Inductance and Magnetic Energy

Self Inductance

Consider a loop or coil of wire carrying a current I. There may be an iron core inside the coil or some other magnetic materials near it, but let's assume that all the magnetic materials involved are linear ($\mathbf{B} = \mu\mu_0\mathbf{H}$). In that case, at any point $\mathbf{r}$ inside or near the coil the magnetic field $\mathbf{B}(\mathbf{r})$ created by the current I in the coil would be proportional to the current, $\mathbf{B}(\mathbf{r}) = \mathbf{f}(\mathbf{r})I$. Now consider the flux Φ of this magnetic field through the coil itself. By linearity,

$$\Phi \;=\; L \times I \tag{1}$$

for some coefficient L which depends on the coil's geometry and the magnetic material inside or near the coil, but not on the current I. This coefficient L is called the *self-inductance* of the coil, which is often shortened to the coil's *inductance* or *inductivity*.

Now let the current through the coil vary with time. As long as this variance is not too rapid, we may use the quasi-static approximation to calculate the magnetic field inside the coil and hence the magnetic flux through the coil, thus

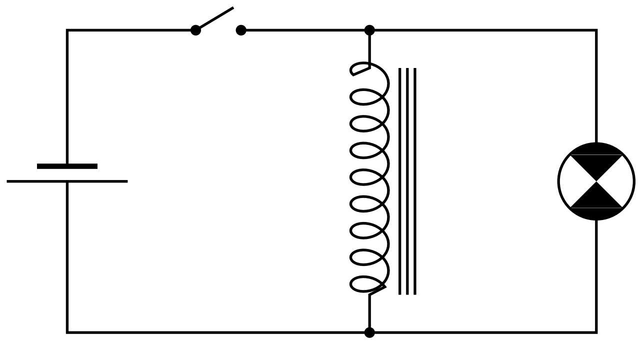
$$\Phi(t) \;=\; L \times I(t). \tag{2}$$

The time-dependence of this magnetic flux induced an EMF in the coil, namely

$$\mathcal{E} \;=\; -\frac{d\Phi}{dt} \;=\; -L \times \frac{dI}{dt} \tag{3}$$

where the minus sign stems from the Lenz rule: *the induced EMF resists changing the current flowing through the coil.*

As an example of this Lenz rule in action, consider the following circuit



When the switch is closed, the light bulb and the coil receive the same voltage from the battery, but since the Ohmic resistance of the coil is much less than the resistance of the bulb, the current through the coil is much stronger than the current through the bulb. In fact, the current through the bulb is rather weak, so the bulb barely light up and stays rather dim. But when the switch is suddenly thrown open, the current which used to flow through the coil cannot stop right away — the coil's self-inductance prevents this according to eq. (3). Instead, this strong current has to flow through the bulb — which makes it flush bright. However, this flash lasts only a short time, as the current through the coil and the bulb decays rather fast.

Let's calculate the time scale and the manner of this decay. For simplicity let's treat the light bulb as a resistor of a constant resistance R_b. The current through the resulting RL circuit follows from the EMF in the coil by the Ohm's Law,

$$\mathcal{E} \;=\; IR_c \;+\; IR_b \;=\; IR \tag{4}$$

where R_c is the Ohmic resistance of the coil and $R = R_b + R_c$ is the net resistance of the RL circuit. At the same time, the EMF follows from the time derivative of the same current according to eq. (3), hence

$$\frac{dI}{dt} \;=\; -\frac{\mathcal{E}}{L} \;=\; -\frac{RI}{L}\,. \tag{5}$$

Solving this differential equation with the initial condition $I(t=0) = I_0$ gives us exponential decay

$$I(t) \;=\; I_0\times\exp(-t/\tau) \tag{6}$$

with the time constant

$$\tau \;=\; \frac{L}{R} \tag{7}$$

For example, in the demo shown in the freshmen E&M class, the big coil has self-inductance about $L\approx 2$ H while the light bulb has resistance about $R\approx 100\ \Omega$, hence a rather short the time constant $\tau\approx$ 0.02 seconds.

BTW, the H in $L=2$ H stands for *Henry,* the MKSA unit of inductance named after American scientist Joseph Henry (1797–1878),

$$1\ \mathrm{H}\times 1\ \mathrm{A} \;=\; 1\ \mathrm{W}\ (\mathrm{Weber}) \;=\; 1\ \mathrm{T}\times 1\ \mathrm{m}^2 \;=\; 1\ \mathrm{V}\times 1\ \mathrm{s}, \tag{8}$$

hence in eq. (7)

$$\frac{1\ \mathrm{H}}{1\ \Omega} \;=\; \frac{1\ \mathrm{V}\cdot\mathrm{s/A}}{1\ \mathrm{V/A}} \;=\; 1\ \mathrm{s}. \tag{9}$$

In Gaussian units, the inductance is defined with an extra factor of c in eq. (1),

$$\Phi \;=\; c\times L\times I \tag{10}$$

to compensate for the $1/c$ factor in the Induction Law so that eq. (3) looks similarly in both unit systems,

$$\mathcal{E} \;=\; -\frac{1}{c}\,\frac{d\Phi}{dt} \;=\; -L\times\frac{dI}{dt}\,. \tag{11}$$

The Gaussian unit if mutual inductance or self-inductance does not have a proper name, but by dimensional analysis it's equivalent to s^2/cm:

$$1\ (\text{Gaussian unit of inductance}) \;=\; \frac{\mathrm{statV}}{(\mathrm{Fr/s})/\mathrm{s}} \;=\; \frac{\mathrm{s}^2}{\mathrm{Fr/statV}} \;=\; \frac{\mathrm{s}^2}{\mathrm{cm}}\,. \tag{12}$$

Now let's calculate the self-inductances of some example coils. For our first example, consider a long thin solenoid without an iron core. As we have learned 3 weeks ago, the magnetic field inside such a solenoid is approximately uniform

$$\mathbf{B} \;=\; \mu_0(N/\ell)I\,\hat{\mathbf{z}} \tag{13}$$

where ℓ is the solenoid's length and N is the number of turns, while the magnetic field outside the solenoid is negligibly small, $\mathbf{B} \approx 0$. Consequently, the magnetic flux through each turn of the solenoid is

$$\Phi_1 \;=\; B \times A \;=\; \mu_0(N/\ell)I \times \pi r^2 \tag{14}$$

where r is the solenoid's radius, and the net flux through the coil is

$$\Phi \;=\; N \times \Phi_i \;=\; N \times \pi r^2 \times \mu_0(N/\ell)I. \tag{15}$$

Identifying this flux as $\Phi = L \times I$, we find the solenoid's self-inductance to be

$$L \;=\; \mu_0 \times N^2 \times \frac{\pi r^2}{\ell}\,. \tag{16}$$

For example, take an $N = 1000$ turn solenoid of length $\ell = 1$ foot ≈ 0.30 m and diameter $2r = 1$ inch (so that $\pi r^2 \approx 5.0\ \text{cm}^2$); the self inductance of this solenoid is

$$L \;=\; (4\pi \cdot 10^{-7}\ \text{H/m}) \times 1000^2 \times (5.0 \cdot 10^{-4}\ \text{m}^2)/(0.30\ \text{m}) \;\approx\; 2.1 \cdot 10^{-3}\ \text{H}. \tag{17}$$

Next, consider a similar solenoid coil but give it an iron core of magnetic permeability $\mu \gg 1$. By Ampere's Law, this core does not change the $\mathbf{H}$ field inside the solenoid — it remains the same $H = (N/\ell)I$ as in a core-less solenoid — but it increases the $\mathbf{B}$ field by the factor of μ, thus

$$\mathbf{B}_{\text{inside}} \;=\; \mu\,\mu_0(N/\ell)I\,\hat{\mathbf{z}}. \tag{18}$$

Consequently, the magnetic flux through the solenoid also increases by the factor of μ to

$$\Phi \;=\; N(\pi r^2) \times B \;=\; N(\pi r^2) \times \mu\mu_0(N/\ell) \times I, \tag{19}$$

which means that the solenoid with an iron core has μ times greater self-inductance than the core-less solenoid,

$$L = \mu\,\mu_0 \times N^2 \times (\pi r^2/\ell). \tag{20}$$

For example, the same 1000 turn solenoid of 1 foot length and 1 inch diameter as in the previous example but with a $\mu = 950$ iron core has self inductance $L \approx 2.0$ Henry.

For our last example, consider a toroidal coil with an iron core. By Ampere's Law, the magnetic field inside such a coil — and in particular inside the core — is

$$\mathbf{B} = \mu\mu_0 \mathbf{H} = \mu\mu_0 \frac{NI}{2\pi s}\,\hat{\phi}\,, \tag{21}$$

while the magnetic field outside the core is negligibly smaller, hence the magnetic flux through the coil is

$$\Phi = N \times \iint B\,d^2A = \mu\mu_0 N^2 \times I \times \iint \frac{d^2A}{2\pi s} \tag{22}$$

where the are integral is over the core's cross-section. Consequently, the self-inductance of the toroidal coil is

$$L = \mu\mu_0 N^2 \times \iint\limits_{\substack{\text{core's cross}\\ \text{section}}} \frac{d^2A}{2\pi s}\,. \tag{23}$$

In particular, for a thin toroid shaped like a bicycle tire (rather than thick like a bagel) we may approximate s in the denominator as a constant — the mean radius R of the toroid — so the integral becomes simply (the core's cross-section A)$/2\pi R$, hence self-inductance

$$L \approx \mu\mu_0 \times N^2 \times \frac{A}{2\pi R}\,, \tag{24}$$

similar to the solenoid of length $\ell = 2\pi R$.

Mutual Inductance and Transformers

Consider two wire coils, with or without iron cores. Or more generally, two wire loops of any geometries, perhaps with some magnetic materials inside or around the loops, but let's assume all such magnetic materials are linear (so inside them $\mathbf{B} = \mu\mu_0\mathbf{H}$). Anyhow, when a current I_1 runs through the coil#1, it creates a magnetic field $\mathbf{B}_1(\mathbf{r})$ inside and around that coil, including also the space inside the other coil#2. Altogether, the current I_1 in the first coil creates a magnetic flux Φ_2 in the second coil. By linearity of the magnetic equations, the magnetic field $\mathbf{B}_1(\mathbf{r})$ at any point $\mathbf{r}$ is proportional to the current I_1, so its flux Φ_2 through the second coil is also proportional to the I_1, thus

$$\Phi_2 \;=\; M_{21} \times I_1 \tag{25}$$

for some current-independent coefficient M_{21} which depends on the two coils' geometries and on the magnetic materials present inside and around them. This coefficient M_{21} is called the *mutual inductance* of the two coils.

Similar to the self-inductance of a coil, the mutual inductance of two coils means that a time-dependent current in one coil induces EMF in the other coil. Indeed, let the current $I_1(t)$ in the first coil vary with time, but slowly enough to use the quasi-static approximation to the magnetic field $\mathbf{B}_1(\mathbf{r}, t)$ it creates. In that case, the flux of this field through the second coil is

$$\Phi_2(t) \;=\; M_{21} \times I_1(t), \tag{26}$$

and the time-dependence of this leads to EMF in the second coil

$$\mathcal{E}_2 \;=\; -\frac{d\Phi_2}{dt} \;=\; -M_{21} \times \frac{dI_1}{dt}\,. \tag{27}$$

For example, the AC current $I_1(t) = I_1^{(0)} \times \cos(\omega t)$ through the first coil induces the AC voltage

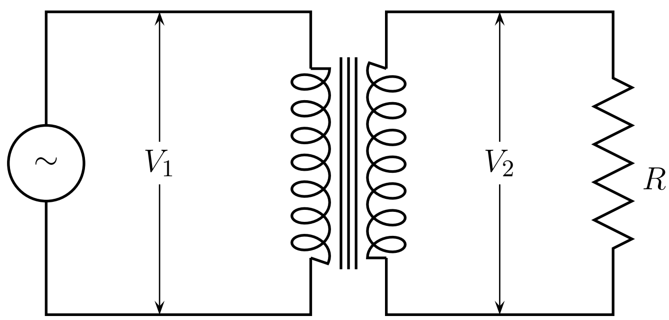
$$\mathcal{E}_2 \;=\; M_{21} \times I_1^{(0)} \times \omega\sin(\omega t) \tag{28}$$

in the second coil. These formulae are very important for the transformers.

Indeed, consider a typical transformer made of two coils wound around the same iron core. The primary coil is connected to an AC power source (such as a wall outlet)

$$V_1(t) \;=\; V_1 \times \sin(\omega t) \tag{29}$$

while the secondary coil is connected to some resistive load, as shown on the following diagram:



(30)

There are AC currents $I_1(t)$ and $I_2(t)$ flowing through the respective coils, and the mutual inductions M_{21} and M_{12} of the two coils — as well as their self-inductions L_1 and L_2 — give rise to the EMFs in both coils. Specifically,

$$\mathcal{E}_1(t) \;=\; -L_1 \times \frac{dI_1}{dt} \;-\; M_{12} \times \frac{dI_2}{dt}, \quad \mathcal{E}_2(t) \;=\; -L_2 \times \frac{dI_2}{dt} \;-\; M_{21} \times \frac{dI_1}{dt}\,. \tag{31}$$

For simplicity, let's assume that the load's resistance R is so large that the current I_2 through the load and the secondary coil is negligibly small compared to the current I_1 through the primary coil, so that eqs. (31) become simply to

$$\mathcal{E}_1(t) \;\approx\; -L_1 \frac{dI_1}{dt}, \quad \mathcal{E}_2(t) \;\approx\; -M_{21} \times \frac{dI_1}{dt}\,. \tag{32}$$

Moreover, in the context of the circuit (30), $\mathcal{E}_2(t)$ is the AC voltage $V_2(t)$ on the load, while $\mathcal{E}_1(t)$ cancels the voltage $V_1(t)$ supplied by the wall outlet. Or rather,

$$V_1(t) \;=\; \mathcal{E}_1(t) \;-\; R_{\text{coil}} \times I_1(t) \tag{33}$$

where R_{coil} is the Ohmic resistance of the primary coil. In practice, this resistance is rather small, so we may approximate $V_1(t) = \mathcal{E}_1(t)$, hence

$$V_1(t) \approx -L_1 \frac{dI_1}{dt}, \quad V_2(t) \approx -M_{21} \times \frac{dI_1}{dt}. \tag{34}$$

In particular, for $V_1(t) = V_1^{(0)} \times \sin(\omega t)$ these relations give us $I_1(t) = I_1^{(0)} \times \cos(\omega t)$ and $V_2(t) = V_2^{(0)} \times \sin(\omega t)$ with amplitudes

$$V_1^{(0)} = \omega L_1 \times I_1^{(0)} \implies I_1^{(0)} = \frac{V_1^{(0)}}{\omega L_1} \tag{35}$$

and

$$V_2^{(0)} = \omega M_{21} \times I_1^{(0)} = \frac{M_{21}}{L_1} \times V_1^{(0)}. \tag{36}$$

Thus, we see that the voltage V_2 on the secondary coil follows the voltage V_1 on the primary coil in a step-up or step-down ratio

$$\frac{V_2}{V_1} = \frac{M_{21}}{L_1} \tag{37}$$

which depends on the mutual inductance of the two coils. In a good transformer, this ratio is simply the ratio of the turn numbers of the two coils,

$$\frac{M_{21}}{L_1} = \frac{N_2}{N_1}, \tag{38}$$

and that's how we get the well-known transformer rule

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}. \tag{39}$$

To see where the relation (38) comes from, consider a transformer made of two coils wound around the same toroidal iron core. In this transformer, the magnetic fluxes through the coil are completely dominated by the flux going through the iron core, and since the primary coil winds N_1 times around that core while the secondary coil winds N_2 times, we have

$$\Phi_1 \;=\; N_1\times\Phi_{\rm core} \quad \text{while} \quad \Phi_2 \;=\; N_2\times\Phi_{\rm core}\,. \tag{40}$$

Consequently, regardless of how the flux through the core is created, the fluxes through the two coils are in a fixed ratio

$$\frac{\Phi_2}{\Phi_1} \;=\; \frac{N_2}{N_1}\,. \tag{41}$$

In particular, when the flux through the core is due to a current I_1 through the primary coil, we get

$$\frac{\Phi_2=M_{21}\times I_1}{\Phi_1=L_1\times I_1} \;=\; \frac{N_2}{N_1} \quad\Longrightarrow\quad \frac{M_{21}}{L_1} \;=\; \frac{N_2}{N_1}\,, \tag{42}$$

as promised in eq. (38). In the same way, when the flux through the core is due to a current I_2 through the secondary coil, we get

$$\frac{\Phi_1=M_{12}\times I_2}{\Phi_2=L_2\times I_2} \;=\; \frac{N_1}{N_2} \quad\Longrightarrow\quad \frac{M_{12}}{L_2} \;=\; \frac{N_1}{N_2}\,, \tag{43}$$

Let's use these ratios to get the mutual inductances M_{21} and M_{12} of the two transformer coils. Earlier in these notes we have calculated the self-inductance of a coil would around a thin toroidal iron core; applying this formula for each coil of the transformer, we get

$$L_1 \;=\; N_1^2\times\mu\mu_0\,\frac{A}{2\pi R} \quad\text{and}\quad L_2 \;=\; N_2^2\times\mu\mu_0\,\frac{A}{2\pi R}\,, \tag{44}$$

and therefore

$$\begin{aligned} M_{21} \;&=\; \frac{N_2}{N_1}\times L_1 \;=\; N_2N_1\mu\mu_0\,\frac{A}{2\pi R}\,,\\ M_{12} \;&=\; \frac{N_1}{N_2}\times L_2 \;=\; N_1N_2\mu\mu_0\,\frac{A}{2\pi R}\,. \end{aligned} \tag{45}$$

Note the symmetry of the mutual inductions between the two transformer coils, $M_{12} = M_{21}$. This is an example of much more general **symmetry theorem:** *for any two wire* *loops or coils, of whatever geometry, in presence or absence of any magnetic materials of whatever shapes, as long as all such magnetic materials are linear,*

$$M_{21} \;=\; M_{21}\,. \tag{46}$$

Let me prove this theorem for the coils without iron cores or any other magnetic materials involved. In this case, the magnetic field due to current in the coil#1 obtains from the Biot–Savart–Laplace formula, or in terms of the vector potential,

$$\mathbf{A}_1(\mathbf{r}) \;=\; \frac{\mu_0 I_1}{4\pi} \oint\limits_{\text{coil\#1}} \frac{d\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|}\,. \tag{47}$$

The magnetic flux Φ_2 through the coil#2 obtains from this vector potential as

$$\Phi_2 \;=\; \iint\limits_{\text{a surface spanning coil\#2}} \mathbf{B}_1 \cdot d^2\mathbf{a} \;=\; \oint\limits_{\text{coil\#2 itself}} \mathbf{A}_1(\mathbf{r}_2)\cdot d\mathbf{r}_2 \tag{48}$$

where the second equality follows by the Stokes theorem. Consequently,

$$\Phi_2 \;=\; \oint\limits_{\text{coil\#2}} d\mathbf{r}_2 \cdot \left(\frac{\mu_0 I_1}{4\pi} \oint\limits_{\text{coil\#1}} \frac{d\mathbf{r}_1}{|\mathbf{r}_1-\mathbf{r}_2|} \right) \;=\; \frac{\mu_0 I_1}{4\pi} \oint\!\!\oint\limits_{\substack{\mathbf{r}_1\in\text{coil\#1}\\ \mathbf{r}_2\in\text{coil\#2}}} \frac{d\mathbf{r}_1\cdot d\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}\,, \tag{49}$$

which means that the mutual inductance M_{21} of the two coils is

$$M_{21} \;=\; \frac{\mu_0}{4\pi} \oint\!\!\oint\limits_{\substack{\mathbf{r}_1\in\text{coil\#1}\\ \mathbf{r}_2\in\text{coil\#2}}} \frac{d\mathbf{r}_1\cdot d\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}\,. \tag{50}$$

This formula is manifestly symmetric between the two coils, thus $M_{21} = M_{12}$, *quod erat demonstrandum.*

To illustrate the usefulness of this symmetry theorem, consider two coaxial solenoidal coils, one inside the other. Specifically, let the first coil be both shorter and narrower than the second coil, and let's put the first coil in the middle of the hollow space inside the second coil. There is no iron or other ferromagnetic materials in this two-coil system.

For simplicity, let's assume that the outer coil's length is much larger than its diameter. In this case, calculating the mutual inductance M_{12} is rather easy: The current I_2 in the second (outer) coil creates a uniform magnetic field

$$\mathbf{B}_2 \;=\; \frac{\mu_0 N_2 I_2}{\ell_2}\,\hat{\mathbf{z}} \tag{51}$$

inside that coil — and in particular, inside the inner coil. Consequently, the flux of this field through the inner coil#1 is

$$\Phi_1 \;=\; B_2 \times \pi r_1^2 N_1 \;=\; \frac{\mu_0 N_2 I_2}{\ell_2} \times \pi r_1^2 N_1 \;=\; \mu_0 N_1 N_2 \times \frac{\pi r_1^2}{\ell_2} \times I_2\,. \tag{52}$$

In terms of the mutual inductance, this means

$$M_{12} \;=\; \mu_0 N_1 N_2 \times \frac{\pi r_1^2}{\ell_2}\,. \tag{53}$$

On the other hand, the direct calculation of the M_{21} mutual inductance is much harder. Indeed, the magnetic field of the current I_1 in the first coil is approximately uniform inside that coil, but become rather complicated near its poles; and since the poles of the first coil are inside the second coil, calculating the net magnetic flux through the second coil becomes quite a challenge. Fortunately, the symmetry theorem allows us to avoid this hard calculation and simply use

$$M_{21} \;=\; M_{12} \;=\; \mu_0 N_1 N_2 \times \frac{\pi r_1^2}{\ell_2}\,. \tag{54}$$

Another useful theorem puts an upper limit on mutual induction of any two coils:

$$M_{12} \;=\; M_{21} \;\le\; \sqrt{L_1 L_2}\,. \tag{55}$$

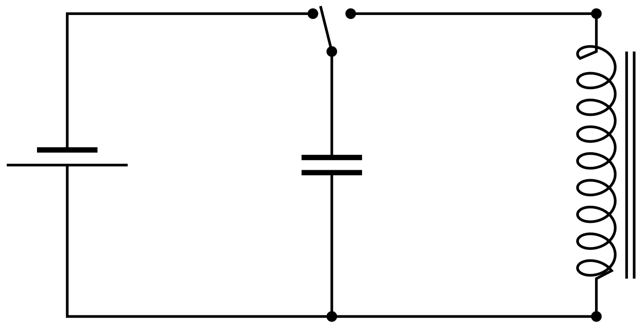
I am not going to prove this theorem in class. Instead, let me simply mention the dimensionless ratio

$$k_{12} \;=\; \frac{M_{12}}{\sqrt{L_1 L_2}} \;\le\; 1 \tag{56}$$

called the *magnetic coupling coefficient* of the two coils. A good transformer should have $k_{12} \approx 1$. On the other hand, for two unrelated coils used independently from each other the coupling coefficient should be as small as possible.

LC Circuit

Consider an LC circuit comprised of an inductor and a capacitor:



The switch starts in the left position to allow the capacitor to charge. At time $t_0 = 0$, the switch moves to the right position, and the capacitor discharges through the inductor. As we shall see in a moment, in this LC circuit the current and the voltage oscillate rather than decay.

For simplicity, let's assume negligible ohmic resistance of the inductor and the rest of the wires. Consequently, the voltage across the inductor is the same as the EMF induced by the time dependence of the current through it, thus

$$V(t) \;=\; \mathcal{E}(t) \;=\; -L \times \frac{dI(t)}{dt}\,. \tag{57}$$

But the same current $I(t)$ also flows through the capacitor, and the voltage across the capacitor is also the same as across the inductor, hence

$$I(t) \;=\; \frac{dQ(t)}{dt} \;=\; C \times \frac{dV(t)}{dt}\,. \tag{58}$$

Combining equations (57) and (58), we get

$$V(t) \;=\; -L\,\frac{d}{dt}\left(I \;=\; C\frac{dV}{dt}\right) \;=\; -LC \times \frac{d^2V}{dt^2}\,, \tag{59}$$

or equivalently

$$\frac{d^2V}{dt^2} \;=\; -\omega^2 V(t) \tag{60}$$

$$\omega = \frac{1}{\sqrt{LC}}\,. \tag{61}$$

Eq. (60) is the harmonic oscillator equation, and its solution is a harmonically oscillating voltage with some amplitude A and initial phase ϕ_0,

$$V(t) \;=\; A\times\cos(\omega t+\phi_0). \tag{62}$$

Consequently, the current (58) through the LC circuit also oscillates harmonically with the amplitude $AC\omega$ and the phase $\phi_0 + 90°$,

$$I(t) \;=\; CA\times\frac{d}{dt}\cos(\omega t+\phi_0) \;=\; -CA\omega\times\sin(\omega t+\phi_0) \;=\; CA\omega\times\cos(\omega t+\phi_0+90°). \tag{63}$$

The amplitudes and the initial phases of the voltage and current oscillations follow from the initial conditions: Before the switch is thrown, the capacitor is charged to the initial voltage V_0 (the same as the battery's voltage), and there is no current through the inductor. Since neither inductor's current nor capacitor's voltage can be instantly changed, it follows that at the time $t_0 = 0$ when the LC circuit forms, we have

$$@t \;=\; 0: \quad V \;=\; V_0 \text{ and } I \;=\; 0, \tag{64}$$

hence $\phi_0 = 0$ and $A = V_0$, thus

$$\begin{aligned} V(t) \;&=\; V_0\times\cos(\omega t), \\ I(t) \;&=\; -V_0C\omega\times\sin(\omega t). \end{aligned} \tag{65}$$

Magnetic Energy

Consider what happens when one tries to increase the current $I(t)$ flowing through an inductor coil. The coil's self-inductance F leads to EMF

$$\mathcal{E}_{\text{coil}} = -L\,\frac{dI}{dt} \tag{66}$$

which resists changing the current and performs negative work

$$dW_{\text{coil}} \;=\; \mathcal{E}_{\text{coil}} \times dQ \;=\; -L\,\frac{dI}{dt} \times I\,dt = \; -L \times I \times dI.$$

Note that this negative work is independent of the time it takes to change the current! This negative work has to be overcome by the positive work of the battery,

$$dW_{\text{battery}} \;=\; -dW_{\text{coil}} \;=\; +LI\,dI, \tag{67}$$

hence for a finite change of the current,

$$W_{\text{net}} \;=\; \int\limits_{I_1}^{I_2} LI\,dI \;=\; \frac{L}{2}\left(I_2^2 - I_1^2\right). \tag{68}$$

This work is *stored as the magnetic energy* of the inductor coil,

$$U_{\text{mag}} \;=\; \frac{LI^2}{2}\,, \tag{69}$$

which may be later used up to power some circuit for a short time, for example the light bulb in the example on the previous page.

Indeed, let's show that the net energy dissipated by the Ohmic resistance R in the RL circuit while the current is exponentially decaying is precisely the magnetic energy (69) stored in the inductor: The dissipated power is

$$P(t) \;=\; R\times I^2(t) \;=\; R\times I_0^2\times\exp(-2t/\tau), \tag{70}$$

hence net dissipated energy

$$W_{\rm net} \;=\int\limits_0^\infty P(t)\,dt \;=\int\limits_0^\infty RI_0^2\,\exp(-2t/\tau)\,dt \;=\; RI_0^2\times\frac{\tau}{2} \tag{71}$$

where $R\times\tau=L$ according to eq. (7), thus

$$W_{\rm net} \;=\; \frac{LI_0^2}{2}\,, \tag{72}$$

— which is precisely the initial energy stored in the inductor according to eq. (69).

For another example, consider the electric and the magnetic energies in an LC circuit. The voltage and the current in such a circuit oscillate as

$$\begin{aligned} V(t) \;&=\; V_0\times\cos(\omega t),\\ I(t) \;&=\; -V_0C\omega\times\sin(\omega t), \end{aligned} \tag{65}$$

hence the electric and the magnetic energies evolve with time as

$$U_{\rm el}(t) \;=\; \tfrac12 CV^2(t) \;=\; \tfrac12 CV_0^2\times\cos^2(\omega t), \tag{73}$$

$$U_{\rm mag}(t) \;=\; \tfrac12 LI^2(t) \;=\; \tfrac12 L(V_0C\omega)^2\times\sin^2(\omega t). \tag{74}$$

Furthermore, since $\omega^2=1/LC$, the maximal magnetic energy of the inductor is

$$U_{\rm mag}^{\rm max} \;=\; \tfrac12 L(V_0C\omega)^2 \;=\; \tfrac12 CV_0^2\times LC\omega^2 \;=\; U_{\rm el}^{\rm max}\times 1, \tag{75}$$

the same as the maximal electric energy of the capacitor. Consequently, the net electric + magnetic energy of the LC oscillator remains constant,

$$U_{\rm net} \;=\; U_{\rm el} \;+\; U_{\rm mag} \;=\; \tfrac12 CV_0^2\times\cos^2(\omega t) \;+\; \tfrac12 CV_0^2\times\sin^2(\omega t) \;=\; \tfrac12 CV_0^2 \;=\; {\rm const}, \tag{76}$$

similar to the net potential + kinetic energy of a mechanical harmonic oscillator.

⋆ ⋆ ⋆

Next, let's relate the magnetic energy $\frac{1}{2}LI^2$ of an inductor coil to the magnetic field in the inductor. For a coil of most general geometry,

$$U = \frac{LI^2}{2} = \frac{I}{2} \times \Phi_{\text{coil}} = \frac{I}{2} \times \oint\limits_{\text{coil}} \mathbf{A} \cdot d\vec{\ell}. \tag{77}$$

We may generalize this formula from a coil made form a thin wire to a thick conductor with some free current $\mathbf{J}_f$ flowing through its volume by simply changing $I d\vec{\ell}$ to $\mathbf{J}_f\, d^3\text{Vol}$, thus

$$U = \frac{1}{2} \iiint \mathbf{A} \cdot \mathbf{J}_f\, d^3\text{Vol}. \tag{78}$$

Moreover, by Ampere's Law $\mathbf{J}_f = \nabla \times \mathbf{H}$, hence

$$\begin{aligned} \mathbf{A} \cdot \mathbf{J}_f &= \mathbf{A} \cdot (\nabla \times \mathbf{H}) \\ &\qquad \langle\!\langle \text{ by Leibniz rule for the } \nabla \cdot (\mathbf{A} \times \mathbf{H}) \rangle\!\rangle \\ &= \nabla \cdot (\mathbf{H} \times \mathbf{A}) + \mathbf{H} \cdot (\nabla \times \mathbf{A}) \\ &= \nabla \cdot (\mathbf{H} \times \mathbf{A}) + \mathbf{H} \cdot \mathbf{B}, \end{aligned} \tag{79}$$

Consequently, the integral (78) for the magnetic energy becomes

$$\begin{aligned} U &= \frac{1}{2} \iiint\limits_{\mathcal{V}} \mathbf{H} \cdot \mathbf{B}\, d^3\text{Vol} + \frac{1}{2} \iiint\limits_{\mathcal{V}} \nabla \cdot (\mathbf{H} \times \mathbf{A})\, d^3\text{Vol} \\ &= \frac{1}{2} \iiint\limits_{\mathcal{V}} \mathbf{H} \cdot \mathbf{B}\, d^3\text{Vol} + \frac{1}{2} \iint\limits_{\mathcal{S}} (\mathbf{H} \times \mathbf{A}) \cdot d^2\mathbf{a}, \end{aligned} \tag{80}$$

where $\mathcal{V}$ is some volume which includes all the current-carrying conductors, and $\mathcal{S}$ is the complete surface of that volume, whatever it is. We can take the volume $\mathcal{V}$ to be as large as we want, so let's make it a ball of very large radius R. In the limit $R \to \infty$, the surface integral in eq. (80) vanishes; indeed, very far from all the currents,

$$\mathbf{A} \propto \frac{1}{R^2}, \quad \mathbf{H} \propto \frac{1}{R^3}, \quad \text{Area}(\mathcal{S}) = 4\pi R^2, \tag{81}$$

hence

$$\iint_{\mathcal{S}} (\mathbf{H} \times \mathbf{A}) \cdot d^2\mathbf{a} \propto \frac{1}{R^3} \xrightarrow[R\to 0]{} 0. \tag{82}$$

In the same $R \to \infty$ limit, the volume integral over $\mathcal{V}$ becomes the integral over the whole space, thus

$$U_{\text{magnetic}} = \frac{1}{2} \iiint_{\substack{\text{whole}\\\text{space}}} \mathbf{H} \cdot \mathbf{B}\, d^3\text{Vol}. \tag{83}$$

Example: toroidal coil.

Earlier in these notes we have calculated the self-inductance of a toroidal coil with an iron core as

$$L = \frac{\mu\mu_0 N^2 A}{2\pi R}, \tag{84}$$

so the net magnetic energy stored in this coil is

$$U = \frac{LI^2}{2} = \frac{\mu\mu_0 N^2 A}{2\pi R} \times \frac{I^2}{2} \tag{85}$$

where I is the current through the coil. The magnetic fields $\mathbf{H}$ and $\mathbf{B}$ created by this coil are negligibly small outside the iron toroid, while inside the toroid

$$\mathbf{H} \approx \frac{NI}{2\pi R}\,\hat{\phi}, \quad \mathbf{B} = \mu\mu_0\mathbf{H}, \quad \Longrightarrow \quad \mathbf{H} \cdot \mathbf{B} \approx \mu\mu_0 \left(\frac{NI}{2\pi R}\right)^2. \tag{86}$$

The energy (83) of these magnetic fields is therefore

$$\begin{aligned}
U_{\text{magnetic}} &= \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{H}\cdot\mathbf{B}\,d^3\text{Vol} \approx \frac{1}{2}\iiint\limits_{\text{toroid}} \mathbf{H}\cdot\mathbf{B}\,d^3\text{Vol} \\
&\approx \frac{1}{2}\big(\text{approx. constant } \mathbf{H}\cdot\mathbf{B}\big)\times\big(\text{volume of the toroid}\big) \\
&= \frac{1}{2}\times\mu\mu_0\left(\frac{NI}{2\pi R}\right)^2\times 2\pi RA \\
&= \frac{I^2}{2}\times\frac{\mu\nu_0 N^2 A}{2\pi R}\,,
\end{aligned} \tag{87}$$

in perfect agreement with eq. (85) for the magnetic energy of the coil.

This example is rather similar to the electric energy stored in a capacitor: we can calculate it as simply

$$U_{\text{capacitor}} = \frac{CV^2}{2} = \frac{Q^2}{2C}\,, \tag{88}$$

or we may calculate the electric tension and displacement fields **E** and **D** inside the capacitor, and then obtain their energy as

$$U_{\text{electric}} = \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}\cdot\mathbf{D}\,d^3\text{Vol}, \tag{89}$$

we would get the same net energy either way.

Note the remarkable similarity between the electric energy (89) and the magnetic energy (83). Microscopically — or in vacuum — these energies become

$$U_{\text{electric}} = \frac{\epsilon_0}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}^2\,d^3\text{Vol}, \quad U_{\text{magnetic}} = \frac{1}{2\mu_0}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{B}^2\,d^3\text{Vol} \tag{90}$$

in MKSA units, or

$$U_{\text{electric}} = \frac{1}{8\pi}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}^2\,d^3\text{Vol}, \quad U_{\text{magnetic}} = \frac{1}{8\pi}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{B}^2\,d^3\text{Vol} \tag{91}$$

in Gaussian units. The similarity between these energies reflect similar behavior of the electric and magnetic fields in vacuum, the only difference being the way the E/B fields couple to the electric charges and currents. Indeed, had the Nature provided us with both electric and magnetic charges and currents, the similarity between the electric and the magnetic fields would be complete!