Electric Charge Conservation and the Continuity Equation

The net electric charge of any closed system is always conserved; whatever happens, this net charge is not going to change. Moreover, the charge conservation is *local:* The electric charge cannot suddenly jump from point A to point B; instead, it must flow as electric current all the way from A to B. Mathematically, this means that for any time-independent volume $\mathcal{V}$ with surface $\mathcal{S}$,

$$\begin{aligned} \frac{d}{dt}Q[\text{inside }\mathcal{V}] \;&=\; I_{\text{inflow}}[\text{through }\mathcal{S}] \;-\; I_{\text{outflow}}[\text{through }\mathcal{S}] \\ &=\; -I^{\text{net}}_{\text{outflow}}[\text{through }\mathcal{S}] \;=\; -\oiint\limits_{\mathcal{S}} \mathbf{J}\cdot\mathbf{d^2A}. \end{aligned} \tag{1}$$

In terms of the time-dependent volume charge density $\rho(\mathbf{r},t)$, the net charge inside volume $\mathcal{V}$ is

$$Q(t)[\text{inside }\mathcal{V}] \;=\; \iiint\limits_{\mathcal{V}} \rho(\mathbf{r},t)\, d^3\text{Vol}, \tag{2}$$

so for any time-independent volume $\mathcal{V}$, the time derivative of this charge is simple the integral of the time derivative of the charge density ρ,

$$\frac{d}{dt}Q[\text{inside }\mathcal{V}] \;=\; \iiint\limits_{\mathcal{V}} \frac{\partial\rho(\mathbf{r},t)}{\partial t}\, d^3\text{Vol}. \tag{3}$$

On the other hand, applying the Gauss theorem to eq. (1), we obtain

$$\frac{d}{dt}Q[\text{inside }\mathcal{V}] \;=\; -\oiint\limits_{\mathcal{S}} \mathbf{J}\cdot\mathbf{d^2A} \;=\; -\iiint\limits_{\mathcal{V}} (\nabla\cdot\mathbf{J})\, d^3\text{Vol}. \tag{4}$$

Comparing this formula to eq. (3), we arrive at

$$\iiint\limits_{\mathcal{V}} \left(\frac{\partial\rho}{\partial t} \,+\, \nabla\cdot\mathbf{J}\right) d^3\text{Vol} \;=\; 0. \tag{5}$$

Moreover, this formula must hold for any time-independent volume $\mathcal{V}$; it does not need to be the whole volume occupied by the charges and the currents, it can be any time-independent part of of the whole volume. And the only way the integral (5) over any such volume would vanish is if the integrand vanishes at every point in space. Thus, *the time-dependence and the position-dependence of the charge density* $\rho(x,y,z,t)$ *and the current density* $\mathbf{J}(x,y,z,t)$ *must be related so as to obey the continuity equation*

$$\frac{\partial\rho(x,y,z,t)}{\partial t} + \nabla\cdot\mathbf{J}(x,y,z,t) = 0 \quad \text{for all} \quad x,y,z,t. \tag{6}$$

This continuity equation is the local form of the Law of Electric Charge Conservation, and it always holds true, for every physical system of charges and currents, everywhere and everywhen, without any exceptions.

Of particular interest to *magnetostatics* are *steady* — *i.e.*, time-independent — currents $\mathbf{J}(x,y,z,\text{ but not }t)$. By the continuity equation (6), a steady currents should be accompanied by a time-dependent charge density

$$\rho(x,y,z,t) = \rho_0(x,y,z) - t\times\big(\nabla\cdot\mathbf{J}(x,y,z)\big). \tag{7}$$

In particular, at any place where $\nabla\cdot\mathbf{J}\neq 0$, the charge density has a steady, linear-in-time build-up, and after a while we would and up with a pretty large ρ. But a large charge density would create a strong electric field $\mathbf{E}$, which would affect the current density $\mathbf{J}$. Moreover, as ρ and hence $\mathbf{E}$ increases with time, its effect on the currents $\mathbf{J}$ would also increase with times, which is obviously incompatible with the steadiness of the current.

Therefore, a steady current $\mathbf{J}(x,y,z)$ must avoid the build-up of the electric charge $\rho(x,y,x,t)$. In light of the continuity equation (6) and hence eq. (7), this means that a steady current must have zero divergence,

$$\text{for a steady } \mathbf{J}(x,y,z), \quad \nabla\cdot\mathbf{J} = 0 \quad \text{at all} \quad x,y,z. \tag{8}$$

This zero divergence is crucial for the Ampere Law: the magnetic field of a steady current obeys

$$\nabla\times\mathbf{B}(x,y,z) = \mu_0\mathbf{J}(x,y,z), \tag{9}$$

which obviously cannot hold true unless $\nabla\cdot\mathbf{J}=0$.

As written, eq. (8) applies to the current densities $\mathbf{J}(x,y,z)$ flowing through the volume of some conductor. For the electric circuits made from thin wires, the zero-divergence rule becomes the Kirchhoff Law for the currents: (1) A steady current I in a wire is uniform along the wire, and (2) at any junction of several wires, the net current flowing into the junction equals to the net current flowing out from the junction,

$$I_{\rm inflow}^{\rm net} \;-\; I_{\rm outflow}^{\rm net} \;=\; 0. \tag{10}$$