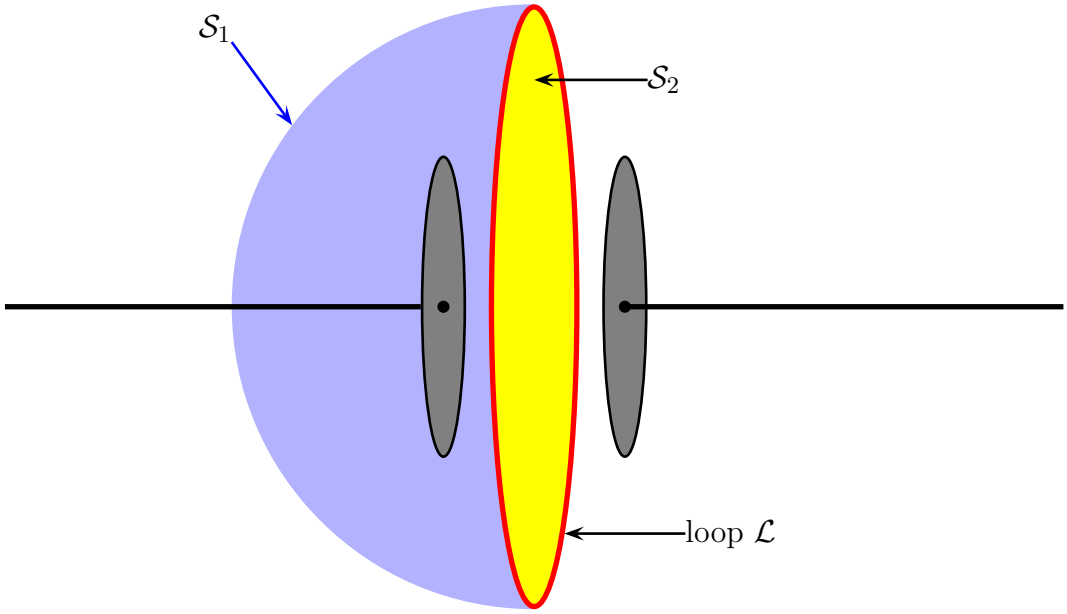
Maxwell–Ampere Law

The original Ampere Law

$$\nabla \times \mathbf{H} \;=\; \mathbf{J}_{\text{free}}\,, \tag{1}$$

$$\oint_{\mathcal{L}} \mathbf{H}\cdot d\vec{\ell} \;=\; I^{\text{net}}_{\text{free}}[\text{through loop } \mathcal{L}] \tag{2}$$

applies to the magnetic fields of steady currents, but it does not work for the time-dependent currents $\mathbf{J}(t)$ with $\nabla\cdot\mathbf{J} \neq 0$. Indeed, consider an AC current $I(t)$ flowing through a capacitor



(3)

Of course, the current does not really flow *through* the capacitor but only through the wires connected to each of the capacitor's plates. Inside the capacitor, there is no electric current; instead, the charges temporarily accumulate on the plates according to

$$\frac{dQ}{dt} \;=\; I(t). \tag{4}$$

Now, let's *try* to apply the Ampere Law (2) to the loop $\mathcal{L}$ on the diagram (3). The current through $\mathcal{L}$ is defined as the net current through any surface spanning the loop, but for a non-steady current different surfaces may give different answers. For example, the net current through the surface $\mathcal{S}_1$ — which crosses one of the wires connected to the capacitor — is the AC current $I(t)$ in the wire, while the net current through the surface $\mathcal{S}_2$ — which goes

between the two plates — is zero. So what should we use on the RHS of eq. (2)? $I(t)$? Zero? Something else?

In 1861, James Clerk Maxwell resolved this issue by adding the *displacement current*

$$\mathbf{J}_d(x,y,z;t) \;=\; \frac{\partial}{\partial t}\,\mathbf{D}(x,y,z;t) \tag{5}$$

to the conduction current $\mathbf{J}_c$ in the Ampere Law:

$$\nabla \times \mathbf{H} \;=\; \mathbf{J}_c \;+\; \mathbf{J}_d\,. \tag{6}$$

The $\mathbf{D}(x,y,z;t)$ in eq. (5) is the electric displacement field

$$\mathbf{D} \;=\; \epsilon_0 \mathbf{E} \;+\; \mathbf{P}, \tag{7}$$

so the displacement current (5) involves the time derivatives of both the electric field $\mathbf{E}$ and of the dielectric polarization $\mathbf{P}$,

$$\mathbf{J}_d \;=\; \epsilon_0\,\frac{\partial \mathbf{E}}{\partial t} \;+\; \frac{\partial \mathbf{P}}{\partial t}\,, \tag{8}$$

except in a vacuum where only the first term is present.

The divergences of the conduction current $\mathbf{J}_c$ and the displacement current $\mathbf{J}_d$ always cancel each other,

$$\nabla \cdot \mathbf{J}_c(x,y,z;t) \;+\; \nabla \cdot \mathbf{J}_d(x,y,z;t) \;=\; 0, \tag{9}$$

and that's what makes the *Maxwell–Ampere equation* (6) mathematically consistent. Likewise, in the global form

$$\oint\limits_{\mathcal{L}} \mathbf{H} \cdot d\vec{\ell} \;=\; \iint\limits_{\mathcal{S}} \big(\mathbf{J}_c + \mathbf{J}_d\big) \cdot d^2\mathbf{a} \tag{10}$$

for *any* surface $\mathcal{S}$ spanning the Ampere loop $\mathcal{L}$.

The divergence cancellation (9) stems from the continuity equation for the conduction current and the free charges,

$$\nabla \cdot \mathbf{J}_c(\mathbf{r},t) \;=\; -\frac{\partial}{\partial t}\,\rho_f(\mathbf{r},t), \tag{11}$$

and the Gauss Law for the electric displacement field,

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) \;=\; \rho_f(\mathbf{r},t), \tag{12}$$

— which works without any modifications for the time-dependent fields and charges. Taking the time derivative of both sides of the Gauss Law equation (12), we obtain

$$\frac{\partial}{\partial t}\,\rho_f(\mathbf{r},t) \;=\; \frac{\partial}{\partial t}\big(\nabla \cdot \mathbf{D}(\mathbf{r},t)\big) \;=\; \nabla \cdot \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} \;=\; \nabla \cdot \mathbf{J}_d(\mathbf{r},t), \tag{13}$$

hence in light of the continuity equation (11),

$$\nabla \cdot \mathbf{J}_c(\mathbf{r},t) \;=\; -\nabla \cdot \mathbf{J}_d(\mathbf{r},t). \tag{14}$$

Therefore, *regardless of how the currents and the charges change with time and place, the combined* conduction + displacement *current* $\mathbf{J}_c + \mathbf{J}_d$ *always has zero divergence,* and that's what makes the Maxwell–Ampere Law (6) mathematically consistent. And indeed, the Maxwell–Ampere Law (6) happens to work for any time-dependent fields and currents.

For an example of the displacement current, consider a parallel-plate capacitor (such as shown on diagram (3)), perhaps with a uniform dielectric between the plates. The displacement field between the plates is uniform (except near the edges of the plates), and its value follows from the plate charges $\pm Q$ via the Gauss Law,

$$\mathbf{D} \;=\; \frac{Q}{\text{plate area } A}\,\hat{\mathbf{z}}\,. \tag{15}$$

Consequently, when the capacitor's charge $Q(t)$ is time-dependent due to the conduction current $I_c(t)$ in the wires connected to the capacitor, the displacement field (15) inside the

capacitor becomes time-dependent, which gives rise to the displacement current

$$\mathbf{J}_d \;=\; \frac{dQ}{dt}\,\frac{\hat{\mathbf{z}}}{A}\,. \tag{16}$$

This displacement current density flows only between the plates, so the net displacement current through the capacitor is

$$I_d \;=\; \mathbf{J}_d \cdot (A\,\hat{\mathbf{z}}) \;=\; \frac{dQ}{dt}\,, \tag{17}$$

which is exactly equal to the conduction current $I_c(t)$ in the wires,

$$I_c(t) \;=\; \frac{dQ}{dt} \;=\; I_d(t). \tag{18}$$

Physically, this means that *the combined conduction + displacement current* $I(t) = I_c(t) + I_d(t)$ *flows without interruption through the wires and through the capacitor. It is this uninterrupted net current* $I(t)$ *which gives rise to the magnetic field surrounding the capacitor and the wires.*

Maxwell Equations

And God said,

$$\nabla \cdot \mathbf{D} \;=\; \rho_{\text{free}}\,, \tag{19.a}$$

$$\nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{19.b}$$

$$\nabla \cdot \mathbf{B} \;=\; 0, \tag{19.c}$$

$$\nabla \times \mathbf{H} \;=\; \mathbf{J}_{\text{free}} \;+\; \frac{\partial \mathbf{D}}{\partial t}\,, \tag{19.d}$$

and there was light.

The four equations (19.a–d) governing the macroscopic electric and magnetic fields are usually called the *Maxwell equations*. Historically, $3\frac{1}{2}$ of these equations were known before James Clerk Maxwell, but he was the first physicist to put them together and discover that they lead to the electromagnetic waves. In particular, he realized that Light is an electromagnetic wave, hence the biblical quote above.

Actually, Maxwell's 1865 article *A Dynamical Theory of the Electromagnetic Field* had about 20 equations, and his 1873 two-volume book *A Treatise on Electricity and Magnetism* had even more. Part of this proliferation was due to not using the vector notations — they were invented a few years after Maxwell's death, so he had to write the vector equations in components. Also, the paper and the book included some extra equations such as the Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, the linear dielectric equation $\mathbf{D} = \epsilon\epsilon_0\mathbf{E}$, *etc.*, *etc.* In any case, Maxwell's *Treatise* contained some tremendously important ideas, but it was poorly organised and very hard to read. It took Oliver Heaviside, Josiah Willard Gibbs, and Heinrich Hertz to eventually (1884?) make Maxwell's work accessible to an average physicist. In fact, it were Hertz and Heaviside who had grouped equations (19.a–d) together, wrote them in the modern form, and called them *the Maxwell's equations.*

In the integral form, the Maxwell equations (19.a–d) become

$$\oiint_{\mathcal{S}} \mathbf{D} \cdot d^2\mathbf{a} \;=\; Q_{\text{free}}[\text{inside } \mathcal{S}], \tag{20}$$

$$\oiint_{\mathcal{S}} \mathbf{B} \cdot d^2\mathbf{a} \;=\; 0, \tag{21}$$

$$\oint_{\mathcal{L}} \mathbf{E} \cdot d\vec{\ell} \;=\; -\frac{d}{dt} \iint_{\mathcal{S}_\mathcal{L}} \mathbf{B} \cdot d^2\mathbf{a}, \tag{22}$$

$$\oint_{\mathcal{L}} \mathbf{H} \cdot d\vec{\ell} \;=\; \iint_{\mathcal{S}_\mathcal{L}} \mathbf{J}_{\text{cond}} \cdot d^2\mathbf{a} \;+\; \frac{d}{dt} \iint_{\mathcal{S}_\mathcal{L}} \mathbf{D} \cdot d^2\mathbf{a}, \tag{23}$$

where $\mathcal{S}$ is any closed surface, $\mathcal{L}$ is any closed loop, and $\mathcal{S}_\mathcal{L}$ is any surface spanning the loop $\mathcal{L}$. These integral equations give rise to the boundary conditions for the fields at a boundary between two dielectric and/or magnetic media. In particular, using a brick-shaped Gaussian surface $\mathcal{S}$ in eqs. (20) and (21) — half the brick in one medium, half the brick in the other — we obtain

$$\mathbf{D}_\perp^{(1)} \;-\; \mathbf{D}_\perp^{(2)} \;=\; \sigma_{\text{free}}\,, \tag{24}$$

$$\mathbf{B}_\perp^{(1)} \;-\; \mathbf{B}_\perp^{(2)} \;=\; 0\,. \tag{25}$$

We have seen these equations before for the *static* $\mathbf{D}$ and $\mathbf{B}$ fields, but now we know they apply without changes to the time-dependent fields.

The other boundary conditions follow from eqs. (22) and (23) for a narrow rectangular Ampere loop $\Delta x \times \Delta z$, with small Δx and much smaller $\Delta z \ll \Delta x$. In the $\Delta z \to 0$ limit, eq. (22) becomes

$$\mathbf{\Delta x} \cdot \left(\mathbf{E}^{(1)} - \mathbf{E}^{(2)}\right) \;=\; 0 \tag{26}$$

since the magnetic flux through the loop vanishes with loop's area as $\Delta z \to 0$. Consequently, for static or dynamical electric fields.

$$\mathbf{E}_{\parallel}^{(1)} \;-\; \mathbf{E}_{\parallel}^{(2)} \;=\; 0 \tag{27}$$

Likewise, in eq. (23) for the same Ampere loop, the $\mathbf{D}$ flux through the loop vanishes in the $\Delta z \to 0$ limit, while the net free current through the loop is reduced to the surface conduction current only, hence

$$\mathbf{H}_{\parallel}^{(1)} \;-\; \mathbf{H}_{\parallel}^{(2)} \;=\; \mathbf{K}_{\rm cond} \times \mathbf{n} \tag{28}$$

where $\mathbf{n}$ is the unit vector $\perp$ to the boundary.

The boundary conditions (24), (25), (27), and (28) are the basis of the theory of reflection and refraction of the electromagnetic waves at boundaries of transparent materials. But that subject will have to wait until the second semester of the Classical Electrodynamics class (PHY 352 L).

As written, the Maxwell equations (19.a–d) govern the macroscopic electric and magnetic fields $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}$ in dielectric/magnetic matter. Microscopically, there are only $\mathbf{E}$ and $\mathbf{B}$ fields, and the Maxwell equations governing them are

$$\nabla \cdot \mathbf{E} \;=\; \frac{1}{\epsilon_0}\,\rho\,, \tag{29.a}$$

$$\nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{29.b}$$

$$\nabla \cdot \mathbf{B} \;=\; 0, \tag{29.c}$$

$$\nabla \times \mathbf{B} \;=\; \mu_0 \mathbf{J} \;+\; \mu_0 \epsilon_0\, \frac{\partial \mathbf{E}}{\partial t}\,. \tag{29.d}$$

Electromagnetic Waves

The most revolutionary aspect of the Maxwell equations (29.a–d) or (19.a–d) is that they allow for the *electromagnetic waves* propagating in the absence of any charges or currents. Or rather, it might take some time-dependent charges or currents to create an electromagnetic wave in the first place; but once created, the wave keeps propagating further and further away without any additional charges or currents.

To see how this works, consider the Maxwell equations in a vacuum, in a total absence of any charges and currents:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{30}$$

The last two equations here are coupled first-order differential equations for the electric and magnetic fields. We can decouple these equations by applying an extra curl:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E}, \end{aligned} \tag{31}$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \\ &= -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{B}. \end{aligned} \tag{32}$$

At the same time, a double curl of a vector field $\mathbf{V}$ is related to its Laplacian according to

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}. \tag{33}$$

In particular, for the electric and magnetic fields which have zero divergences (in the vacuum), we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}, \qquad \nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B}. \tag{34}$$

Consequently, the decoupled second-order equations (31) and (32) become

$$\begin{aligned} \nabla^2 \mathbf{E}(x,y,z,t) &= +\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E}(x,y,z,t), \\ \nabla^2 \mathbf{B}(x,y,z,t) &= +\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{B}(x,y,z,t). \end{aligned} \tag{35}$$

Physically, this means that each component $E_x, E_y, E_z, B_x, B_y, B_z$ of the electric or magnetic field in the vacuum obeys the *wave equation*

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \text{Field}(x,y,z,t) = 0, \tag{36}$$

with wave speed

$$\frac{1}{v^2} = \mu_0\epsilon_0 \quad \Longrightarrow \quad v = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c = 299\,792\,458 \text{ m/s}. \tag{37}$$

Back in Maxwell's time, the measurements of speed of light in the vacuum were a few percent off, about $3.15 \cdot 10^8$ m/s. The Coulomb constant $1/(4\pi\epsilon_0)$ was also a few percent off, so plugging it into eq. (37), Maxwell got $v \approx 3.11 \cdot 10^8$ m/s. The sheer coincidence between the experimental speed of light in the vacuum and the theoretical speed of the EM waves (also in the vacuum) immediately suggested to Maxwell that light is an electromagnetic wave. Later, with better measurements, both speeds were corrected by a few percent, but the Maxwell's conclusion stands: the light is an electromagnetic wave.

Let me conclude these notes with the speed of light in a linear dielectric/magnetic medium. In the absence of *free* charges or currents, the macroscopic Maxwell equations (19.a–d) become

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{1}{\epsilon\epsilon_0} \nabla \cdot \mathbf{D} = 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \mu\mu_0 \frac{\partial \mathbf{D}}{\partial t} = \mu\mu_0\epsilon\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{38}$$

Consequently,

$$\begin{aligned}\nabla^2\mathbf{E} &= \nabla(\nabla\cdot\mathbf{E}) - \nabla\times(\nabla\times\mathbf{E}) = -\nabla\times(\nabla\times\mathbf{E}) \\ &= -\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) = +\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = +\frac{\partial}{\partial t}\left(\mu\mu_0\epsilon\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\right) \\ &= +\mu\mu_0\epsilon\epsilon_0\,\frac{\partial^2}{\partial t^2}\,\mathbf{E},\end{aligned} \tag{39}$$

and likewise

$$\begin{aligned}\nabla^2\mathbf{B} &= \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) = -\nabla\times(\nabla\times\mathbf{B}) \\ &= -\nabla\times\left(\mu\mu_0\epsilon\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\right) = -\mu\mu_0\epsilon\epsilon_0\,\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = -\mu\mu_0\epsilon\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \\ &= +\mu\mu_0\epsilon\epsilon_0\,\frac{\partial^2}{\partial t^2}\,\mathbf{B}.\end{aligned} \tag{40}$$

Thus, each component of the electric or magnetic field obeys the wave equation

$$\left(\frac{1}{v^2}\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\mathrm{Field}(x,y,z,t) = 0, \tag{41}$$

for wave speed

$$\frac{1}{v^2} = \mu\mu_0\epsilon\epsilon_0 \quad\Longrightarrow\quad v = \frac{1}{\sqrt{\mu\epsilon}\sqrt{\mu_0\epsilon_0}} = \frac{c}{\sqrt{\mu\epsilon}}\,. \tag{42}$$