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# Image charge Method

works for space regions of ~~ex~~:

a) half-space, $z \geq 0$, any $x, y$

b) spherical cavity

c) outside of a sphere.

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Setup: infinite conducting plate

top surface @ $z = 0$.

Know all charges above the plate

forex: 1 point charge $Q$ @ 0

$(x, y, z) = (0, 0, a)$ $a > 0$.

@ $z = 0$ (and all $x, y$), $V(x, y, 0) = 0$

some unknown charges on the top surface.

Below the plate: don't know & don't care

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Math: want to find $V(x, y, z)$

(at $z \geq 0$, all $x, y$)

such that:

1) $-\vec{\nabla}^2 V(x, y, z) = \frac{1}{\epsilon_0} \rho(x, y, z)$

forex $-\vec{\nabla}^2 V = \frac{Q}{\epsilon_0} \delta(x) \delta(y) \delta(z-a)$

Note: $-\vec{\nabla}^2 V = \frac{\rho}{\epsilon_0}$ only for $z > 0$.

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2) $V(x,y,z) \equiv 0$ @ $z=0$

3) $V(\vec{r}) \to 0$ for $|\vec{r}| \to \infty$

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~~Sol~~ Treat the $z=0$ plane as a mirror.

put an image charge $Q_I = -Q$

@ the mirror image of the real charge $Q$.

$\oplus Q$ @ $(0,0,+a)$

$\ominus Q_I$ @ $(0,0,-a)$.

@ $z \geq 0$ only,
$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{(x^2+y^2+(z-a)^2)^{1/2}} - \frac{Q}{4\pi\varepsilon_0}\frac{1}{(x^2+y^2+(z+a)^2)^{1/2}}$$

conditions:

3): @ $\vec{r} \to \infty$ in any direction

$V \to 0$.

2) @ $z=0$, $V(x,y,0)=0$

(2 terms cancel)

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Poisson eq-n

$$-\epsilon_0 \vec{\nabla}^2 V(x,y,z) = +Q\,\delta(x)\delta(y)\delta(z-a)$$
$$-Q\,\delta(x)\delta(y)\delta(z+a).$$

but for $z>0$, $2^{nd}$ term $\equiv 0$.

$\rightarrow$ in the $z>0$ region in question

$$-\epsilon_0 \vec{\nabla}^2 V(x,y,z) = +Q\,\delta(x)\delta(y)\delta(z-a)$$
$$\equiv \rho(x,y,z).$$

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bottom line: This $V(x,y,z)$ is a solution $\Rightarrow$ the solution.

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The image charge is not real.
instead there is $\sigma(x,y)$ on the surface that yields the same $V(x,y,z)$ as the image charge @ $z>0$.

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Surface charges screen the $\vec{E}$[real charge] from inside the metal @ $z<0$

$$\rightarrow \left(V[\text{real }Q] + V[\text{surface}]\right) @ (x,y,z) = 0$$
@ $z<0$

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by $z \to -z$ refl. symmetry

$$V[\text{surface}](x,y,+z) = V[\text{surface}](x,y,-z)$$

@ $z>0$, $V[\text{surf.}](x,y,+z) = V[\text{surf}](x,y,-z)$

$$= -V[\text{real } Q](x,y,-z)$$

$$= +V[\text{image}](x,y,+z)$$

for $Q_{\text{image}} = -Q_{\text{real}}$.

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Let's calculate $\sigma(x,y)$.

$\sigma(x,y) = \epsilon_0 \operatorname{disc}(E_z)$ @ $z=0$.

$z>0$ but $z\to 0$

$$E_z(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{z-a}{(x^2+y^2+(z-a)^2)^{3/2}}$$

$$-\frac{Q}{4\pi\epsilon_0}\cdot\frac{z+a}{(x^2+y^2+(z+a)^2)^{3/2}}$$

$$\xrightarrow[z\to 0]{} \frac{Q}{4\pi\epsilon_0}\frac{-a}{(x^2+y^2+a^2)^{3/2}} - \frac{Q}{4\pi\epsilon_0}\frac{+a}{(x^2+y^2+a^2)^{3/2}}$$

$$= \frac{-2Qa}{4\pi\epsilon_0}\frac{1}{(x^2+y^2+a^2)^{3/2}}$$

$z<0 \to E_z = 0$.

$$\epsilon_0 \operatorname{disc}(E_z) = \frac{-Qa}{2\pi}\frac{1}{(x^2+y^2+a^2)^{3/2}}$$

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surface charge density

$$\sigma = \frac{-Qa}{2\pi} \frac{1}{(x^2+y^2+a^2)^{3/2}}$$

Net surface charge

$$Q_{net}\,[\text{surface}] = \iint dx\,dy\, \sigma(x,y) = -Q$$

$= +Q_{image}$.

$$Q_{net} = \int_0^\infty ds \times 2\pi s \times \sigma(s)$$

$$= \frac{-Qa}{2\pi} \underbrace{\int_0^\infty ds \times 2\pi s \times \frac{1}{(s^2+a^2)^{3/2}}}_{\frac{2\pi}{a}}$$

$$= -Q.$$