\mu_0 IN}{L}\,. \tag{68}$$

Now let's remember the solenoid has finite length and consider one of its ends, or rather one end of the iron core:



Inside the core, the magnetic field lines are approximately parallel to the solenoid's axis, so at the end of the core they are almost perpendicular to the boundary. Consequently, just outside the core's end

$$B^{\perp}_{\text{outside}} = B^{\perp}_{\text{inside}} = \mu\,\frac{\mu_0 IN}{L}\,. \tag{69}$$

Thus, the iron core amplifies the magnetic field μ-fold not only inside the core but also outside-the-core but near-the-poles, and hence also everywhere outside the electromagnet!

For a numeric example, take a solenoid with $N/L = 1000$ loops per meter powered by a current $I = 1$ A. This solenoid creates the magnetic intensity field $H = 1000$ A/m, so without the core the magnetic induction field would be $B \approx 1.3 \cdot 10^{-3}$ Tesla, quite weak. However, with an iron core of relative permeability $\mu = 1000$, the magnetic field increases thousandfold, to a rather strong $B \approx 1.3$ Tesla.

Another advantage of the iron core is that we can have it stick out from the solenoid for some distance, and even bend around in a curve, and most of the magnetic flux would follow such a bent core to its end. Consequently, we can put the poles of the electromagnet wherever we need them rather than have them stuck at the end of the solenoidal coil. But these engineering aspects of the iron core go beyond this class, so let me stop here.

Appendix: Proving the Lemma

Near the beginning of these notes I have used a Lemma of vector calculus saying that *for any vector field* $\mathbf{T}(\mathbf{r})$ *and any volume* $\mathcal{V}$ *with surface* $\mathcal{S}$*,*

$$\iiint\limits_{\mathcal{V}}(\nabla\times\mathbf{T})d^3\mathrm{Vol} \;=\; -\iint\limits_{\mathcal{S}}\mathbf{T}\times\mathbf{d^2A} \;=\; -\iint\limits_{\mathcal{S}}(\mathbf{T}(\mathbf{r})\times\mathbf{n}(\mathbf{r}))d^2A \tag{70}$$

where $\mathbf{n}(\mathbf{r})$ *is a unit vector normal to to the surface* $\mathcal{S}$ *at point* $\mathbf{r}$. This lemma was part of the homework set#2 back in January, but let me repeat the proof here, just in case you have forgotten.

Pick any constant vector $\mathbf{c}$, then by the triple product rule $\mathbf{c}\cdot(\nabla\times\mathbf{T})=\nabla\cdot(\mathbf{T}\times\mathbf{c})$ and therefore

$$\begin{aligned}
\mathbf{c}\cdot\iiint\limits_{\mathcal{V}}(\nabla\times\mathbf{T})d^3\mathrm{Vol} &= \iiint\limits_{\mathcal{V}}\mathbf{c}\cdot(\nabla\times\mathbf{T})d^3\mathrm{Vol} \;=\; \iiint\limits_{\mathcal{V}}\nabla\cdot(\mathbf{T}\times\mathbf{c})d^3\mathrm{Vol}\\
&\quad\langle\!\langle\text{ by Gauss theorem }\rangle\!\rangle\\
&= \iint\limits_{\mathcal{S}}(\mathbf{T}\times\mathbf{c})\cdot\mathbf{d^2A}\\
&\quad\langle\!\langle\text{ by the triple product rule }(\mathbf{T}\times\mathbf{c})\cdot\mathbf{d^2A}=(\mathbf{d^2A}\times\mathbf{T})\cdot\mathbf{c}\,\rangle\!\rangle\\
&= \iint\limits_{\mathcal{S}}\big((\mathbf{d^2A}\times\mathbf{T})\cdot\mathbf{c}\big) \;=\; \mathbf{c}\cdot\iint\limits_{\mathcal{S}}\mathbf{d^2A}\times\mathbf{T}.
\end{aligned} \tag{71}$$

Since $\mathbf{c}$ here is an arbitrary constant vector, this implies

$$\iiint\limits_{\mathcal{V}}(\nabla\times\mathbf{T})d^3\mathrm{Vol} \;=\; \iint\limits_{\mathcal{S}}\mathbf{d^2A}\times\mathbf{T} \;=\; -\iint\limits_{\mathcal{S}}\mathbf{T}\times\mathbf{d^2A} \;=\; -\iint\limits_{\mathcal{S}}(\mathbf{T}\times\mathbf{n})\,d^2A \tag{70}$$

Quod erat demonstrandum.