⋆ Please do not waste your time copying parts of the textbook, the class notes, or the homework solutions, or re-deriving formulae I have derived in class. Simply write down the equations you need and apply them to the problem at hand.

1. [25 points] Consider three electric fields:

$$\mathbf{E}_1 = \alpha(2xy + y^2)\hat{\mathbf{x}} + \alpha(2xy + x^2)\hat{\mathbf{y}} - 3\alpha z^2\,\hat{\mathbf{z}}, \tag{1}$$

$$\mathbf{E}_2 = \alpha(x^2y + y^3)\hat{\mathbf{x}} - \alpha(x^3 + xy^2)\hat{\mathbf{y}}, \tag{2}$$

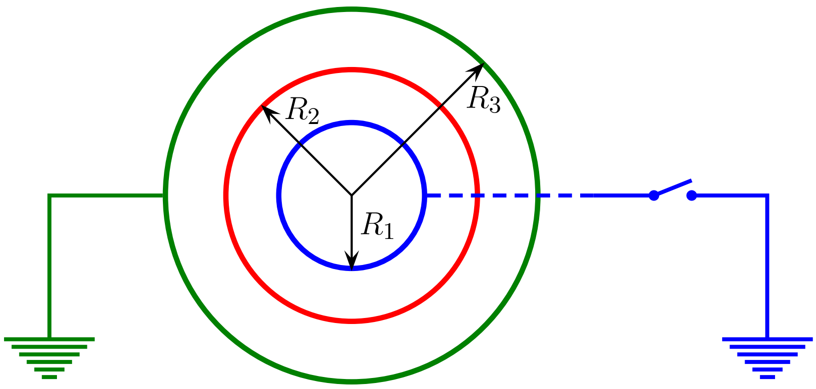
$$\mathbf{E}_3 = \alpha yz\,\hat{\mathbf{x}} + \alpha xz\,\hat{\mathbf{y}} + \alpha xy\,\hat{\mathbf{z}}, \tag{3}$$

for some overall constant α.

(a) [15 pt] Find which of the these three fields $\mathbf{E}(x, y, z)$ may happen in electrostatics and which may not.

(b) [10 pt] For each allowed electrostatic field, find the potential $V(x, y, z)$ and the electric charge density $\rho(x, y, z)$.

2. [30 points] Consider 3 coaxial metal shells as shown on the figure below; the shells are very thin and very long, $L \gg R_3 > R_2 > R_1$. The outer shell is grounded, the middle shell is insulated, and the inner shall can be insulated or grounded, depending on the switch.



Initially, the switch is open and the inner shell is insulated. Then both the inner and the middle shell are given respective charges $Q_1 = \lambda_1 L$ and $Q_2 = \lambda_2 L$.

(a) [5 pt] Find the charge $Q_3 = \lambda_3 L$ induced on the grounded outer shell?

(b) [7 pt] Find the electric fields inside, outside, and between the shells?

(c) [8 pt] Calculate the potentials of the three shells as functions of the charges Q_1 and Q_2 and show that

$$
\begin{aligned}
V_3 &= 0, && (4)\\
V_2 &= \frac{Q_1+Q_2}{2\pi\epsilon_0 L}\times\ln\frac{R_3}{R_2}\,, && (5)\\
V_1 &= \frac{Q_1}{2\pi\epsilon_0 L}\times\ln\frac{R_3}{R_1} + \frac{Q_2}{2\pi\epsilon_0 L}\times\ln\frac{R_3}{R_2}\,. && (6)
\end{aligned}
$$

Now let's close the switch and connect the inner shell to the ground. (While the middle shell remains insulated and the outer shell remains grounded.)

(d) [10 pt] After the grounding of the inner shell, what would be the charges Q'_1 and Q'_3 on the inner and the outer shells and the potential V'_2 of the middle shell?
Hint: use eq. (6).

3. [25 points] An unknown system of electric charges generates a spherically symmetric potential

$$
V(r) = V_0 \times \begin{cases} r/R & \text{for } r \leq R \\ R/r & \text{for } r \geq R \end{cases} \tag{7}
$$

for some constants V_0 and $R > 0$.

(a) [15 pt] Find the electric field $\mathbf{E}(\mathbf{r})$ (everywhere in space) and the electric charge distribution which generates the potential (7). Make sure to look for both volume and surface charges, and check if there are also line or point charges.

(b) [10 pt] Find the net electrostatic energy of the system.

4. [20 points] A cubic box of size $a \times a \times a$ has no electric charges inside it. The four sides and the bottom of the box are conducting and grounded, while the top square is non-conducting and charged. The potential across the top square happens to be

$$
V(x, y; z = a) = (V_0 = \text{const}) \times \sin(3\pi x/a) \times \sin(4\pi y/a). \tag{8}
$$

Use the separation-of-variables method to find the potential $V(x, y, z)$ everywhere inside the cube. Note: don't justify the method itself or re-solve any equation I have solved in class. Instead, simply recombine the formulae I've derived in class to the problem at hand.

⋆ For extra credit [10 points]:
Find the charge density $\sigma(x, y)$ on the inner surface of the box's bottom square.