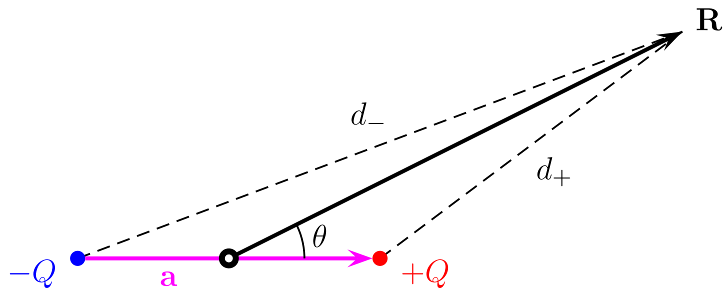
ELECTRIC DIPOLES

A Simple Dipole

A simple electric dipole is pair of opposite point charges $+Q$ and $-Q$ at a short distance a from each other. The potential generated by such a dipole is obviously

$$V(\mathbf{R}) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{d_+} - \frac{1}{d_-}\right) \tag{1}$$

where $d_\pm$ are the distances from the respective charges to the point $\mathbf{R}$. Let's put the coordinate origin at the mid-point of the dipole as on the diagram below



(2)

Working through this geometry, we get

$$d_\pm^2 = \left(\mathbf{R} - \frac{\mathbf{a}}{2}\right)^2 = R^2 \mp Ra\cos\theta + \frac{a^2}{4}, \tag{3}$$

$$d_- - d_+ = \frac{d_-^2 - d_+^2}{d_- + d_+} = \frac{2Ra\cos\theta}{d_- + d_+}, \tag{4}$$

$$\frac{1}{d_+} - \frac{1}{d_-} = \frac{d_- - d_+}{d_- d_+} = \frac{2Ra\cos\theta}{d_- d_+ (d_- + d_+)}, \tag{5}$$

where at the large distances from the dipole, $R \gg a$, we may approximate the denominator here as

$$d_- d_+ (d_- + d_+) \approx 2R^3 \quad \Longrightarrow \quad \frac{1}{d_+} - \frac{1}{d_-} \approx \frac{a\cos\theta}{R^2}. \tag{6}$$

Thus, at large distances from the electric dipole, the potential it generates becomes

$$V(R,\theta) = \frac{Q}{4\pi\epsilon_0}\frac{a\cos\theta}{R^2}, \tag{7}$$

or in vector notations

$$V(\mathbf{R}) \;=\; \frac{Q\mathbf{a}\cdot\widehat{\mathbf{R}}}{4\pi\epsilon_0\,|\mathbf{R}|^2} \;=\; \frac{Q\mathbf{a}\cdot\mathbf{R}}{4\pi\epsilon_0\,|\mathbf{R}|^3}\,. \tag{8}$$

Note that this potential depends on the charges $\pm Q$ and the distance between them only via their product $p = Qa$ called *the dipole moment.* Or in vector notations — where $\mathbf{a}$ runs from the $-Q$ charge to the $+Q$ charge, — the dipole moment vector is $\mathbf{p} = Q\mathbf{a}$, and the potential is

$$V(\mathbf{R}) \;=\; \frac{\mathbf{p}\cdot\widehat{\mathbf{R}}}{4\pi\epsilon_0\,|\mathbf{R}|^2} \;=\; \frac{\mathbf{p}\cdot\mathbf{R}}{4\pi\epsilon_0\,|\mathbf{R}|^3}\,. \tag{9}$$

Multi–Charge Dipoles

Now consider a system of several point charges Q_i located at respective points $\mathbf{r}_i$, whose net charge happens to vanish,

$$Q_{\rm net} \;=\; \sum_i Q_i \;=\; 0. \tag{10}$$

Let's look at the net potential generated by this system of charges

$$V(\mathbf{R}) \;=\; \frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{|\mathbf{R}-\mathbf{r}_i|} \tag{11}$$

at distances far away from the system,

$$|\mathbf{R}| \;\gg\; \text{all of the } |\mathbf{r}_i|. \tag{12}$$

In this limit, we may approximate

$$|\mathbf{R}-\mathbf{r}_i|^2 \;=\; \mathbf{R}^2 \;-\; 2\mathbf{R}\cdot\mathbf{r} \;+\; \mathbf{r}_i^2 \;\approx\; R^2\times\left(1 - \frac{2\widehat{\mathbf{R}}\cdot\mathbf{r}_i}{R} \;+\; O\left(\frac{r_i^2}{R^2}\right)\right), \tag{13}$$

$$|\mathbf{R}-\mathbf{r}_i| \;\approx\; R\times\left(1 - \frac{\widehat{\mathbf{R}}\cdot\mathbf{r}_i}{R} \;+\; O\left(\frac{r_i^2}{R^2}\right)\right), \tag{14}$$

$$\frac{1}{|\mathbf{R}-\mathbf{r}_i|} \;\approx\; \frac{1}{R}\times\left(1 + \frac{\widehat{\mathbf{R}}\cdot\mathbf{r}_i}{R} \;+\; O\left(\frac{r_i^2}{R^2}\right)\right), \tag{15}$$

and therefore

$$\begin{aligned}
V(\mathbf{R}) &= \frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{R}\times\left(1 + \frac{\widehat{\mathbf{R}}\cdot\mathbf{r}_i}{R} + O\left(\frac{r_i^2}{R^2}\right)\right)\\
&= \frac{1}{4\pi\epsilon_0}\frac{1}{R}\times\sum_i Q_i + \frac{1}{4\pi\epsilon_0}\frac{1}{R^2}\times\sum_i Q_i\mathbf{r}_i\cdot\widehat{\mathbf{R}} + \frac{1}{4\pi\epsilon_0}\frac{1}{R^3}\times\sum_i Q_i\times O(\mathbf{r}_i^2)\\
&= \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{net}}}{R} + \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\text{net}}\cdot\widehat{\mathbf{R}}}{R^2} + \frac{1}{4\pi\epsilon_0}\frac{O(Qr^2)}{R^3},
\end{aligned} \tag{16}$$

where in the second term on the bottom line

$$\mathbf{p}_{\text{net}} = \sum_i Q_i\mathbf{r}_i\,. \tag{17}$$

For a system of zero net charge, the first term on the bottom line of eq. (16) vanishes, while the third term becomes much smaller than the second term at the large distances from the system. Therefore, at large distances from the system the net potential becomes the dipole potential

$$V(\mathbf{R}) \approx \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\text{net}}\cdot\widehat{\mathbf{R}}}{R^2} \tag{18}$$

where $\mathbf{p}_{\text{net}}$ is the net electric dipole moment of the system. In particular, for a system of just two point charges $+Q$ and $-Q$, eq. (17) for this net dipole moment yields

$$\mathbf{p}_{\text{net}} = (+Q)\mathbf{r}_+ + (-Q)\mathbf{r}_- = Q(\mathbf{r}_+ - \mathbf{r}_-) = Q\mathbf{a}, \tag{19}$$

which is precisely the dipole moment of a pure dipole.

Finally, for a compact system of continuous charges, eq. (17) generalizes to

$$\mathbf{p}_{\text{net}} = \iiint \rho(\mathbf{r})\,\mathbf{r}\,d^3\text{vol}(\mathbf{r}) + \begin{pmatrix}\text{similar contributions from}\\ \text{surface and line charges}\end{pmatrix}. \tag{20}$$

Similar to systems of discrete charges, far away from a compact continuous charge system

$$V(\mathbf{R}) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{net}}}{R} + \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_{\text{net}}\cdot\widehat{\mathbf{R}}}{R^2} + \frac{1}{4\pi\epsilon_0}\frac{O(Qr^2)}{R^3}, \tag{21}$$

and if the net charge

$$Q_{\text{net}} = \iiint \rho(\mathbf{r})\, d^3\text{vol}(\mathbf{r}) \;+\; \text{surface and line charges} \tag{22}$$

happens to vanish, then the dominant contribution to the potential far away from the system is due to the net dipole moment,

$$V(\mathbf{R}) \approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_{\text{net}} \cdot \widehat{\mathbf{R}}}{R^2}. \tag{23}$$

Example:

Consider a solid ball of radius a filled with some matter with a non-uniform charge density

$$\rho(r,\theta,\phi) = kr\sin\theta\sin\phi \tag{24}$$

for some constant k. Let's find the dipole moment vector of this ball.

In spherical coordinates,

$$d^3\text{vol} = r^2\, dr \times \sin\theta\, d\theta \times d\phi, \tag{25}$$

$$\mathbf{r}(r,\theta,\phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta), \tag{26}$$

while $\rho(r,\theta,\phi)$ is as in eq. (24), hence

$$\mathbf{p} = \iiint\limits_{\text{ball}} d^3\text{vol}\, \rho\, \mathbf{r}$$

$$= \int_0^a dr\, r^2 \int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\phi\, kr\sin\theta\sin\phi(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta),$$

hence in components

$$p_x = \int_0^a dr\, kr^4 \times \int_0^\pi d\theta\, \sin^3\theta \times \int_0^{2\pi} d\phi\, \sin\phi\cos\phi, \tag{27}$$

$$p_y = \int_0^a dr\, kr^4 \times \int_0^\pi d\theta\, \sin^3\theta \times \int_0^{2\pi} d\phi\, \sin^2\phi,$$

$$p_z = \int_0^a dr\, kr^4 \times \int_0^\pi d\theta\, \sin^2\theta\cos\theta \times \int_0^{2\pi} d\phi\, \sin\phi.$$

Taking the ϕ integrals, we immediately see that the $\phi \to 2\pi - \phi$ symmetry kills all the integrals containing odd powers of $\sin\phi$, thus

$$\int_0^{2\pi} d\phi\, \sin\phi\cos\phi \;=\; 0, \qquad \int_0^{2\pi} d\phi\, \sin\phi \;=\; 0, \tag{28}$$

and hence $p_x = 0$ and $p_z = 0$. As to the y component of the dipole moment,

$$\int_0^{2\pi} d\phi\, \sin^2\phi \;=\; \pi, \tag{29}$$

hence

$$p_y \;=\; k\pi \times \int_0^a dr\, r^4 \times \int_0^\pi d\theta\, \sin^3\theta, \tag{30}$$

where

$$\int_0^a dr\, r^4 \;=\; \frac{a^5}{5} \tag{31}$$

while

$$\int_0^\pi d\theta\, \sin^3\theta \;=\; \int_{-1}^{+1} d\cos\theta \times (\sin^2\theta = 1 - \cos^2\theta) \;=\; 2 \;-\; \tfrac{2}{3} \;=\; \tfrac{4}{3}\,, \tag{32}$$

thus altogether

$$p_y \;=\; \pi k \times \frac{a^5}{5} \times \frac{4}{3} \;=\; \frac{4\pi}{15}\, ka^5. \tag{33}$$

Caveat:

The net dipole moment

$$\mathbf{p}_{\rm net} \;=\; \sum_i Q_i \mathbf{r}_i \quad \text{or} \quad \mathbf{p}_{\rm net} \;=\; \iiint d^3\mathrm{vol}(\mathbf{r})\,\rho(\mathbf{r})\mathbf{r} \tag{34}$$

involves radius vectors **r** to the charge locations **r** from some coordinate origin **O**, so it seems to depend on the choice of this coordinate origin **O**. But actually, for systems of zero net charge the net dipole moment is origin-independent.

Proof: Let's change the coordinate origin from **O** to **O′**, and let $\mathbf{d} = \overline{\mathbf{OO}'}$ is the vector from the old origin to the new. This re-coordinatization changes the radius vectors of all the charges in the system as

$$\mathbf{r}_i \;\to \mathbf{r}'_i \;=\; \mathbf{r}_i \;-\; \mathbf{d}. \tag{35}$$

Consequently, the new net dipole moment of the system becomes

$$\mathbf{p}'_{\rm net} \;=\; \sum_i Q_i \mathbf{r}'_i \;=\; \sum_i Q_i(\mathbf{r}_i - \mathbf{d}) \;=\; \sum_i Q_i \mathbf{r}_i \;-\; \mathbf{d}\sum_i Q_i \;=\; \mathbf{p}_{\rm net} \;-\; \mathbf{d}Q_{\rm net}\,. \tag{36}$$

In particular, if $Q_{\rm net} = 0$ — and only if $Q_{\rm net} = 0$ — then $\mathbf{p}'_{\rm net} = \mathbf{p}_{\rm net}$. Thus, the net dipole moment of a system is independent on the choice of the coordinate origin if and only if the system has zero net charge.

For example, the net dipole moment of a neutral molecule like H_2O is well defined regardless of the coordinate origin, but the net dipole moment of a molecular ion can be defined only relative to a particular origin **O**. The usual choice of such origin is the ion's center of mass, but sometimes other choices may be more convenient.

Electric Field of a Dipole

Now that we know the potential

$$V(\mathbf{R}) \;\approx\; \frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{p}\cdot\widehat{\mathbf{R}}}{|\mathbf{R}|^2}\,. \tag{23}$$

of an electric dipole moment $\mathbf{p}$, let's calculate the electric field $\mathbf{E} = -\nabla V$. Let's work in the coordinate system where the z axis points in the direction of the dipole moment, or in spherical coordinates, the $\Theta = 0$ axis points in the direction of $\mathbf{p}$. In these coordinates

$$V(R,\Theta,\Phi) \;=\; \frac{p}{4\pi\epsilon_0}\,\frac{\cos\Theta}{R^2}\,, \tag{37}$$

hence

$$\mathbf{E} \;=\; \frac{p}{4\pi\epsilon_0}\left(\frac{2\cos\Theta}{R^3}\,\nabla R \,+\, \frac{\sin\theta}{R^2}\,\nabla\Theta\right) \;=\; \frac{p}{4\pi\epsilon_0}\,\frac{1}{R^3}\Big(2\cos\theta\,\widehat{\mathbf{R}} \,+\, \sin\theta\,\widehat{\boldsymbol{\Theta}}\Big), \tag{38}$$

where $\widehat{\mathbf{R}}$ and $\widehat{\boldsymbol{\Theta}}$ are unit vectors in the radial and the meridional direction. These unit vectors themselves depend on Θ and Φ, so let's translate them to the Cartesian components as

$$\begin{aligned} \widehat{\mathbf{R}} \;&=\; \sin\Theta\cos\Phi\,\hat{\mathbf{x}} \;+\; \sin\Theta\sin\Phi\,\hat{\mathbf{y}} \;+\; \cos\Theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\Theta}} \;&=\; \cos\Theta\cos\Phi\,\hat{\mathbf{x}} \;+\; \cos\Theta\sin\Phi\,\hat{\mathbf{y}} \;-\; \sin\Theta\,\hat{\mathbf{z}}. \end{aligned} \tag{39}$$

Consequently

$$2\cos\Theta\,\widehat{\mathbf{R}} + \sin\Theta\,\hat{\boldsymbol{\Theta}} \;=\; 3\sin\Theta\cos\Theta(\cos\Phi\,\hat{\mathbf{x}} + \sin\Phi\,\hat{\mathbf{y}}) + (2\cos^2\Theta - \sin^2\Theta \;=\; 3\cos^2\Theta - 1)\,\hat{\mathbf{z}}, \tag{40}$$

and therefore

$$\begin{aligned} E_x(R,\Theta,\Phi) \;&=\; \frac{p}{4\pi\epsilon_0}\,\frac{3\sin\Theta\cos\Theta\cos\Phi}{R^3}\,, \\ E_y(R,\Theta,\Phi) \;&=\; \frac{p}{4\pi\epsilon_0}\,\frac{3\sin\Theta\cos\Theta\sin\Phi}{R^3}\,, \\ E_z(R,\Theta,\Phi) \;&=\; \frac{p}{4\pi\epsilon_0}\,\frac{3\cos^2\Theta - 1}{R^3}\,. \end{aligned}$$

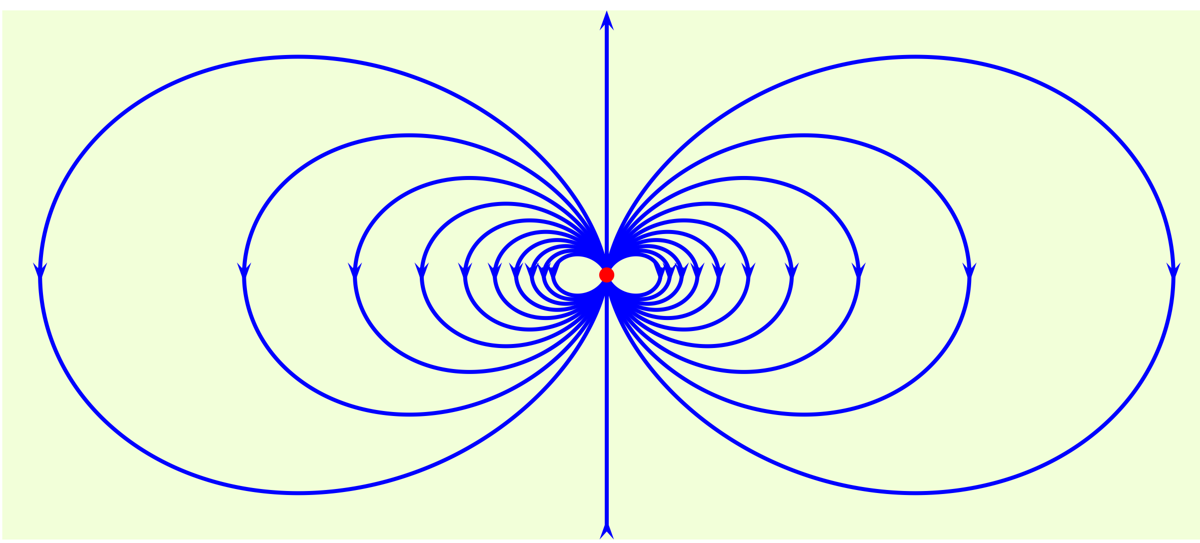
In terms of the (X, Y, Z) coordinates

$$\begin{aligned}
E_x(x, y, z) &= \frac{p}{4\pi\epsilon_0} \frac{3XZ}{(X^2 + Y^2 + Z^2)^{5/2}}, \\
E_y(x, y, z) &= \frac{p}{4\pi\epsilon_0} \frac{3XZ}{(X^2 + Y^2 + Z^2)^{5/2}}, \\
E_z(x, y, z) &= \frac{p}{4\pi\epsilon_0} \frac{2Z^2 - X^2 - X^2}{(X^2 + Y^2 + Z^2)^{5/2}},
\end{aligned} \tag{41}$$

or in vector notations,

$$\mathbf{E}(\mathbf{R}) = \frac{3(\mathbf{p} \cdot \widehat{\mathbf{R}})\widehat{\mathbf{R}} - \mathbf{p}}{4\pi\epsilon_0 |\mathbf{R}|^3}. \tag{42}$$

Note that along the dipole axis the electric field points in the direction of the dipole moment $\mathbf{p}$, while in the plane $\perp$ to the dipole axis the field points in the opposite direction from the dipole moment. To get a more general picture of the dipole's electric field, here is the diagram of the electric field lines in the xz plane:



Force and Torque on a Dipole

Consider an electric dipole $\mathbf{p}$ placed into an external electric field $\mathbf{E}(\mathbf{r})$ generated by some other sources. In general, this electric field exerts a torque on the dipole, and a non-uniform field also exerts a net force. In this section we shall see how this works.

Let's start with a simple dipole in a uniform electric field $\mathbf{E} = \text{const}$. The field exerts force $\mathbf{F}_+ = +Q\mathbf{E}$ on the charge $+Q$ and exactly opposite force $\mathbf{F}_- = -Q\mathbf{E}$ on the other charge $-Q$, so the net force on the dipole vanishes,

$$\mathbf{F}_{\text{net}} = 0 \qquad \langle\!\langle \text{ in a uniform electric field } \rangle\!\rangle. \tag{43}$$

However, the two forces act at different places, so together they generate net torque

$$\vec{\tau} = \mathbf{r}_+ \times \mathbf{F}_+ + \mathbf{r}_- \times \mathbf{F}_- = (\mathbf{r}_+ - \mathbf{r}_-) \times Q\mathbf{E} = Q(\mathbf{r}_+ - \mathbf{r}_-) \times \mathbf{E}, \tag{44}$$

or in terms of the dipole moment $\mathbf{p} = Q(\mathbf{r}_+ - \mathbf{r}_-)$,

$$\vec{\tau} = \mathbf{p} \times \mathbf{E}. \tag{45}$$

This torque vanishes when the dipole moment $\mathbf{p}$ is parallel to the electric field $\mathbf{E}$. Otherwise, the torque twists the dipole trying to make it align with the field, $\mathbf{p} \to \mathbf{p}' \upuparrows \mathbf{E}$.

Likewise, take any system of several electric charges — or a continuous charge distribution — with zero net charge. When placed in a uniform electric field, the net force on this system vanishes while the net torque is exactly as in eq. (45) in terms of the net dipole moment $\mathbf{p}_{\text{net}}$ of the system. Indeed,

$$\mathbf{F}_{\text{net}} = \sum_i \mathbf{F}_{\text{on}\,Q_i} = \sum_i Q_i\mathbf{E} = Q_{\text{net}}\mathbf{E} \longrightarrow 0 \quad \text{for} \quad Q_{\text{net}} = 0, \tag{46}$$

while

$$\vec{\tau}_{\text{net}} = \sum_i \mathbf{r}_i \times Q_i\mathbf{E} = \left(\sum_i \mathbf{r}_i Q_i\right) \times \mathbf{E} = \mathbf{p}_{\text{net}} \times \mathbf{E}. \tag{47}$$

In a non-uniform electric field, the net force on a dipole generally does not vanish. Indeed, for a simple dipole (just 2 charges $+Q$ and $-Q$),

$$\mathbf{F}_{\rm net}\ =\ (+Q)\mathbf{E}(\mathbf{r}_+)\ +\ (-Q)\mathbf{E}(\mathbf{r}_i)\ =\ Q\big(\mathbf{E}(\mathbf{r}_+)\ -\ \mathbf{E}(\mathbf{r}_-)\big)\ \neq\ 0\quad\text{(in a non-uniform field).}\tag{48}$$

But suppose the distance a between the two charges of a dipole is rather short while the electric field varies from place to place on a much longer distance scale. In this case, we may expand the difference between the fields acting on the two charges in a power series in $\mathbf{a}=\mathbf{r}_+-\mathbf{r}_-$. Indeed, let $\mathbf{r}_\pm=\mathbf{r}\pm\frac{1}{2}\mathbf{a}$ where $\mathbf{r}$ is the center of the dipole; then

$$\mathbf{E}(\mathbf{r}_\pm)\ =\ \mathbf{E}(\mathbf{r})\ \pm\ (\tfrac{1}{2}\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}}\ +\ \tfrac{1}{2}(\tfrac{1}{2}\mathbf{a}\cdot\nabla)^2\mathbf{E}\Big|_{@\mathbf{r}}\ \pm\ \tfrac{1}{6}(\tfrac{1}{2}\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}}\ +\ \cdots,\tag{49}$$

hence the difference

$$\mathbf{E}(\mathbf{r}_+)\ -\ \mathbf{E}(\mathbf{r}_-)\ =\ (\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}}\ +\ \tfrac{1}{24}(\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}}\ +\ \cdots,\tag{50}$$

and therefore the net force on the dipole

$$\mathbf{F}^{\rm net}\ =\ Q(\mathbf{a}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}}\ +\ \frac{Q}{24}(\mathbf{a}\cdot\nabla)^3\mathbf{E}\Big|_{@\mathbf{r}}\ +\ \cdots.\tag{51}$$

For a physical dipole with a finite distance a between the two charges, we must generally take into account all the subleading terms in this expansion. But for an ideal dipole — negligibly small size $a\to 0$ but finite dipole moment $p=Q\times a$ due to very large charges $\pm Q\to\pm\infty$, — the subleading terms in eq. (51) become negligible, and only the leading first term remains important. Indeed, in the limit of $a\to 0$, $Q\to\infty$, while $p=Qa$ remains finite, the subleading terms in eq. (51) — which are proportional to $Q\times a^n=p\times a^{n-1}$ with $n>1$ — become negligible compared to the finite leading term $\propto Q\mathbf{a}=\mathbf{p}$. Thus, the net force on an ideal dipole is simply

$$\mathbf{F}^{\rm net}\ =\ (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{R}).\tag{52}$$

Similarly, consider a dipole comprised of several point charges Q_i, or perhaps from a continuous charge distribution, but all charges are located very close to some central point $\mathbf{r}_0$,

$$\text{all } \mathbf{r}_i \;=\; \mathbf{r}_0 \;+\; \Delta\mathbf{r}_i, \qquad |\Delta\mathbf{r}_i| \;\leq\; a, \tag{53}$$

while the electric field varies on a much longer distance scale than a. Then

$$\begin{aligned}\mathbf{E}(\mathbf{r}_i) \;&=\; \mathbf{E}(\mathbf{r}_0) \;+\; (\Delta\mathbf{r}_i\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_0} \;+\; \tfrac{1}{2}(\Delta\mathbf{r}_i\cdot\nabla)^2\mathbf{E}\Big|_{@\mathbf{r}_0} \;+\; \cdots \\ &=\; \mathbf{E}(\mathbf{r}_0) \;+\; (\Delta\mathbf{r}_i\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_0} \;+\; O(a^2\nabla\nabla\mathbf{E}),\end{aligned} \tag{54}$$

and hence

$$\begin{aligned}\mathbf{F}_{\text{net}} \;&=\; \sum_i Q_i\left(\mathbf{E}(\mathbf{r}_i) \;=\; \mathbf{E}(\mathbf{r}_0) \,+\, (\Delta\mathbf{r}_i\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_0} \,+\, O(a^2\nabla\nabla\mathbf{E})\right) \\ &=\; \left(\sum_i Q_i\right)\mathbf{E}(\mathbf{r}_0) \;+\; \left(\left(\sum_i Q_i\Delta\mathbf{r}_i\right)\cdot\nabla\right)\mathbf{E}\Big|_{@\mathbf{r}_0} \;+\; O(Qa^2\nabla\nabla\mathbf{E}) \\ &=\; Q_{\text{net}}\mathbf{E}(\mathbf{r}_0) \;+\; (\mathbf{p}_{\text{net}}\cdot\nabla)\mathbf{E}\Big|_{@\mathbf{r}_0} \;+\; O(Qa^2\nabla\nabla\mathbf{E}).\end{aligned} \tag{55}$$

On the bottom line here, the first term vanishes when the system in question has zero net charge, while the last term becomes negligibly small in the ideal dipole limit: $a \to 0$ (and hence all $\Delta\mathbf{r} \to 0$) while $Q_i \to \infty$ such that the net dipole moment stays finite. Thus, in the ideal dipole limit we get

$$\mathbf{F}^{\text{net}} \;=\; (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{R}), \tag{52}$$

exactly as for a simple 2-charge dipole.

Similar to the net force, the net potential energy of a dipole in the external field obtains as

$$U^{\text{net}} \;=\; (+Q)V(\mathbf{r}_+) \;+\; (-Q)V(\mathbf{r}_-), \tag{56}$$

or for a multi-charge dipole

$$U^{\text{net}} \;=\; \sum_i Q_i V(\mathbf{r}_i). \tag{57}$$

Again, for compact charge system of small size a and a slowly varying potential $V(\mathbf{r})$, we have

$$V(\mathbf{r}_i) \;=\; V(\mathbf{r}_0) \;+\; (\Delta\mathbf{r}_i\cdot\nabla)V\Big|_{@\mathbf{r}_0} \;+\; O(a^2\nabla\nabla V), \tag{58}$$

hence

$$\begin{aligned}
U^{\rm net} &= \sum_i Q_i\left(V(\mathbf{r}_0) + (\Delta\mathbf{r}_i\cdot\nabla)V\Big|_{@\mathbf{r}_0} + O(a^2\nabla\nabla V)\right)\\
&= \left(\sum_i Q_i\right)V(\mathbf{r}_0) \;+\; \left(\left(\sum_i Q_i\Delta\mathbf{r}_i\right)\cdot\nabla\right)V\Big|_{@\mathbf{r}_0} \;+\; O(Qa^2\nabla\nabla V)\\
&= Q_{\rm net}V(\mathbf{r}_0) \;+\; (\mathbf{p}_{\rm net}\cdot\nabla)V\Big|_{@\mathbf{r}_0} \;+\; O(Qa^2\nabla\nabla V)\\
&\longrightarrow\; (\mathbf{p}_{\rm net}\cdot\nabla)V\Big|_{@\mathbf{r}_0} \quad \text{for an ideal dipole with } Q_{\rm net}=0 \text{ and } a\to 0\\
&= -\mathbf{p}\cdot\mathbf{E}(\mathbf{r}_0).
\end{aligned} \tag{59}$$

Thus, an ideal dipole with moment $\mathbf{p}$ located at point $\mathbf{r}$ has net potential energy

$$U(\mathbf{r},\mathbf{p}) \;=\; -\mathbf{p}\cdot\mathbf{E}(\mathbf{r}). \tag{60}$$

The potential energy (60) accounts for the mechanical work of the force (52) when the dipole is moved around and also for the work of the torque (45) when the dipole is rotated. Consequently, *both the force* (52) *and the torque* (45) *are conservative.* To see how this works, consider infinitesimal displacements and rotations of the dipole,

$$\mathbf{r} \;\to\; \mathbf{r} \;+\; \vec{\alpha}, \quad \mathbf{p} \;\to\; \mathbf{p} \;+\; \vec{\varphi}\times\mathbf{p} \tag{61}$$

for some infinitesimal vectors $\vec{\alpha}$ and $\vec{\varphi}$. The work of the force (52) and the torque (45) due to such combined displacement and rotation is

$$\delta W \;=\; \vec{\alpha}\cdot\mathbf{F} \;+\; \vec{\varphi}\cdot\vec{\tau} \;=\; \vec{\alpha}\cdot\big[(\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{r})\big] \;+\; \vec{\varphi}\cdot\big[\mathbf{p}\times\mathbf{E}(\mathbf{r})\big], \tag{62}$$

so let's check that the infinitesimal variation of the energy (60) agrees with

$$\delta W \;=\; -\delta U(\mathbf{r},\mathbf{p}). \tag{63}$$

Indeed,

$$\begin{aligned} -\delta U &= +\delta\mathbf{p}\cdot\mathbf{E}(\mathbf{r}) \;+\; \mathbf{p}\cdot\big[\delta\mathbf{E}(\mathbf{r}) \;=\; (\delta\mathbf{r}\cdot\nabla)\mathbf{E}(\mathbf{r})\big] \\ &= (\vec{\varphi}\times\mathbf{p})\cdot\mathbf{E}(\mathbf{r}) \;+\; \mathbf{p}\cdot\big[(\vec{\alpha}\cdot\nabla)\mathbf{E}(\mathbf{r})\big], \end{aligned} \tag{64}$$

where the first term has form $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$, thus

$$1^{\rm st}\,\text{term} \;=\; (\vec{\varphi}\times\mathbf{p})\cdot\mathbf{E}(\mathbf{r}) \;=\; \vec{\varphi}\cdot(\mathbf{p}\times\mathbf{E}(\mathbf{r})) \;\equiv\; \vec{\varphi}\times\tau, \tag{65}$$

which is precisely the torque term in the work (62). As to the second term in eq. (64),

$$\begin{aligned} 2^{\rm nd}\,\text{term} &= \mathbf{p}\cdot\big[(\vec{\alpha}\cdot\nabla)\mathbf{E}(\mathbf{r})\big] \;=\; -\mathbf{p}\cdot\big[(\vec{\alpha}\cdot\nabla)\nabla V(\mathbf{r})\big] \\ &= -(\vec{\alpha}\cdot\nabla)(\mathbf{p}\cdot\nabla)V(\mathbf{r}) \;=\; -\vec{\alpha}\cdot\big[(\mathbf{p}\cdot\nabla)\nabla V(\mathbf{r})\big] \\ &= +\vec{\alpha}\cdot\big[(\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{r})\big] \;\equiv\; \vec{\alpha}\cdot\mathbf{F}, \end{aligned} \tag{66}$$

which is precisely the force term in the work (62). And this proves that the force (52) and the torque (45) on the dipole are indeed conservative and their work is accounted by the potential energy (60).

To be precise, the torque (45) is *the torque relative to the dipole center* $\mathbf{r}$. In a non-uniform electric field, the torque relative to some other point $\mathbf{r}_0$ has an extra term due to the net force (52) on the dipole, thus

$$\vec{\tau}^{\rm net} \;=\; (\mathbf{r}-\mathbf{r}_0)\times\mathbf{F}^{\rm net} \;+\; \vec{\tau}^{\text{relative to }\mathbf{r}} \;=\; (\mathbf{r}-\mathbf{r}_0)\times(\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{r}) \;+\; \mathbf{p}\times\mathbf{E}(\mathbf{r}). \tag{67}$$

This net torque may also be obtained from the potential energy U — or rather its infinitesimal variation under simultaneous rotations of the dipole moment vector $\mathbf{p}$ and of radius vector $\mathbf{r}-\mathbf{r}_0$ of the dipole from the reference point $\mathbf{r}_0$. But let me skip the proof of this statement.

Instead, I shall let you work out the consequences of eq. (67) in a couple of homework problems in set#7. In those problems, you have 2 dipoles exerting forces and torques on each other. When you calculate the torque on each dipole relative to its center, you'll find that both torques are in the same clockwise direction. To resolve this apparent contradiction with the Law of Angular Momentum Conservation, you have to account for the counterclockwise torque of the force between the dipoles. Then, when you calculate all torques relative to the same pivot point, you will add up with a zero net torque and hence conserved angular momentum.