Ohm's Law

Ohm's Law — which is perhaps the best known law in all of Physics — applies to most conducting bodies — regardless if they conduct electricity well or poorly, or even so poorly they are called insulators. According to Ohm's Law, the current I through a body and the voltage V across that body are linearly related to each other,

$$V = R \times I. \tag{1}$$

Moreover, when the conducting body is made of a uniform material and has uniform cross-section, the *resistance* R is related to the body's length L and the cross-section area A by a simple formula

$$R = \frac{L}{A} \times \rho, \tag{2}$$

where the *electric resistivity* ρ depends only on the material in questions (as well as its temperature and pressure), but not on the geometry of the conducting body. The inverse of the resistivity

$$\sigma = \frac{1}{\rho} \tag{3}$$

is called the *electric conductivity.*

Ohm's linear relation (1) between the voltage and the current works surprisingly well over a huge range of voltages and currents. In comparison, a similar linear relation between the flow rate of some fluid in a pipe and the pressure difference between the pie's ends works only in the laminar regime, and even then the linearity is only approximate. Moreover, the hydrodynamic resistance of a pipe and the electric resistance of a wire depend on the pipe's or wire's diameter in a different way:

- Hydrodynamic resistance of a pipe is proportional to $1/(\text{diameter})^4$, *cf.* Poiseuille's Law.
- Electric resistance of a wire is proportional to $1/(\text{diameter})^2$, *cf.* Pouillet's equation (2).

This difference between the electric and the hydrodynamic resistances implies that the charge carriers in a conductor do not flow like a cohesive fluid but rather *diffuse* through the conductor. The diffusion work locally — every tiny (but macroscopic) bit of conductor conducts independently of any other bit, — and in every tiny bit, the current density $\mathbf{J}(x,y,z)$ is proportional to the (macroscopic) electric field $\mathbf{E}(x,y,z)$ in that bit,

$$\mathbf{J}(x,y,z) \;=\; \sigma\times\mathbf{E}(x,y,z). \tag{4}$$

This formula is the *local form of the Ohm's Law*. Indeed, for a uniform wire where each bit obeys eq. (4), the net current through the wire is

$$I \;=\; A\times J \;=\; A\times\sigma\times E \;=\; A\times\sigma\times\frac{V}{L}\,. \tag{5}$$

or in other words,

$$I \;=\; \frac{V}{R} \quad\text{for}\quad R \;=\; \frac{L}{A\sigma} \;=\; \frac{L}{A}\times\rho. \tag{6}$$

Drude–Lorentz Model

The earliest explanation of the local Ohm Law (4) for the metals was proposed by Paul Drude in 1900 — right after the discovery of the electron by J. J. Thomson in 1897 — and improved by Hendrik Antoon Lorentz in 1905. In this purely classical model — called the *Drude–Lorentz model,* or simply the *Drude model,* — a metal consists of motionless ions arranged in a crystalline lattice and of free electrons flying between the ions as molecules in a gas. The free electrons frequently collide with each other and with the ions, but the electron-electron collision do not matter as they preserve the net electron momentum and hence the net electric current. In the other hand, in the electron-ion collision, the ion stays motionless while the electron is equally likely to bounce back or continue forward. For simplicity, the Drude–Lorentz models disregards the strong microscopic electric field between the ions, so the free electrons are assumed to fly as free particles between their collisions with the ions. Here is the video of the Drude–Lorenz model.

Now let's apply a macroscopic electric field $\mathbf{E}$ to the Drude–Lorentz model of a metal. This fields pushes each electron with a force $\mathbf{F} = -e\mathbf{E}$ and gives it acceleration

$$\mathbf{a} = -\frac{e}{m_e}\mathbf{E}. \tag{7}$$

Or rather, this is the electrons' acceleration between collisions. But during each collision of an electron with an ion, the electrons velocity vector randomly changes its direction: perhaps it continues forwards, perhaps bounces back, perhaps sideways — all new directions are equally probably. Altogether, the velocity of an electron is

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a} \times t = \mathbf{v}_0 - \frac{e}{m_e}\mathbf{E} \times t \tag{8}$$

where $\mathbf{v}_0$ is the velocity this electron acquired after its last collision with an ion and t is the time since that collision. Averaging this velocity vector over all the electrons in a macroscopic piece of the conductor, we obtain the *electron drift velocity* as

$$\langle\mathbf{v}\rangle = \langle\mathbf{v}_0\rangle - \frac{e}{m_e}\mathbf{E} \times \langle t\rangle\,. \tag{9}$$

Moreover, since in an electron-ion collision the electron is equally likely to bounce in all directions, the average velocity vector right after the last collision is zero,

$$\langle\mathbf{v}_0\rangle = 0. \tag{10}$$

Also, at any particular moment in time, average over the electrons of the times t they had since their last collision with an ion is simply $\frac{1}{2}$ the *mean time between collisions*,

$$\langle t\rangle = \frac{\tau}{2}\,. \tag{11}$$

Therefore,

$$\langle\mathbf{v}\rangle = 0 - \frac{e}{m_e}\mathbf{E} \times \frac{\tau}{2} = -\frac{e\tau}{2m_e}\mathbf{E} \tag{12}$$

— in the absence of macroscopic electric field the electron gas as a whole stays at rest, but in a macroscopic field $\mathbf{E}$, the electron gas starts drifting with velocity (12). Consequently, there is electric current

$$\mathbf{J} \;=\; -en_e\,\langle\mathbf{v}\rangle \tag{13}$$

where n_e is the *free electron density, i.e.*, the number of free electrons in a unit volume. Altogether, we have

$$\mathbf{J} \;=\; -en_e \times \frac{-e\tau}{2m_e}\,\mathbf{E} \;=\; +\frac{e^2 n_e \tau}{2m_e}\,\mathbf{E}, \tag{14}$$

which explains the local Ohm's Law (4). Moreover, the Drude–Lorentz model gives us the metal's conductivity as

$$\sigma \;=\; \frac{e^2}{2m_e} \times n_e \times \tau. \tag{15}$$

Finally, in the classical Drude–Lorentz model, the mean time τ between electron-ion collisions can be obtained as

$$\tau \;=\; \frac{\ell}{v_T} \;=\; \frac{\text{mean free path of an electron}}{\text{thermal avg. speed of an electron}}\,. \tag{16}$$

The mean free path ℓ — the average distance an electron travels between collisions with the ions — is fixed by the ion size and the ion lattice geometry, while the thermal avg. speed of an electron increases with temperature as $v_T \sim \sqrt{3kT/m_e} \propto \sqrt{T}$. Consequently,

$$\tau \;\propto\; \frac{1}{\sqrt{T}} \tag{17}$$

and hence metal's conductivity (15) decreases with temperature as $1/\sqrt{T}$. Or in terms of the resistivity $\tau = 1/\sigma$,

$$\rho \;\sim\; \sqrt{T}. \tag{18}$$

In reality, eq. (15) works OK under reasonable assumptions about the mean free path ℓ and the free electron density in a metal, but the temperature dependence (18) of the resistivity is quite off the mark. Instead, the dependence is approximately linear,

$$\rho(T) \;\approx\; \rho_0 \;+\; \alpha T, \tag{19}$$

where the ρ_0 terms is rather small and become important only at cryogenic temperatures.

To explain eq. (19) we turn to the quantum version of the Drude–Lorentz model known as the *Drude–Sommerfeld model*. In this model, the electrons form quantum degenerate gas obeying Fermi–Dirac statistics rather than the classical Boltzmann statistics. You will learn about such degenerate gases in the Statistical Mechanics class PHY 369; for the moment, let me simply say that the average speed $v_{\rm avg}$ of a quantum electron in a metal is approximately temperature-independent and much higher than the classical thermal speed v_T.

Another quantum feature aspect of the electron gas is that in a perfect lattice of motionless ions, the electrons do not collide with the ions! Instead, they simply fly through the lattice, although their wave functions become modified from the simple plane waves into Bloch waves. However, in an imperfect lattice, the electrons do collide with impurities and crystal defects. Also, at finite temperatures, the ions vibrate, and their vibrations can be described by a quantum gas of *phonons*, and the phonons cal collide with the electrons and change the directions of their velocities. At high-enough temperatures (*i.e.*, not cryogenic), the phonon density is proportional to the temperature, hence the mean free path of an electron between collisions behaves as

$$\frac{1}{\ell}\;\sim\;\text{const}\times T\;+\;\text{const}' \tag{20}$$

where the second term is due to collisions with impurities and crystal defects. Consequently, the mean time between collisions behaves as

$$\frac{1}{\tau}\;=\;\frac{v_{\rm avg}}{\ell}\;=\;\text{const}\times T\;+\;\text{const}', \tag{21}$$

so the electric resistivity of the metal behaves as

$$\rho\;=\;\frac{1}{\sigma}\;=\;\frac{2m_e}{e^2n_e}\times\frac{1}{\tau}\;=\;\text{const}\times T\;+\;\text{const}', \tag{22}$$

— in perfect agreement with eq. (19).

CONDUCTION IN OTHER MATERIALS

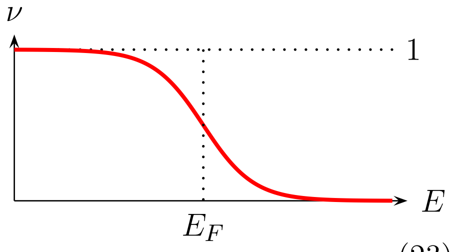
Now that we understand the electric conduction in metals, let's consider other kinds of materials — the semi-metals like graphite, the semiconductors like silicon, and the insulators like glass or teflon. For simplicity, let me skip the electrolytes, the plasmas, the superconductors, and other exotic kinds of materials and focus on ordinary solids at ordinary temperatures.

The hugely different resistivities of the metals, the semiconductors, and the insulators — ranging from $\rho = 1.6 \cdot 10^{-8}\ \Omega \cdot \mathrm{m}$ for the silver metal to $\rho > 10^{+18}\ \Omega \cdot \mathrm{m}$ for the teflon — stem from very different densities of the charge carriers — free electrons or holes. To understand them we need a bit of quantum theory of solids.

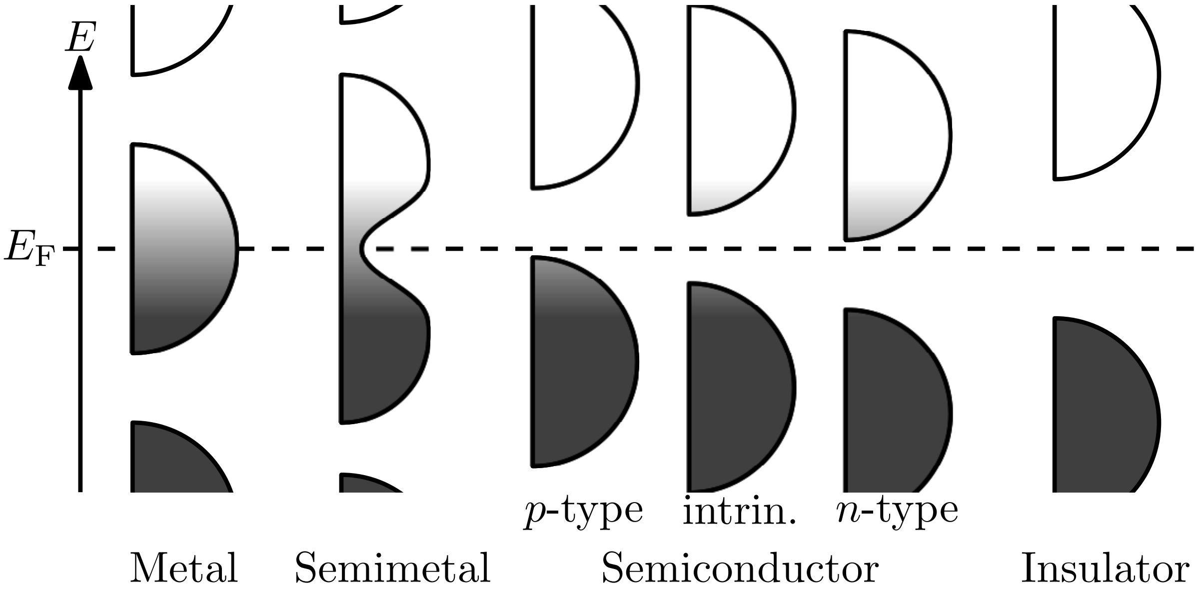
In a standalone atom, the electrons cannot have arbitrary energies. Instead, they must occupy discrete quantum states with specific discrete energy levels; the energies between these levels are impossible. When several atoms are brought in close contact — like the neighboring atoms in a crystal — the energy levels split into sublevels, typically N sublevels per level for N atoms. For a macroscopic crystal with an approximately infinite number of atoms, the sublevels merge into continuous *energy bands*. An electron may have energy within any of these bands, but the energies in the gaps between the bands are not allowed.

The electrons seek to minimize their energies, but they can't all fall into the lowest-energy state because of the Pauli principle. Instead, the electrons 'fill up' the band structure starting from the bottom and stopping when all the electrons are accommodated; the top energy level filled by the electrons is called the *Fermi energy* or the *Fermi level.* Or rather, the Fermi level E_F is the top filled-up level at zero temperature. At finite temperature, we get partially-filled energy levels within a few kT of the Fermi level; the filling fraction behaves like

$$\nu = \frac{\#\text{electrons}}{\#\text{quantum states}} = \frac{1}{1+\exp\bigl((E-E_F)/kT\bigr)} \tag{23}$$



Depending on a material, the Fermi level may lie with a continuous band of energies or between the bands. The following picture (borrowed from Wikipedia) shows a few examples:



Here, height is energy while width denotes the density of available states for that energy level. The color follows the filling fraction (23): black for completely filled, white for empty, with shades of gray for partial filling. In metals and semimetals, the Fermi level E_F lies inside a continuous band. In semiconductors the Fermi level is in a gap between the bands gap, but there is a nearby band which gets thermally populated by free electrons or holes. Finally, in insulators the Fermi level lies in the middle of a wide gap, so the bands below are filled-up while the bands above are empty.

Within an energy band, the electrons energy depends on its momentum, although the dependence is much more complicated than $p^2/2m$. So if a band is completely filled up, the Pauli principle does not allow the electrons to change their momenta, which means they cannot be accelerated by the macroscopic electric field. Thus, the completely filled bands do not contribute to the electric conduction. Likewise, the completely empty bands also do not contribute. Instead, the conduction stems from the partially filled levels near the Fermi level.

- In metals, there are a big lot of energy level right near the Fermi level, hence high electric conductivity. In semimetals, there few fewer levels right at the E_F, hence lower conductivity.

- In semiconductors, there are no allowed energy levels right at the Fermi level, but the thermal excitation puts some electrons in the band just above the E_F or makes some holes (un-filled levels) in the band just below the E_F, and these excited electrons or holes provide for n-type or p-type electric conductivity. The abundance of such excited electrons or holes rises sharply with temperatures, so in the semiconductors, the conductivity increases rather than decreases with temperature.
- In insulators, the Fermi level lies in the middle of a wide gap between the band. Similar to the semiconductors, there are a few thermally excited electrons in the band above the E_F and a few holes in the band below the E_F, but wider gap makes for a higher energy cost of such excited electrons or holes, so their abundance is exponentially lower than in semiconductors. Consequently, the insulators have much smaller electric conductivities than the semiconductors.

Implications of the Local Ohm's Law

Consider a steady current flowing through a uniform conductor. Regardless of the conductor shape, at every point inside the conductor we have

$$\mathbf{E}(\mathbf{r}) \;=\; \rho\mathbf{J}(\mathbf{r}). \tag{4}$$

Also, the density of a steady current must be divergence-less,

$$\nabla\cdot\mathbf{J} \;=\; 0, \tag{24}$$

Therefore, in the middle of a uniform conductor, the electric field is also divergence-less,

$$\nabla\cdot\mathbf{E} \;=\; \nabla\cdot(\rho\mathbf{J}) \;=\; \rho(\nabla\cdot\mathbf{J}) \;=\; 0, \tag{25}$$

and the electric potential $V(\mathbf{r})$ obeys the Laplace equation

$$\nabla^2 V(\mathbf{r}) \;=\; -\nabla\cdot\mathbf{E}(\mathbf{r}) \;=\; 0. \tag{26}$$

Now consider the surfaces of a conductor. If a surface is not in contact with another conductor (a non-contact surface), then the electric current cannot flow through that boundary. In terms of the local current density $\mathbf{J}(\mathbf{r})$, this means the component $\perp$ to the boundary must vanish, $J^\perp = 0$. By the local Ohm's Law (4), this meas that the normal component of the electric field must also vanish, $E^\perp = 0$. Thus, at a non-contact surface of a conductor, the electric field is parallel to the surface.

The above rules — $\nabla^2 V = 0$ in the bulk of a conductor and $E^\perp = 0$ at its non-contact surfaces — give rise to the **uniqueness theorem** for conductors of most general shapes: *If we specify the potential at all contact surfaces of a conductor, there is a unique solution to the $V(\mathbf{r})$ throughout the conductor, and hence for the electric field and the current throughout the conductor.*

For example, consider a uniform conductor in the shape of a cylinder of length L and radius a. The outer surface of the cylinder is non-contact while the top and the bottom disks are in contact with much better conductors, hence uniform potential V_t at the top disk and a different uniform potential V_b at the bottom disk. The potential $V(s,\phi,z)$ in this conducting cylinder obeys the following conditions:

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{for } s<a \text{ and } 0<z<L, \\ E_s = -\frac{\partial V}{\partial s} &= 0 \quad \text{for } s=a, \\ V &= V_b \quad \text{for } z=0, \\ V &= V_t \quad \text{for } z=H, \end{aligned} \tag{27}$$

and there is an obvious solution

$$V(s,\phi,z) = V(z \text{ only} = V_b + \frac{V_t - V_b}{L}\times z. \tag{28}$$

By the uniqueness theorem, this is the only solution and there is no need to look for other, more complicated solution. Thus, the cylindrical conductor has a uniform electric field

$$\mathbf{E} = \frac{V_b - V_t}{L}\,\hat{\mathbf{z}} \tag{29}$$

and a uniform current

$$\mathbf{J} = \frac{\mathbf{E}}{\rho} = \frac{V_b - V_t}{L\rho}\,\hat{\mathbf{z}}. \tag{30}$$

The overall current is therefore

$$I \;=\; \pi a^2 \times J_z \;=\; \pi a^2 \times \frac{V_b - V_t}{L\rho}\,, \tag{31}$$

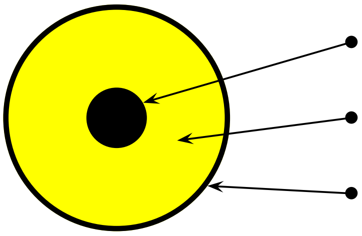
In perfect agreement with the Ohm's Law for the overall resistance

$$R \;=\; \frac{\Delta V = V_b - V_t}{I} \;=\; \frac{L\rho}{\pi a^2}\,. \tag{32}$$

In the limit of $L \gg a$, the cylinder in this example becomes a straight uniform wire. We may extend the above solution to curving wires, as long as the radius of curvature is much larger than the wire's radius a. In this case, working with one bit of wire's length at a time, we find: (1) at every point of the wire, the **E** and **J** vectors point along the wire; (2) their magnitudes are uniform along the wire,

$$E \;=\; \frac{\Delta V}{L}, \quad J \;=\; \frac{E}{\rho} \;=\; \frac{I}{\pi a^2}\,. \tag{33}$$

For another example, consider a long coaxial cable with a leaky insulator between the inner and the outer wires:



- Inner wire: radius a, fixed constant potential V_1.
- Insulator: uniform conductivity $\sigma \neq 0$.
- Outer wire: (inner) radius b, fixed constant potential V_2.

In this case, we are interested in the current and the electric field in the insulator, whose contact surfaces are the inner and the other wire, while the non-contact surfaces are the two ends of the cable. In the limit of the infinitely long cable, these non-contact surfaces go away. We also get a translational symmetry of the cable, which together with the rotational symmetry tells us that

$$V(s,\phi,z)\;=\;V(s\text{ only}). \tag{34}$$

Hence, solving for $\nabla^2 V=0$ inside the insulator we immediately obtains

$$\frac{d^2V}{ds^2}\;+\;\frac{1}{s}\,\frac{dV}{ds}\;=\;0\quad\Longrightarrow\quad V(s)\;=\;\alpha\;+\;\beta\ln(s) \tag{35}$$

for some constants α and β. Fitting them to the boundary conditions at the inner and the outer wires, we obtain

$$V(s)\;=\;V_1\;+\;(V_2-V_1)\,\frac{\ln(s/a)}{\ln(b/a)}\,, \tag{36}$$

and hence

$$\mathbf{E}\;=\;\frac{V_1-V_2}{\ln(b/a)}\,\frac{\hat{\mathbf{s}}}{s}\,,\quad \mathbf{J}\;=\;\sigma\,\frac{V_1-V_2}{\ln(b/a)}\,\frac{\hat{\mathbf{s}}}{s}\,. \tag{37}$$

The net current through the leaky insulator is

$$I\;=\;L\times 2\pi s\times J_s(s)\;=\;2\pi L\times\sigma\times\frac{\Delta V}{\ln(b/a)}\;=\;\frac{\Delta V}{R} \tag{38}$$

for the resistance against leakage

$$R\;=\;\frac{\ln(b/a)}{2\pi L\sigma}\,. \tag{39}$$

There are more examples in your homework due 11/30, in particular problems **7.1** and **7.3**(a).