Gradient, Divergence, Curl
and Related Formulae

The gradient, the divergence, and the curl are first-order differential operators acting on fields. The easiest way to describe them is via a vector *nabla* whose components are partial derivatives WRT Cartesian coordinates (x, y, z):

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{1}$$

The ∇ operator is a true vector in the following sense: Consider any two Cartesian coordinate systems (x, y, z) and (x', y', z') at some angles to each other; in general, they are related to each other as

$$\begin{aligned}
x' &= T_{x'x} \times x + T_{x'y} \times y + T_{x'z} \times z, \\
y' &= T_{y'x} \times x + T_{y'y} \times y + T_{y'z} \times z, \\
z' &= T_{z'x} \times x + T_{z'y} \times y + T_{z'z} \times z,
\end{aligned} \tag{2}$$

for some orthogonal 3×3 matrix T. Then the components of the ∇ operator in the (x', y', z') system — *i.e.*, the partial derivatives $\partial/\partial x'$, $\partial/\partial y'$, and $\partial/\partial z'$ — are related to the partial derivatives WRT x, y, z precisely as components of any other vector:

$$\begin{aligned}
\frac{\partial}{\partial x'} &= T_{x'x} \times \frac{\partial}{\partial x} + T_{x'y} \times \frac{\partial}{\partial y} + T_{x'z} \times \frac{\partial}{\partial z}, \\
\frac{\partial}{\partial y'} &= T_{y'x} \times \frac{\partial}{\partial x} + T_{y'y} \times \frac{\partial}{\partial y} + T_{y'z} \times \frac{\partial}{\partial z}, \\
\frac{\partial}{\partial z'} &= T_{z'x} \times \frac{\partial}{\partial x} + T_{z'y} \times \frac{\partial}{\partial y} + T_{z'z} \times \frac{\partial}{\partial z}.
\end{aligned} \tag{3}$$

Consequently, just like a regular vector — say, $\mathbf{A}$ — has different components in the two coordinate systems but they assemble to the same vector as

$$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} = A_{x'}\hat{\mathbf{x}}' + A_{y'}\hat{\mathbf{y}}' + A_{z'}\hat{\mathbf{z}}' = \text{same } \mathbf{A}, \tag{4}$$

for the ∇ operator we also have

$$\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} = \hat{\mathbf{x}}'\frac{\partial}{\partial x'} + \hat{\mathbf{y}}'\frac{\partial}{\partial y'} + \hat{\mathbf{z}}'\frac{\partial}{\partial z'} = \text{same } \nabla. \tag{5}$$

THE GRADIENT: Acting with the ∇ operator on a scalar field $S(x, y, z)$ produces a vector field

$$\nabla S \;=\; \frac{\partial S}{\partial x}\,\hat{\mathbf{x}} \;+\; \frac{\partial S}{\partial y}\,\hat{\mathbf{y}} \;+\; \frac{\partial S}{\partial z}\,\hat{\mathbf{z}} \;=\; \mathbf{grad}\,S(x, y, z) \tag{6}$$

called the *gradient* of S. Physically, the gradient measures how much S is changing with the location; it points in the direction of steepest increase of S while its magnitude is the rate of that increase. At the local maxima, local minima, or other stationary points of S, the gradient vanishes.

LEIBNIZ RULE for the gradient of a product of two scalar fields $S_1(x, y, z)$ and $S_2(x, y, z)$:

$$\nabla\big(S_1 S_2\big) \;=\; S_1 \nabla S_2 \;+\; S_2 \nabla S_1. \tag{7}$$

CHAIN RULE for the gradient of a function of another function of the 3 coordinates:

$$\textbf{If } P(x, y, z) \;=\; f\big(S(x, y, z)\big), \quad \textbf{then } \nabla P \;=\; \frac{dP}{dS}\,\nabla S. \tag{8}$$

As an example, let's apply this rule to a spherically symmetric field

$$P(x, y, z) \;=\; P(r\,\text{only}), \quad r \;=\; |\mathbf{r}| \;=\; \sqrt{x^2 + y^2 + z^2}. \tag{9}$$

By the chain rule, the gradient of any such field is

$$\nabla P(r) \;=\; \frac{dP}{dr}\,\nabla r \;=\; \frac{dP}{dr}\,\hat{\mathbf{r}} \tag{10}$$

where the second equality follows from the gradient of the radius r being the unit vector $\hat{\mathbf{r}}$ in the radial direction. This should be obvious from the physical meaning of the gradient, but let's prove this mathematically. Let's start with the gradient of $r^2 = x^2 + y^2 + z^2$:

$$\nabla(r^2) \;=\; \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\big(x^2 + y^2 + z^2\big) \;=\; 2x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 2z\,\hat{\mathbf{z}} \;=\; 2\mathbf{r}. \tag{11}$$

At the same time, by the chain rule,

$$\nabla(r^2) \;=\; \frac{dr^2}{dr}\,\nabla r \;=\; 2r\,\nabla r, \tag{12}$$

hence

$$\nabla r \;=\; \frac{1}{2r}\,\nabla(r^2) \;=\; \frac{1}{2r}\,(2\mathbf{r}) \;=\; \hat{\mathbf{r}}, \tag{13}$$

quod erat demonstrandum.

As an example of eq. (10), the gradient of any power of the radius is

$$\nabla\left(r^n\right) \;=\; nr^{n-1}\,\hat{\mathbf{r}}. \tag{14}$$

THE DIVERGENCE of a vector field $\mathbf{V}(x,y,z)$ is a scalar field $\mathbf{div}\,\mathbf{V}(x,y,z)$ which measures how much $\mathbf{V}$ spreads out at each point (x,y,z) (or for a negative divergence, how much $\mathbf{V}$ converges to the point (x,y,z)). Algebraically, the divergence is the scalar product (dot product) of the ∇ operator and the vector field on which it acts:

$$\mathbf{div}\,\mathbf{V}(x,y,z) \;=\; \nabla\cdot\mathbf{V} \;=\; \frac{\partial}{\partial x}V_x \;+\; \frac{\partial}{\partial y}V_y \;+\; \frac{\partial}{\partial z}V_z\,. \tag{15}$$
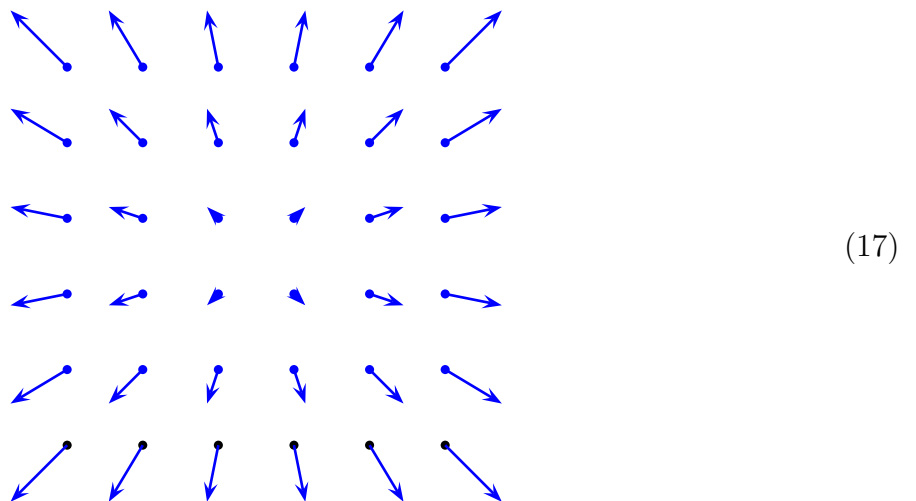
Example: A vector field parallel to the x axis spreading out in x direction, $\mathbf{V}(x,y,z) \;=\; cx\,\hat{\mathbf{x}}$ (for a constant c)
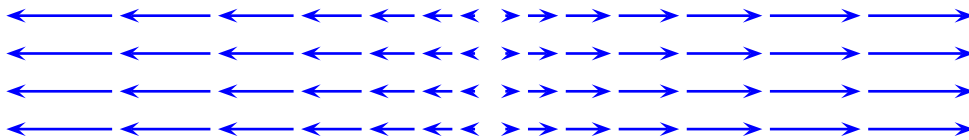
The divergence of this field is

$$\nabla\cdot\mathbf{V} \;=\; \frac{\partial V_x}{\partial x} \;+\; \frac{\partial V_y}{\partial y} \;+\; \frac{\partial V_z}{\partial z} \;=\; \frac{\partial cx}{\partial x} \;+\; \frac{\partial 0}{\partial y} \;+\; \frac{\partial 0}{\partial z} \;=\; c \;+\; 0 \;+\; 0 \;=\; c. \tag{16}$$

Another example: A field pointing and spreading out in the radial direction, $\mathbf{V} = c\mathbf{r}$,

$$\tag{17}$$

The divergence of this field is

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \frac{\partial cx}{\partial x} + \frac{\partial cy}{\partial y} + \frac{\partial cz}{\partial z} = c + c + c = 3c. \tag{18}$$

Similar to the Leibniz rule for the gradient, there is a Leibniz rule for the divergence of a product of a scalar field $S(x, y, z)$ and a vector field $\mathbf{V}(x, y, z)$:

$$\nabla \cdot (S\mathbf{V}) = (\nabla S) \cdot \mathbf{V} + S(\nabla \cdot \mathbf{V}). \tag{19}$$

This rule is particularly useful for calculating divergences of spherically symmetric radial fields

$$\mathbf{V}(x, y, z) = V(r \,\text{only}) \,\hat{\mathbf{r}} = \frac{V(r)}{r} \mathbf{r} \tag{20}$$

Indeed, by the Leibniz rule (19),

$$\nabla \cdot \left(\mathbf{V} = \frac{V(r)}{r} \mathbf{r} \right) = \left(\nabla \left(\frac{V(r)}{r} \right) \right) \cdot \mathbf{r} + \frac{V(r)}{r} (\nabla \cdot \mathbf{r}) \tag{21}$$

where in the second term on the RHS $\nabla \cdot \mathbf{r} = 3$, *cf.* the example (18). As to the first term on the RHS,

$$\nabla \left(\frac{V(r)}{r} \right) = \frac{d(V/r)}{dr} \hat{\mathbf{r}} = \left(\frac{V'}{r} - \frac{V}{r^2} \right) \hat{\mathbf{r}} \tag{22}$$

and hence

$$\left(\nabla \left(\frac{V(r)}{r} \right) \right) \cdot \mathbf{r} = \left(\frac{V'}{r} - \frac{V}{r^2} \right) \times \left(\hat{\mathbf{r}} \cdot \mathbf{r} = r \right) = \frac{dV}{dr} - \frac{V}{r}. \tag{23}$$

Plugging this formula and $\nabla \cdot \mathbf{r} = 3$ into eq. (21), we arrive at

$$\nabla \cdot (\mathbf{V} = V(r) \,\hat{\mathbf{r}}) = \frac{dV}{dr} - \frac{V}{r} + \frac{V}{r} \times 3 = \frac{dV}{dr} + 2\frac{V}{r}. \tag{24}$$

In particular, for $V(r) = kr^n$ (for a constant k),

$$\nabla \cdot (kr^n \,\hat{\mathbf{r}}) = nkr^{n-1} + 2kr^{n-1} = (n+2)kr^{n-1}. \tag{25}$$

Note: the Coulomb electric field of a point charge has this form for $n = -2$. Consequently,

$$\text{for } \mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2}, \quad \nabla \cdot \mathbf{E} = ((n = -2) + 2 = 0) \times \frac{Q}{4\pi\epsilon_0} \times \frac{1}{r^3} = 0, \tag{26}$$

the Coulomb electric field seems to be divergence-less! However, this formula is valid only outside the charge, at $\mathbf{r} \neq \mathbf{0}$, but right on top of the charge, at $\mathbf{r} = \mathbf{0}$, the electric field itself is

singular and its divergence is actually infinite. As we shall see later in class,

$$\text{for } \mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\hat{\mathbf{r}}}{r^2}\,,\qquad \nabla\cdot\mathbf{E}(\mathbf{r}) \;=\; \frac{Q}{\epsilon_0}\times\delta^{(3)}(\mathbf{r}). \tag{27}$$

THE CURL or *rotor* of a vector field $\mathbf{V}(x,y,z)$ is another vector field $\mathbf{curl\,V}(x,y,z)$ which measures the vorticity of the $\mathbf{V}$ field. Specifically, the x component of the $\mathbf{curl\,V}$ measure the vorticity of the $\mathbf{V}$ field in the yz plane (and thus around the x axis), and likewise for the y and z components.

Algebraically, $\mathbf{curl\,V}(x,y,z)$ obtains as a vector product of the ∇ and the vector field $\mathbf{V}$ on which it acts:

$$
\begin{aligned}
\mathbf{curl\,V} \;=\; \mathbf{rot\,V} \;&=\; \nabla\times\mathbf{V} \\
&=\; \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\times\left(\hat{\mathbf{x}}V_z + \hat{\mathbf{y}}V_y + \hat{\mathbf{z}}V_z\right) \\
&=\; \det\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix} \\
&=\; \hat{\mathbf{x}}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) \;+\; \hat{\mathbf{y}}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) \;+\; \hat{\mathbf{z}}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right).
\end{aligned}
\tag{28}
$$

Example: the radially symmetric spreading out field (17) $\mathbf{V}(\mathbf{r}) = c\mathbf{r}$ has zero curl. Indeed, the x component of the curl evaluates to

$$\bigl(\nabla\times(c\mathbf{r})\bigr)_x \;=\; \frac{\partial(c\mathbf{r})_z}{\partial y} - \frac{\partial(c\mathbf{r})_y}{\partial z} \;=\; \frac{\partial cz}{\partial y} - \frac{\partial cy}{\partial z} \;=\; 0 - 0 \;=\; 0, \tag{29}$$

and likewise, the y component and the z component of $\nabla\times(c\mathbf{r})$ also vanish.

Another example: a vector field $\mathbf{V} = cx\hat{\mathbf{y}} - cy\hat{\mathbf{x}}$ swirling in the xy plane like this

$$(30)$$
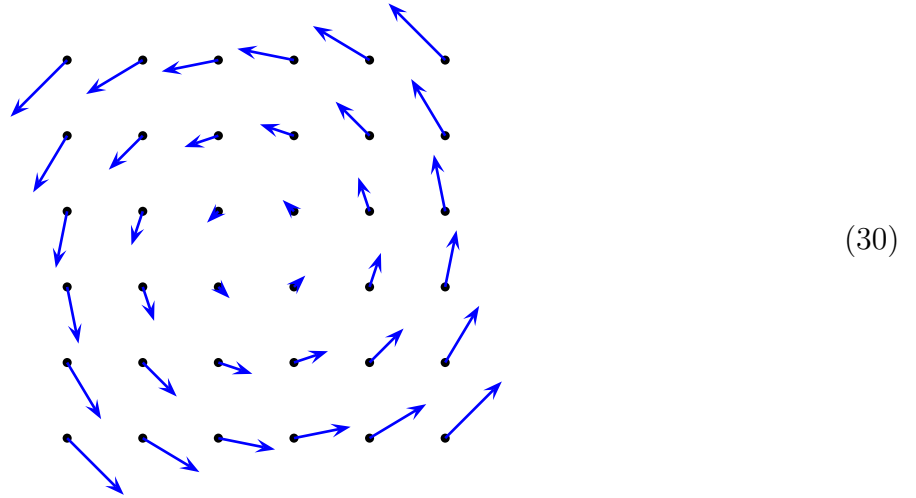
has a non-trivial curl in the z direction:

$$\left(\nabla \times (cx\hat{\mathbf{y}} - cy\hat{\mathbf{x}})\right)_z \;=\; \frac{\partial(V_y = cx)}{\partial x} \;-\; \frac{\partial(V_x = -cy)}{\partial y} \;=\; (+c) \;-\; (-c) \;=\; 2c, \qquad (31)$$

while the x and y components of the curl vanish:

$$\begin{aligned}
\left(\nabla \times (cx\hat{\mathbf{y}} - cy\hat{\mathbf{x}})\right)_x &= \frac{\partial(V_z = 0)}{\partial y} - \frac{\partial(V_y = +cx)}{\partial z} = 0 - 0 = 0, \\
\left(\nabla \times (cx\hat{\mathbf{y}} - cy\hat{\mathbf{x}})\right)_y &= \frac{\partial(V_x = -cy)}{\partial z} - \frac{\partial(V_z = 0)}{\partial x} = 0 - 0 = 0.
\end{aligned} \qquad (32)$$

LEIBNIZ RULE for the curl of a product of a scalar field $S(x, y, z)$ and a vector field $\mathbf{V}(x, y, z)$:

$$\nabla \times \left(S\mathbf{V}\right) \;=\; \left(\nabla S\right) \times \mathbf{V} \;+\; S\left(\nabla \times V\right), \qquad (33)$$

Using this rule, it is easy to show that any spherically symmetric radial field $\mathbf{V}(x, y, z) = V(r)\hat{\mathbf{r}}$

has zero curl. Indeed,

$$\mathbf{V} = \frac{V(r)}{r}\,\mathbf{r},$$

$$\nabla \times \mathbf{V} = \left(\nabla \frac{V}{r}\right) \times \mathbf{r} + \frac{V}{r}\left(\nabla \times \mathbf{r}\right)$$

$$\text{where } \nabla \times \mathbf{r} = 0,$$

$$\text{and } \nabla \frac{V}{r} = \frac{d(V/r)}{dr}\,\hat{\mathbf{r}}, \tag{34}$$

$$\text{hence } \nabla \times \mathbf{V} = \left(\frac{d(V/r)}{dr}\,\hat{\mathbf{r}}\right) \times \mathbf{r} + 0$$

$$= \frac{d(V/r)}{dr}\left(\hat{\mathbf{r}} \times \mathbf{r} = 0\right) + 0$$

$$= 0.$$

OTHER LEIBNIZ RULES:

(a) Divergence of a vector product of two vector fields $\mathbf{A}(x, y, z)$ and $\mathbf{B}(x, y, z)$,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \tag{35}$$

(b) Curl of a cross product of two vector fields,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}). \tag{36}$$

(c) Gradient of a dot product of two vector fields:

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}). \tag{37}$$

In the last two formulae here $(\mathbf{A} \cdot \nabla)$ and $(\mathbf{B} \cdot \nabla)$ are not the divergences but rather the differential operators

$$(\mathbf{A} \cdot \nabla) = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z},$$

$$(\mathbf{B} \cdot \nabla) = B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}, \tag{38}$$

which then act on other fields. For example,

$$
\begin{aligned}
(\mathbf{A} \cdot \nabla)S \;&=\; A_x \frac{\partial S}{\partial x} \;+\; A_y \frac{\partial S}{\partial y} \;+\; A_z \frac{\partial S}{\partial z}, \\
(\mathbf{A} \cdot \nabla)\mathbf{B} \;&=\; \left(A_x \frac{\partial B_x}{\partial x} \;+\; A_y \frac{\partial B_x}{\partial y} \;+\; A_z \frac{\partial B_x}{\partial z} \right) \hat{\mathbf{x}} \;+\; \left(A_x \frac{\partial B_y}{\partial x} \;+\; A_y \frac{\partial B_y}{\partial y} \;+\; A_z \frac{\partial B_y}{\partial z} \right) \hat{\mathbf{y}} \\
&\quad +\; \left(A_x \frac{\partial B_z}{\partial x} \;+\; A_y \frac{\partial B_z}{\partial y} \;+\; A_z \frac{\partial B_z}{\partial z} \right) \hat{\mathbf{z}}.
\end{aligned}
\tag{39}
$$

SECOND DERIVATIVES

The gradient, the divergence, and the curl are first-order differential operators for the fields. By acting with two such operators — one after the other — we can make interesting second-order differential operators.

Thanks to $\nabla \times \nabla = 0$, two combinations of first-order operators are identical zeros: (a) The curl of a gradient of any scalar field vanishes identically,

$$
\text{for any } S(x, y, z), \quad \mathbf{curl}(\mathbf{grad}(S)) \;=\; \nabla \times (\nabla S) \;\equiv\; 0.
\tag{40}
$$

(b) The divergence of a curl of any vector field vanishes identically,

$$
\text{for any } \mathbf{V}(x, y, z), \quad \mathbf{div}(\mathbf{curl}(\mathbf{V})) \;=\; \nabla \cdot (\nabla \times \mathbf{V}) \;\equiv\; 0.
\tag{41}
$$

Proof: Let's start with the curl of a gradient. For the z component of the curl, we have

$$
\left(\nabla \times (\nabla S) \right)_z \;=\; \frac{\partial}{\partial x}(\nabla S)_y \;-\; \frac{\partial}{\partial y}(\nabla S)_x \;=\; \frac{\partial}{\partial x}\frac{\partial S}{\partial y} \;-\; \frac{\partial}{\partial y}\frac{\partial S}{\partial x} \;=\; 0
\tag{42}
$$

because the partial derivatives WRT x, y, z can be taken in any order without changing the result. Likewise, the x and y components of the curl of a gradient vanish for the same reason:

$$
\begin{aligned}
\left(\nabla \times (\nabla S) \right)_x \;&=\; \frac{\partial}{\partial y}\frac{\partial S}{\partial z} \;-\; \frac{\partial}{\partial z}\frac{\partial S}{\partial y} \;=\; 0, \\
\left(\nabla \times (\nabla S) \right)_y \;&=\; \frac{\partial}{\partial z}\frac{\partial S}{\partial x} \;-\; \frac{\partial}{\partial x}\frac{\partial S}{\partial z} \;=\; 0.
\end{aligned}
\tag{43}
$$

Finally, the divergence of a curl of a vector field is

$$
\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{V}) &= \frac{\partial}{\partial x}\big(\nabla \times \mathbf{V}\big)_x + \frac{\partial}{\partial y}\big(\nabla \times \mathbf{V}\big)_y + \frac{\partial}{\partial z}\big(\nabla \times \mathbf{V}\big)_z \\
&= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \\
&\quad \langle\!\langle \text{ regrouping terms } \rangle\!\rangle \\
&= \left(\frac{\partial}{\partial x}\frac{\partial V_z}{\partial y} - \frac{\partial}{\partial y}\frac{\partial V_z}{\partial x}\right) + \left(\frac{\partial}{\partial y}\frac{\partial V_x}{\partial z} - \frac{\partial}{\partial z}\frac{\partial V_x}{\partial y}\right) + \left(\frac{\partial}{\partial z}\frac{\partial V_y}{\partial x} - \frac{\partial}{\partial x}\frac{\partial V_y}{\partial z}\right) \\
&= 0 + 0 + 0 = 0.
\end{aligned}
\tag{44}
$$

Quod erat demonstrandum.

Now let's turn to the non-vanishing combinations of the derivative operators. Consider the divergence of a gradient of a scalar field:

$$
\mathbf{div}(\mathbf{grad}\,S) = \nabla \cdot (\nabla S) = \frac{\partial}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial}{\partial y}\frac{\partial S}{\partial y} + \frac{\partial}{\partial z}\frac{\partial S}{\partial z} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)S. \tag{45}
$$

The second-order differential operator inside the () on the RHS is called the *Laplace operator* or simply the *Laplacian*; it is usually denoted by the triangle $\triangle$,

$$
\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{46}
$$

Algebraically, the Laplacian is the scalar square of the ∇ vector,

$$
\triangle = \nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}. \tag{47}
$$

Since the ∇ operator acts as a true vector under coordinate transforms, the Laplacian is a true scalar — it has the same form in all Cartesian coordinate systems,

$$
\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{ same } \triangle. \tag{48}
$$

Hence, the Laplacian operator acting on a scalar field as in eq. (45) produces a bone fide scalar field.

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We may also act with the Laplacian operator (46) on a vector field $\mathbf{V}(x, y, z)$ — by acting on each component V_x, V_y, V_z separately — and produce a vector field

$$\triangle \mathbf{V} \;=\; \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) \hat{\mathbf{x}} + \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right) \hat{\mathbf{y}} + \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) \hat{\mathbf{z}}.$$
$$(49)$$

Such Laplacian of a vector field also obtains from combining the gradient of the divergence of that vector field with a curl of the curl,

$$\triangle \mathbf{V} \;=\; \nabla(\nabla \cdot \mathbf{V}) \;-\; \nabla \times (\nabla \times \mathbf{V}) \;=\; \mathbf{grad}(\mathbf{div\,V}) \;-\; \mathbf{curl}(\mathbf{curl\,V}). \qquad (50)$$

Formally, this formula obtains from the $\mathbf{BAC - CAB}$ rule for the double cross product of 3 vectors where two of them are ∇ and the third is $\mathbf{V}$:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \;=\; \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) \;-\; \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \;=\; \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) \;-\; (\mathbf{A} \cdot \mathbf{B})\mathbf{C},$$
$$\text{let } \mathbf{A} \;=\; \mathbf{B} \;=\; \nabla, \quad \mathbf{C} \;=\; \mathbf{V}(x, y, z), \qquad (51)$$
$$\nabla \times (\nabla \times \mathbf{V}) \;=\; \nabla(\nabla \cdot \mathbf{V}) \;-\; (\nabla \cdot \nabla)\mathbf{V} \;=\; \nabla(\nabla \cdot \mathbf{V}) \;-\; \triangle \mathbf{V}.$$

More rigorously, we should redo the vector algebra here plugging the appropriate derivatives for the components of the ∇ vector and making explicit use of partial derivatives commuting with each other, but I am not doing it in these notes.

Physically, the Laplacian of the electric potential is very important in electrostatics, while the Laplacians of the electric and the magnetic fields are very important for the propagation of the electromagnetic waves.
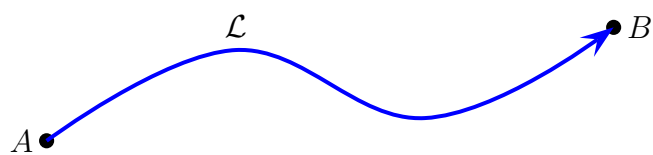
Integrals and Fundamental Theorems

For the functions of one variable, the derivatives and the integrals are related by the *Newton's Fundamental theorem of calculus:*

$$\text{for } f'(x) = \frac{df}{dx}, \quad \int_A^B f'(x)\, dx = f(B) - f(A). \tag{52}$$

There are similar theorems relating the gradients, the divergences, and the curls of fields to the line, surface, and volume integrals.

LINE INTEGRAL OF A VECTOR FIELD

Consider a vector field $\mathbf{V}(x, y, z)$ integrated along some line $\mathcal{L}$, straight or curved,

$$I = \int_{\mathcal{L}} \mathbf{V} \cdot d\vec{\ell} = \int_{\mathcal{L}} (V_x\, dx + V_y\, dy + V_z\, dz). \tag{53}$$

Note: the infinitesimal length element $d\vec{\ell}$ is treated as a vector (in the direction tangent to the line), and the integrand is the scalar product of that $d\vec{\ell}$ with the vector field. For a line described by a vector-valued function $\mathbf{r}(\xi)$ of a single coordinate ξ parametrizing the line, the line element is $d\vec{\ell} = d\mathbf{r}(\xi) = (d\mathbf{r}/d\xi)d\xi$, hence

$$\int_{\mathcal{L}} \mathbf{V} \cdot d\vec{\ell} = \int_{\xi_A}^{\xi_B} \mathbf{V}(\mathbf{r}(\xi)) \cdot \frac{d\mathbf{r}}{d\xi}\, d\xi. \tag{54}$$

Physics example of a line integral: the mechanical work of a position dependent force $\mathbf{F}(x, y, z)$.

$$W = \int_{\text{trajectory}} \mathbf{F} \cdot d\vec{\ell}. \tag{55}$$

SURFACE INTEGRAL OF A VECTOR FIELD $\mathbf{V}(x, y, z)$ is the *flux* of $\mathbf{V}$ through some surface $\mathcal{S}$,

$$\Phi = \iint_{\mathcal{S}} \mathbf{V} \cdot d^2\mathbf{A}$$

$$= \iint_{\mathcal{S}} (\mathbf{V} \cdot \vec{n}) \, d^2 A.$$
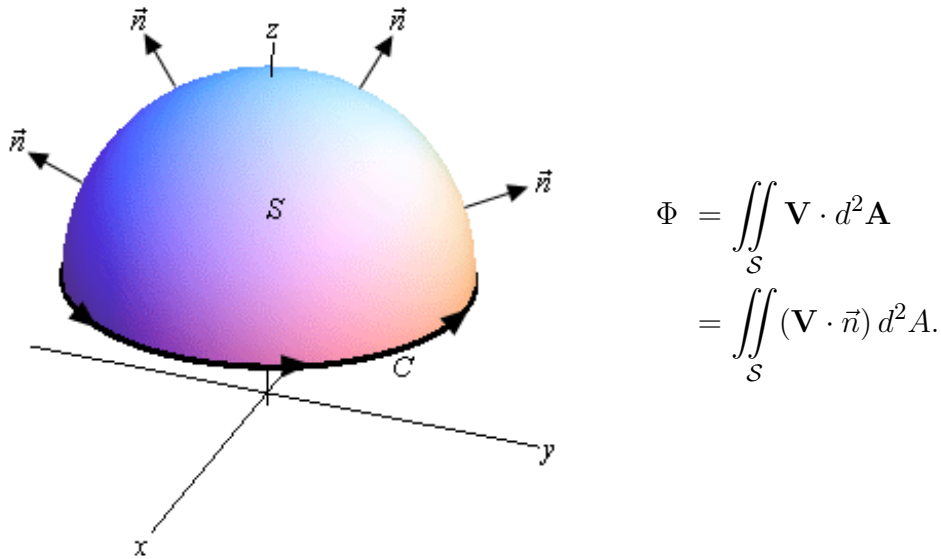
Again, we treat the infinitesimal area element $d^2\mathbf{A}$ as a vector in the direction $\vec{n}$ normal to the surface $\mathcal{S}$, and the integrand is the dot product of that infinitesimal area vector $d^2\mathbf{A}$ with the vector field $\mathbf{V}$. For a surface described by a vector-valued function $\mathbf{r}(\xi, \eta)$ of two independent coordinates ξ and η parametrizing the surface, the area element is

$$d^2\mathbf{A} = \left(\frac{\partial \mathbf{r}}{\partial \xi} \, d\xi\right) \times \left(\frac{\partial \mathbf{r}}{\partial \eta} \, d\eta\right) = \left(\frac{\partial \mathbf{r}}{\partial \xi} \times \frac{\partial \mathbf{r}}{\partial \eta}\right) d\xi \, d\eta. \tag{56}$$

For example, for a sphere parametrized by the latitude θ and the longitude ϕ,

$$\frac{\partial \mathbf{r}}{\partial \theta} = R \, \hat{\theta}, \quad \frac{\partial \mathbf{r}}{\partial \phi} = R \sin\theta \, \hat{\phi}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = R^2 \sin\theta \left(\hat{\theta} \times \hat{\phi}\right) = R^2 \sin\theta \, \hat{\mathbf{r}}, \tag{57}$$

hence $d^2\mathbf{A} = \hat{\mathbf{r}} \, R^2 \sin\theta \, d\theta \, d\phi$.

Finally, the VOLUME INTEGRAL OF A SCALAR FIELD

$$\iiint_{\mathcal{V}} S \, d^3\text{Vol} = \iiint_{\mathcal{V}} S(x, y, z) \, dx \, dy \, dz \tag{58}$$

does not involve any vectors, it's just the ordinary 3D integral of $S(x, y, z)$ over some volume $\mathcal{V}$ of 3D space. But the devil is in the details — the integration boundaries for the x, y, z variables could be rather complicated, depending on the exact shape of the volume $\mathcal{V}$.

The **gradient theorem** concerns line integrals of gradients:

$$\int\limits_{\text{line } \mathcal{L}} (\nabla S) \cdot d\vec{\ell} \;=\; S(\text{finish}(\mathcal{L})) \;-\; S(\text{start}(\mathcal{L})). \tag{59}$$

Note: in general, a line integral of a vector field over a line $\mathcal{L}$ depends on the entire geometry of the line. But when that vector field happens to be be the gradient of some scalar field $S(x, y, z)$, the integral depends only on ending points of the line, regardless of any intermediate points of the line.
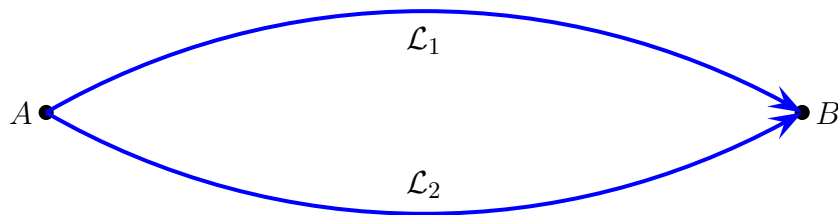
$$\text{for a generic } \mathbf{V}(x, y, z), \quad \int\limits_{\mathcal{L}_1} \mathbf{V} \cdot d\vec{\ell} \;\neq\; \int\limits_{\mathcal{L}_2} \mathbf{V} \cdot d\vec{\ell}, \tag{60}$$

$$\text{but for } \mathbf{V} = \nabla S(x, y, z), \quad \int\limits_{\mathcal{L}_1} (\nabla S) \cdot d\vec{\ell} \;=\; \int\limits_{\mathcal{L}_2} (\nabla S) \cdot d\vec{\ell} \;=\; S(B) \;-\; S(A).$$

In particular, integrating a gradient over a closed loop yields zero,
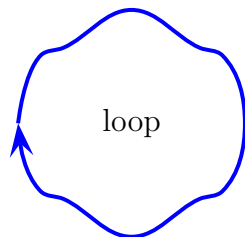
$$\oint\limits_{\text{loop}} (\nabla S) \cdot d\vec{\ell} \;=\; 0. \tag{61}$$

The **curl theorem** — also known as the **Stokes' theorem** — concerns surface integrals (fluxes) of curls. Let $\mathcal{S}$ be a surface *spanning* some closes loop $\mathcal{C}$ as on the figure below; in other words, $\mathcal{S}$ is a finite surface with a single *boundary* and the loop $\mathcal{C}$ is that boundary. Then the flux through $\mathcal{S}$ of a curl $\nabla \times \mathbf{V}$ of some vector field $\mathbf{V}(x, y, z)$ equals the line integral of the $\mathbf{V}$

field itself over the boundary loop $\mathcal{C}$:
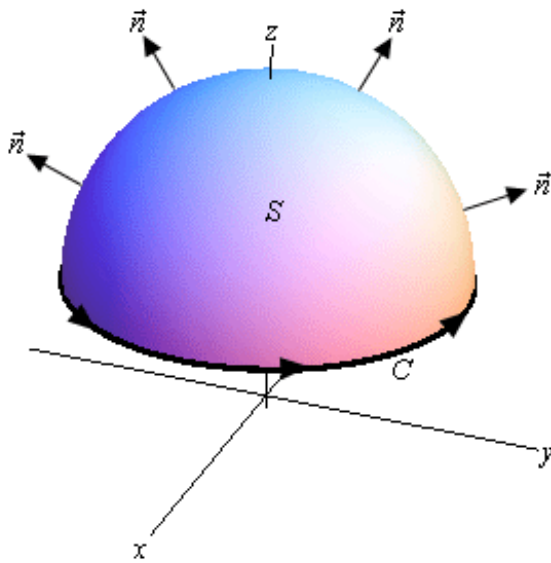
$$\iint_{\mathcal{S}} (\nabla \times \mathbf{V}) \cdot d^2\mathbf{A} \;=\; \oint_{\mathcal{C}} \mathbf{V} \cdot d\vec{\ell}. \qquad (62)$$

Again the flux of a general vector field through a surface depends on the entire surface, but when that vector field happens to be the curl of another vector field, the flux depends only on the surface's boundary. Thus for two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ spanning the same loop $\mathcal{C}$ — like the hemisphere and the disk on the figure below —
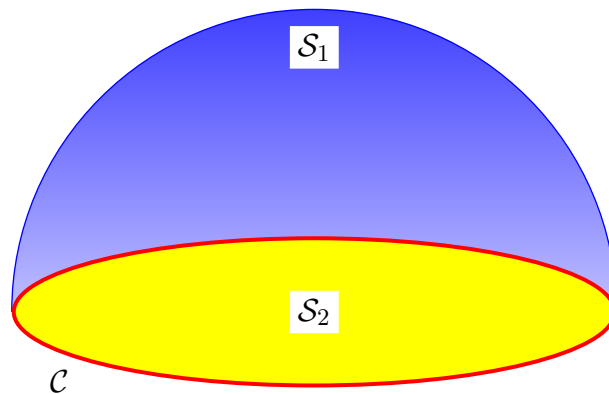
$$\text{for a generic } \mathbf{V}(x,y,z), \quad \iint_{\mathcal{S}_1} \mathbf{V} \cdot d^2\mathbf{A} \;\neq\; \iint_{\mathcal{S}_2} \mathbf{V} \cdot d^2\mathbf{A}, \qquad (63)$$

$$\text{but for } \mathbf{V} = \nabla \times \mathbf{V}'(x,y,z), \quad \iint_{\mathcal{S}_1} (\nabla \times \mathbf{V}') \cdot d^2\mathbf{A} \;=\; \iint_{\mathcal{S}_2} (\nabla \times \mathbf{V}') \cdot d^2\mathbf{A} \;=\; \oint_{\mathcal{C}} \mathbf{V}' \cdot d\vec{\ell}.$$

The **divergence theorem** — also known as the **Gauss' theorem**, the **Green's theorem**, or the **Ostrogradsky's theorem** — concerns the volume integrals of the divergences of vector fields. Let $\mathcal{V}$ be some finite volume of space and let $\mathcal{S}$ be its complete surface. Then the volume integral over $\mathcal{V}$ of the divergence $\nabla \cdot \mathbf{V}$ of some vector field $\mathbf{V}(x, y, z)$ equals to the flux of the $\mathbf{V}$ field itself through the surface $\mathcal{S}$ of $\mathcal{V}$:

$$\iiint\limits_{\mathcal{V}} (\nabla \cdot \mathbf{V}) \, d^3\text{Vol} \;=\; \iint\limits_{\mathcal{S}} \mathbf{V} \cdot d^2\mathbf{A}. \tag{64}$$

EXAMPLE: Let $\mathcal{S}$ be a sphere of radius R while $\mathcal{V}$ is the solid ball filling that sphere, and let $\mathbf{V}(x, y, z) = f(r)\hat{\mathbf{r}}$ be a spherically symmetric radial vector field. As we saw earlier in these notes in eq. (24), the divergence of such a field is

$$\nabla \cdot \big(f(r)\hat{\mathbf{r}}\big) \;=\; \frac{df}{dr} \;+\; \frac{2f}{r}. \tag{65}$$

To integrate this divergence over the volume of the ball, we use spherical coordinates in which the volume element is

$$d^3\text{Vol} \;=\; r^2 \, dr \, d^2\Omega \quad \longrightarrow \quad 4\pi r^2 \, dr \quad \text{(once we integrate over the directions).} \tag{66}$$

Consequently, the integral of the divergence over the solid ball evaluates to

$$\iiint\limits_{\text{ball}} (\nabla \cdot \mathbf{V}) \, d^3\text{Vol} = \int_0^R \left(\frac{df}{dr} + \frac{2f}{r}\right) \times 4\pi r^2 \, dr \;=\; \int_0^R \frac{d}{dr}\Big(4\pi r^2 \times f(r)\Big) \, dr \;=\; 4\pi R^2 f(R). \tag{67}$$

On the other hand, the flux of the radial $\mathbf{V}$ field through the sphere is simply

$$\iint\limits_{\text{sphere}} \big(\mathbf{V} = f(r)\hat{\mathbf{r}}\big) \cdot d^2\mathbf{A} \;=\; f(R) \times \text{Area(sphere)} \;=\; f(R) \times 4\pi R^2. \tag{68}$$

By inspection, this flux is equal to the volume integral (67), in perfect agreement with the Gauss Theorem.

Applications to Electrostatics — the Potential

In electrostatics, the electric field lines never form any closed loops. This is a visualization of a stronger rule: the line integral of a static electric field over any closed loop is zero,

$$\forall \text{loop } \mathcal{C}, \quad \oint_{\mathcal{C}} \mathbf{E} \cdot d\vec{\ell} \;=\; 0. \tag{69}$$
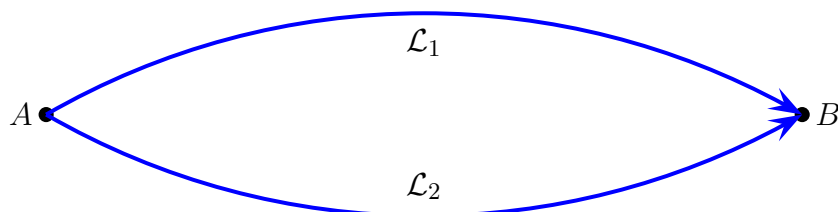
By the Stokes' theorem, this means that the curl $\nabla \times \mathbf{E}$ has zero flux through any surface $\mathcal{S}$ spanning any closed loop $\mathcal{C}$,

$$\forall \mathcal{S}, \quad \iint_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot d^2\mathbf{A} \;=\; 0. \tag{70}$$

In particular, the flux through any infinitesimal square — of any orientation, and located anywhere in space — must vanish, which means that the curl $\nabla \times \mathbf{E}(x, y, z)$ must vanish identically everywhere in space. Thus, the *static* electric field has identically zero curl,

$$\nabla \times \mathbf{E}(x, y, z) \;\equiv\; 0 \quad \forall x, y, z. \tag{71}$$

Another consequence of the vanishing-loop-integrals rule (69) is that the integral of $\mathbf{E}(x, y, z)$ over any open line depends only on the starting and finishing points of that line:

$$\int_{\mathcal{L}_1} \mathbf{E} \cdot d\vec{\ell} \;-\; \int_{\mathcal{L}_2} \mathbf{E} \cdot d\vec{\ell} \;=\; \overset{\text{closed loop}}{\oint_{\mathcal{C}=\mathcal{L}_1-\mathcal{L}_2}} \mathbf{E} \cdot d\vec{\ell} \;=\; 0. \tag{72}$$

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Physically, the line integrals here are related to the work of the electrostatic force $\mathbf{F} = q\mathbf{E}(x, y, z)$ on a moving probe charge q,

$$W \;=\; q \times \int\limits_{\text{trajectory}} \mathbf{E} \cdot d\vec{\ell}. \tag{73}$$

Thus, eq. (72) means that the work of moving a probe charge from a point A to a point B is the same regardless of a particular path the charge takes from A to B. This makes the electrostatic force *conservative*. Consequently, the work of this force is related to a potential energy $U(x, y, z)$ — which obviously have the form

$$U(x, y, z) \;=\; q \times V(x, y, z) \tag{74}$$

for some q-independent *electric potential $V(x, y, z)$*.

The potential difference between any two points A and B obtains by integrating the electric field over any line from A to B,

$$V(A) \;-\; V(B) \;=\; \int\limits_{A}^{B} \mathbf{E} \cdot d\vec{\ell}, \quad \text{same integral for any path from } A \text{ to } B. \tag{75}$$

The potential itself is ambiguous — we may add a position-independent constant to the $V(x, y, z)$ without changing any of the differences. To remove the ambiguity, one usually picks a *reference point $\mathcal{O}$* where the potential is set to zero by fiat. Then at any other point A,

$$V(A) \;=\; \int\limits_{A}^{\mathcal{O}} \mathbf{E} \cdot d\vec{\ell} \quad \text{along any path from } A \text{ to } \mathcal{O}. \tag{76}$$

Example: Consider the Coulomb field of a point charge at $\mathbf{r} = \vec{0}$,

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{kQ}{|r|^2}\, \hat{\mathbf{r}} \tag{77}$$

where $k = 1/4\pi\epsilon_0$ (in the MKSA units). Then along any line in space,

$$\mathbf{E} \cdot d\vec{\ell} \;=\; \frac{kQ}{|r|^2}\, \hat{\mathbf{r}} \cdot d\mathbf{r} \;=\; \frac{kQ}{|r|^2}\, d|\mathbf{r}| \;=\; d\left(\frac{-kQ}{|r|}\right), \tag{78}$$

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and hence

$$\int\limits_A^{\mathcal{O}} \mathbf{E} \cdot d\vec{\ell} = \int\limits_A^{\mathcal{O}} d\left(\frac{-kQ}{r(x,y,z)}\right) = \frac{-kQ}{r(\mathcal{O})} - \frac{-kQ}{r(A)}, \tag{79}$$

regardless of the directions of the points A and $\mathcal{O}$ relative to the charge Q, only their distances from the Q affect this integral. For the electric potential, this means

$$V(A) = +\frac{kQ}{r(A)} - \frac{kQ}{r(\mathcal{O})}. \tag{80}$$

For simplicity, let's move the reference point $\mathcal{O}$ (where the potential vanishes) to the infinity, $r(\mathcal{O}) = \infty$, then the electric potential due to a point charge Q become simply

$$V[\text{point charge}] = \frac{Q}{4\pi\epsilon_0} \times \frac{1}{r}. \tag{81}$$

The superposition principle for the electric field of multiple charges carries over to the electric potential. Thus, for a system of several point charges Q_i located at points $\mathbf{R}_i$, the net electric potential is simply

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{Q_i}{|\mathbf{r} - \mathbf{R}_i|}. \tag{82}$$

Note that its easier to add up the scalar potentials due to each charge than to add up the electric field vectors. This is particularly true when the source charges are continuous rather than point-like, and the sum (80) becomes an integral. For example, for a volume change of density $\rho(x, y, z)$, the electric potential is

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(x',y',z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \, dx' \, dy' \, dz'. \tag{83}$$

This is a painful triple integral, but at least we need only one such integral rather than three triple integrals we would need for calculating the electric field:

$$
\begin{aligned}
E_x(x,y,z) &= \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(x',y',z') \times (x-x')}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}} \, dx' \, dy' \, dz', \\
E_y(x,y,z) &= \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(x',y',z') \times (y-y')}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}} \, dx' \, dy' \, dz', \\
E_z(x,y,z) &= \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(x',y',z') \times (z-z')}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}} \, dx' \, dy' \, dz'.
\end{aligned}
\tag{84}
$$

In light of the formulae like (82) and (83) for the electric potential, the obvious question is: Given the electric potential $V(x, y, z)$, how do we reconstruct the electric field $\mathbf{E}(x, y, z)$ from this potential? The answer is very simple: take the gradient (and flip the sign),

$$\mathbf{E}(x, y, z) = -\nabla V(x, y, z). \tag{85}$$

Proof: By the gradient theorem,

$$\int_A^B (-\nabla V) \cdot d\vec{\ell} = V(A) - V(B) = \int_A^B \mathbf{E} \cdot d\vec{\ell}. \tag{86}$$

Both equalities here do not depend on the specific path from A to B, so let's put point B infinitesimally close to point A, $\mathbf{r}(B) = \mathbf{r}(A) + d\mathbf{r}$ and take the straight-line path from A to B. In this case, the integrals simplify to integrand $\cdot\, d\mathbf{r}$, thus

$$(-(\nabla V)@A) \cdot d\mathbf{r} = (\mathbf{E}@A) \cdot d\mathbf{r}. \tag{87}$$

This equality must hold true for any infinitesimal $d\mathbf{r}$, which requires the vector equality

$$-\nabla V = \mathbf{E} \quad \text{at the same point } A, \text{ for any such } A. \tag{88}$$

In other words, $\mathbf{E}(x, y, z) \equiv -\nabla V(x, y, z)$ as vector fields, *quod erat demonstrandum.*

EXAMPLE: Let's find the potential $V(x, y, z)$ due to a uniformly charged thin spherical shell or radius R, and then use that potential to find the electric field.

Adapting eq. (83) to a surface rather than volume charge density, we have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma\, d^2 A}{|\mathbf{r} - \mathbf{r}'|}, \tag{89}$$

where for the spherical shell in question

$$\sigma = \frac{Q_{\text{net}}}{4\pi R^2} = \text{const.} \tag{90}$$

By the spherical symmetry of the system, the potential V depends only on the radial distance r from the center of the shell but not on the angular coordinates θ and ϕ (in the spherical coordinates), $V(r, \theta, \phi) = V(r\, \text{only})$. So without loss of generality, let's measure the potential along the positive half of the z axis, or in spherical coordinates ($\theta = 0$, ϕ undefined).

The charges on the spherical shell are located at fixed $r' = R$ while their angular coordinates θ' and ϕ' span all possible directions, thus

$$V(r) = \frac{\sigma}{4\pi\epsilon_0} \iint\limits_{\text{sphere}} \frac{d^2A = R^2 \sin\theta'\, d\theta'\, d\phi'}{\text{distance}}.$$

(91)

The distance (between the point where we measure the potential and the charge) follows from the cosine theorem:

$$\text{distance}^2 = |\mathbf{r} - \mathbf{r}'|^2 = \mathbf{r}^2 + \mathbf{r}'^2 - 2\mathbf{r}\cdot\mathbf{r}' = r^2 + R^2 - 2rR\cos\theta'$$

(92)

because $|\mathbf{r}'| = R$ and the angle between the $\mathbf{r}$ and the $\mathbf{r}'$ vectors is θ'. Consequently,

$$
\begin{aligned}
V(r) &= \frac{\sigma}{4\pi\epsilon_0} \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \frac{R^2 \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \\
&= \frac{\sigma R^2}{4\pi\epsilon_0} \int_0^\pi \frac{d\theta'\, \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \times \left(\int_0^{2\pi} d\phi' = 2\pi \right) \\
&= \frac{2\pi\sigma R^2}{4\pi\epsilon_0} \int_{\theta'=0}^{\theta'=\pi} \left(\frac{-d\cos\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} = \frac{+1}{rR} \times d\sqrt{R^2 + r^2 - 2Rr\cos\theta'} \right) \\
&= \frac{\frac{1}{2}Q_{\text{net}}}{4\pi\epsilon_0} \times \frac{1}{rR} \times \left. \sqrt{R^2 + r^2 - 2Rr\cos\theta'} \right|_{\theta'=0}^{\theta'=\pi} \\
&= \frac{Q_{\text{net}}}{4\pi\epsilon_0} \times \frac{1}{2rR} \times \left(\sqrt{R^2 + r^2 + 2Rr} - \sqrt{R^2 + r^2 - 2rR} \right).
\end{aligned}
$$

(93)

Furthermore, $R^2 + r^2 \pm 2Rr = (R \pm r)^2$, but when we take a square root of that, we must take the positive square root. Thus, $\sqrt{R^2 + r^2 + 2Rr} = R + r$ but

$$\sqrt{R^2 + r^2 - 2Rr} = \sqrt{(R-r)^2} = |R - r| = \begin{cases} r - R & \text{outside the shell,} \\ R - r & \text{inside the shell,} \end{cases}$$

(94)

and therefore
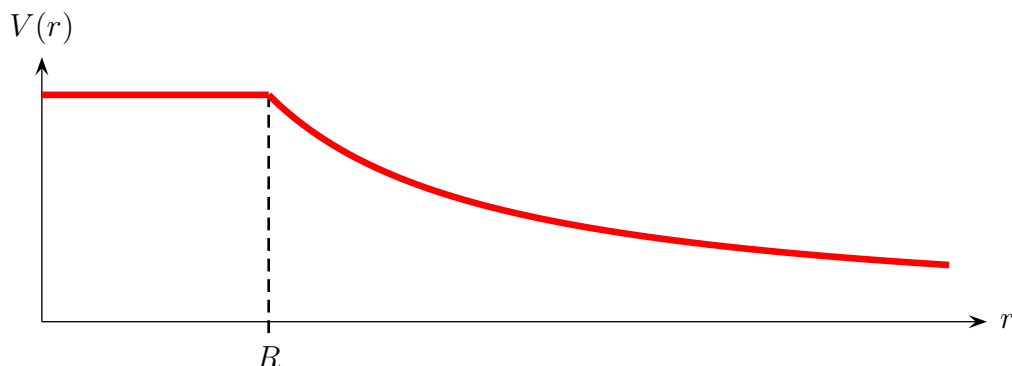
$$\sqrt{R^2 + r^2 + 2Rr} - \sqrt{R^2 + r^2 - 2rR} = (R+r) - |R-r| = \begin{cases} 2R & \text{outside the shell,} \\ 2r & \text{inside the shell.} \end{cases}$$

(95)

Plugging this formula back into eq. (93) for the electric potential, we arrive at

$$V(r) = \frac{Q_{\text{net}}}{4\pi\epsilon_0} \times \frac{1}{r} \quad \text{outside the shell,}$$

(96)

$$\text{but } V(r) \;=\; \frac{Q_{\text{net}}}{4\pi\epsilon_0}\times\frac{1}{R}\quad\text{inside the shell.}\tag{97}$$

Graphically,

Given the potential, the electric field obtains as (minus) its gradient. According to eq. (97), inside the shell the potential V is constant, independent of r or the other two coordinates, so its gradient is zero and the electric field vanishes! On the other hand, outside the shell the potential depends on the radius as $1/r$, hence the electric field is

$$\mathbf{E} \;=\; -\nabla V(r) \;=\; -\frac{dV}{dr}\,\hat{\mathbf{r}} \;=\; -\frac{d}{dr}\left(\frac{Q}{4\pi\epsilon_0}\,\frac{1}{r}\right)\hat{\mathbf{r}} \;=\; +\frac{Q}{4\pi\epsilon_0}\,\frac{1}{r^2}\,\hat{\mathbf{r}}\tag{98}$$

— the same as the Coulomb field of the point charge at the center of the shell.

Earlier in class we have found the same electric field — zero inside the shell and similar to the point charge's outside the shell — from the Gauss Law. But it's nice to re-derive this field by a direct calculation.

Applications to Electrostatics — the Gauss Law

In the integral form, the Gauss Law relates the flux of the electric field $\mathbf{E}(x,y,z)$ through any *closed* surface $\mathcal{S}$ to the net electric charge Q surrounded by that surface,

$$\Phi_E(\mathcal{S}) \;=\; \iint\limits_{\mathcal{S}}\mathbf{E}\cdot d^2\mathbf{A} \;=\; \frac{1}{\epsilon_0}\,Q_{\text{net}}[\text{surrounded by }\mathcal{S}].\tag{99}$$

Applying the Gauss Theorem to the electric field flux on the LHS here, we relate it to the

volume integral of the electric field's divergence,

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{E}) \, d^3\text{Vol} \;=\; \iint_{\mathcal{S}} \mathbf{E} \cdot d^2\mathbf{A} \;=\; \frac{1}{\epsilon_0} Q_{\text{net}}[\text{in } \mathcal{V}]. \tag{100}$$

For a continuous charge distribution with volume density $\rho(x, y, z)$, the net charge on the RHS obtains as a volume integral

$$Q_{\text{net}}[\text{in } \mathcal{V}] \;=\; \iiint_{\mathcal{V}} \rho \, d^3\text{Vol}, \tag{101}$$

hence

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{E}) \, d^3\text{Vol} \;=\; \frac{1}{\epsilon_0} \iiint_{\mathcal{V}} \rho \, d^3\text{Vol}. \tag{102}$$

Note the volume integrals on both sides of this equation are over the same volume $\mathcal{V}$, whatever it is. Moreover, the equation must hold for any such volume, and this is possible only when the integrands are equal everywhere in space, point by point:

$$\forall x, y, z: \quad \nabla \cdot \mathbf{E}(x, y, z) \;=\; \frac{1}{\epsilon_0} \rho(x, y, z). \tag{103}$$

This is the Gauss Law in the differential form (in MKSA units).

Altogether, the *static* electric field has zero curl while its divergence is proportional to the local electric charge density,

$$\nabla \times \mathbf{E}(x, y, z) \;\equiv\; 0, \quad \nabla \cdot \mathbf{E}(x, y, z) \;\equiv\; \frac{1}{\epsilon_0} \rho(x, y, z). \tag{104}$$

By comparison, the *static* magnetic field has zero divergence while its curl is proportional to the local electric current density,

$$\nabla \cdot \mathbf{B}(x, y, z) \;\equiv\; 0, \quad \nabla \times \mathbf{B}(x, y, z) \;=\; \mu_0 \, \mathbf{J}(x, y, z). \tag{105}$$

I shall address the magnetic field later in this class; for now, let's focus on the electrostatics.

Let's recast the electrostatic equations (104) in terms of the electric potential $V(x, y, z)$. Note that the very existence of the potential — or rather, the fact that the $\mathbf{E}(x, y, z)$ field is a (minus) gradient of some scalar potential — guarantees zero curl,

$$\mathbf{E} \;=\; -\nabla V \quad \Longrightarrow \quad \nabla \times \mathbf{E} \;=\; 0. \tag{106}$$

Conversely, the Poincare Lemma says (among other things) that if a vector field has zero curl everywhere, throughout the whole 3D space, then that field is a gradient of some scalar field. Thus, the zero-curl equation for the electric field assures the existence of the electric potential,

$$\nabla \times \mathbf{E}(x, y, z) \;\equiv\; 0 \; \forall \, x, y, z \quad \Longrightarrow \quad \exists V(x, y, z) \text{ such that } \mathbf{E}(x, y, z) \;=\; -\nabla V(x, y, z). \tag{107}$$

As to the Gauss Law, in terms of the potential, the divergence of the electric field becomes

$$\nabla \cdot \mathbf{E} \;=\; \nabla \cdot (-\nabla V) \;=\; -\nabla^2 V \;=\; -\triangle V \tag{108}$$

where $\triangle$ is the Laplacian operator

$$\triangle \;=\; \nabla^2 \;=\; \frac{\partial^2}{\partial x^2} \;+\; \frac{\partial^2}{\partial y^2} \;+\; \frac{\partial^2}{\partial z^2} \,. \tag{109}$$

Hence, the Gauss Law for the electric field becomes the Poisson equation for the potential,

$$\triangle V(x, y, z) \;=\; -\frac{1}{\epsilon_0} \, \rho(x, y, z). \tag{110}$$

In particular, in empty regions of space where there are no charges, the potential obeys the *Laplace equation*

$$\triangle V(x, y, z) \;=\; 0. \tag{111}$$

We shall spend quite a few weeks learning how to solve the Poisson equation and the Laplace equation for various boundary conditions.

Summary

Let me conclude these notes with the summary of essential electrostatic equations in both MKSA and Gauss units.

- Gauss Law in the integral form:

$$\iint_{\mathcal{S}} \mathbf{E} \cdot d^2\mathbf{A} \;=\; Q_{\mathrm{net}}[\text{inside } \mathcal{S}] \times \begin{cases} 4\pi & \text{in Gauss units,} \\ \frac{1}{\epsilon_0} & \text{MKSA units.} \end{cases} \tag{112}$$

- Gauss Law in the differential form:

$$\nabla \cdot \mathbf{E}(x,y,z) \;=\; \rho(x,y,z) \times \begin{cases} 4\pi & \text{in Gauss units,} \\ \frac{1}{\epsilon_0} & \text{MKSA units.} \end{cases} \tag{113}$$

- Poisson equation:

$$\triangle V(x,y,z) \;=\; -\rho(x,y,z) \times \begin{cases} 4\pi & \text{in Gauss units,} \\ \frac{1}{\epsilon_0} & \text{MKSA units.} \end{cases} \tag{114}$$

- Coulomb potential of a point charge Q:

$$V(x,y,z) \;=\; \frac{Q}{r} \times \begin{cases} 1 & \text{in Gauss units,} \\ \frac{1}{4\pi\epsilon_0} & \text{in MKSA units.} \end{cases} \tag{115}$$

- Coulomb electric field of a point charge Q:

$$V(x,y,z) \;=\; \frac{Q\,\hat{\mathbf{r}}}{r^2} \times \begin{cases} 1 & \text{in Gauss units,} \\ \frac{1}{4\pi\epsilon_0} & \text{in MKSA units.} \end{cases} \tag{116}$$

⋆ Finally, some equations have similar form in both systems of units:

$$\nabla \times \mathbf{E}(x,y,z) \;\equiv\; 0, \tag{117}$$

$$\mathbf{E}(x,y,z) \;=\; -\nabla V(x,y,z), \tag{118}$$

$$V(A) \;-\; V(B) = \int_{A}^{B} \mathbf{E} \cdot d\vec{\ell}. \tag{119}$$