⋆ Please do not waste your time copying parts of the textbook, the class notes, or the homework solutions, or re-deriving formulae I have derived in class. Simply write down the equations you need and apply them to the problem at hand.

1. [25 points] Electric charge $+Q$ is uniformly distributed along a circular ring of radius R, while a point charge $-Q$ sits at the center of the ring. Consider the electrostatic potential $V(r, \theta, \phi)$ of this charge system and the multipole expansion of this potential at long distances $r \gg R$.

   (a) [10 pt] What is the leading multipole of this expansion? That is, what is the lowest–ℓ multipole with a non-zero multipole moment?

   (b) [10 pt] Calculate the leading multipole moment. Use the axial symmetry of the system in question.

   (c) [5pt] Write down the potential $V(r, \theta, \phi)$ at large $r \gg R$ in the leading multipole approximation.

2. [20 points] A capacitor is made of two coaxial tubes of radii a (the inner tube) and b (the outer tube) and length $L \gg b > a$. The dielectric filling the space between the tubes has non-uniform dielectric constant

$$\epsilon(s, \phi, z) \;=\; \epsilon(s \text{ only}) \;=\; \frac{b^2}{s^2}. \tag{1}$$

   The capacitor is charged: the inner tube has free charge $+Q$ while the outer tube has free charge $-Q$.

   (a) [10 pt] Find the electric displacement field $\mathbf{D}$, the electric tension field $\mathbf{E}$, and the dielectric polarization $\mathbf{P}$ between the two tubes.

   (b) [10 pt] Find the voltage between the tubes, the capacitance of the capacitor, and the net electrostatic energy of the system.

   (⋆) For extra credit [10 points]:

   Find all the bound charges in the volume and on the surfaces of the dielectric and verify that the net bound charge is zero.

3. [30 points] Consider two infinitely long straight wires that are **not parallel** to each other. Specifically, wire#1 with current I_1 runs along the z axis while wire#2 with current I_2 spans

$$\mathbf{r} = a\,\hat{\mathbf{x}} + \ell\sin\theta\,\hat{\mathbf{y}} + \ell\cos\theta\,\hat{\mathbf{z}}, \quad \text{fixed } a, \theta, \quad \text{variable } \ell \text{ from } -\infty \text{ to } +\infty. \quad (2)$$

(a) [15 pt] Write down the magnetic field $\mathbf{B}_1(x, y, z)$ of the first wire and calculate the force $d\mathbf{F}$ it exerts on the element $d\ell$ of the second wire.

(b) [15 pt] Calculate the net force between the two wires. Note: the *net* force rather than the force per unit length.

Here are some formulae to help your calculations:

$$\frac{\hat{\phi}}{s} = \frac{x\,\hat{\mathbf{y}} - y\,\hat{\mathbf{x}}}{x^2 + y^2}, \quad \int_{-\infty}^{+\infty}\frac{d\ell}{a^2 + c^2\ell^2} = \frac{\pi}{ac}, \quad \int_{-\infty}^{+\infty}\frac{\ell\,d\ell}{a^2 + c^2\ell^2} = 0. \quad (3)$$

4. [25 points] Find the magnetic field of a helical current density

$$\mathbf{J}(x, y, z) = J_0\Big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\Big) \quad \langle\!\langle \text{constant } J_0, \text{ constant } k \rangle\!\rangle \quad (4)$$

flowing through the whole space. Note several symmetries of this current, including invariance under simultaneous translations in z direction and rotations around the z axis.

(a) [6 pt] Can this current be steady?

(b) [7 pt] Without taking any integrals or solving any differential equations, use the symmetries of the current density (4) to argue that the vector potential should have form

$$\mathbf{A}(x, y, z) = A_0\Big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\Big) \quad (5)$$

for some constant A_0.

(c) [7 pt] Check that the vector potential (5) obeys the Poisson equation $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ for the appropriate value of A_0 and find that value.

(d) [5 pt] Finally, write down the magnetic field $\mathbf{B}$ for the vector potential (5).