SEPARATION OF VARIABLES METHOD

There is no universal method for solving the boundary problems for the electrostatic potential $V(\mathbf{r})$. Instead, there are many special methods for particular types of boundaries or boundary conditions. In particular, the separation of variables method works for boundaries that have simple geometries (a coordinate = const) in some kind of a coordinate system. For example:

- A rectangular box of size $a \times b \times c$, with boundaries at
$$x = 0, \quad x = a, \quad y = 0, \quad y = b, \quad z = 0, \quad z = c \tag{1}$$
in the Cartesian coordinate system (x, y, z).

- A spherical cavity, or a shell between 2 concentric spheres, with boundaries at
$$r = a \quad \text{and} \quad r = b \tag{2}$$
in the spherical coordinate system (r, θ, ϕ).

- A cylindrical cavity, with boundaries at
$$s = R, \quad z = 0, \quad z = L \tag{3}$$
in the cylindrical coordinate system (s, ϕ, z).

In general, the separation-of-variables method starts by looking for solutions of the Laplace equation $\nabla^2 V(\mathbf{r}) = 0$ subject to some of the boundary conditions (but not all of them!) of the form
$$\begin{aligned} V(x,y,z) &= f(x) \times g(y) \times h(z), \\ \text{or} \quad V(r,\theta,\phi) &= f(r) \times g(\theta) \times h(\phi), \\ \text{or} \quad V(s,\phi,z) &= f(s) \times g(\phi) \times h(z), \end{aligned} \tag{4}$$
where f, g, h are 3 independent functions of the individual coordinates. Eventually, one finds an infinite series of such solutions, and then one looks at their linear combination that also obeys the remaining boundary conditions. And that's all the general description I am going to give to

this method. Instead, in these notes we shall see may examples how the separation-of-variables method works in practice for different kinds of boundaries in different coordinate systems, in 2D and in 3D.

Separation of Variables in Cartesian Coordinates

2D Example: Infinite Slot

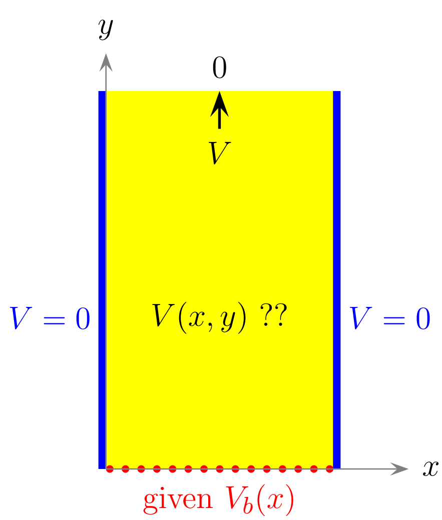
Let's start with a 2D example where the potential $V(x,y)$ depends only on the x and y coordinates but not on the z. Specifically, consider an infinite slot

$$0 \leq x \leq a, \quad 0 \leq y < \infty, \quad -\infty < z < +\infty \tag{5}$$

between 2 conducting and grounded walls (where $V = 0$) at $x = 0$ and at $x = a$. There are no electric charges within the slot, but there are some unknown charges outside the slot, and also unknown surface charges on the wall. On the other hand, somebody have measured the potential at the front boundary $y = 0$ of the slot and found that it depends only on the x coordinate across the slot but not on the vertical z coordinate,

$$@y = 0, \quad V(x, 0, z) \;=\; \text{known } V_b(x \text{ only}). \tag{6}$$

Note that the slot's geometry is invariant under translations in the z direction, $z \to z + \text{const}$, so since the boundary conditions for the potential are also independent on z, then the whole potential inside the slot should be z-independent. Thus, we have a two-dimensional problem for $V(x,y)$ inside the yellow band $0 \leq x \leq a$, $0 \leq y < +\infty$ on this diagram:



(7)

Mathematically, we are looking for $V(x,y)$ — for the (x,y) inside the yellow band only — which obeys the following 4 conditions:

[1] V obeys the 2D Laplace equation,

$$\triangle V(x,y) \;=\; \frac{\partial^2 V}{\partial x^2} \;+\; \frac{\partial^2 V}{\partial y^2} \;=\; 0; \tag{8}$$

[2] V vanishes at the left wall and the right wall (since they are conducting and grounded),

$$\text{for } (x=0 \text{ or } x=a) \text{ and any } y, \quad V(x,y)=0; \tag{9}$$

[3] V asymptotically approaches zero deep inside the slot,

$$\text{for } y \to +\infty \text{ and any} x, \quad V \to 0; \tag{10}$$

[4] At the from wall $x=0$, the potential has the measured value,

$$\text{at } y=0, \quad V(x,0) \;=\; \text{given } V_b(x). \tag{11}$$

In the separation-of-variables method, we start by looking at the solutions of the homogeneous conditions [1], [2], and [3], — but not [4], — that have a particular simple form

$$V(x,y) \;=\; f(x)\times g(y) \tag{12}$$

for two independent functions $f(x)$ and $g(y)$ of the individual coordinates. In terms of such a product, the boundary conditions [2] and [3] become

$$f(x=0) \;=\; f(x=a) \;=\; 0, \tag{13}$$

$$g(y) \;\to 0 \quad \text{for } y \to +\infty, \tag{14}$$

while the Laplace equation [1] becomes

$$\triangle V(x,y) \;=\; f''(x)\times g(y) \;+\; f(x)\times g''(y) \;=\; 0. \tag{15}$$

Dividing the Laplacian on the LHS by the potential $V = fg$ itself, we get

$$\frac{\triangle V}{V} \;=\; \frac{f''(x)}{f(x)} \;+\; \frac{g''(y)}{g(y)}\,, \tag{16}$$

which means we need

$$\frac{f''(x)}{f(x)} \;+\; \frac{g''(y)}{g(y)} \;=\; 0 \quad \text{for all } x \text{ and } y. \tag{17}$$

But on the LHS of this equation, the first term depends only on the x while the second term depends only on the y, so the only way these two terms may add up to zero for all x and all y if both terms are constants! Thus, eq. (17) implies

$$\frac{f''(x)}{f(x)} \;=\; -C, \qquad \frac{g''(y)}{g(y)} \;=\; +C, \qquad \text{for the same constant } C. \tag{18}$$

Altogether, the $f(x)$ function should obey

$$f''(x) \;+\; Cf(x) \;=\; 0, \quad f(x=0) \;=\; f(x=a) \;=\; 0, \tag{19}$$

while the $g(y)$ function obeys

$$g''(y) \;-\; Cg(y) \;=\; 0, \quad g(y) \xrightarrow[y\to+\infty]{} 0. \tag{20}$$

Now let's solve eq. (19) for the $f(x)$. The differential equation $f'' + Cf = 0$ has general solutions

$$f(x) \;=\; \alpha\cos(kx) \;+\; \beta\sin(kx) \qquad \text{for positive } C \;=\; +k^2, \tag{21}$$

$$f(x) \;=\; \alpha\cosh(\kappa x) \;+\; \beta\sinh(\kappa x) \qquad \text{for negative } C \;=\; -\kappa^2, \tag{22}$$

but the boundary condition $f(x = 0) = 0$ eliminate the cos or cosh terms, thus (up to an overall constant factor)

$$f(x) \;=\; \sin(kx) \quad \text{or} \quad f(x) \;=\; \sinh(\kappa x). \tag{23}$$

Furthermore, the solutions of the $\sinh(\kappa x)$ type do not have any zeros besides $x = 0$, which conflicts with the other boundary condition $f(x = 0) = a$. On the other hand, the solutions of the $\sin(kx)$ type have zeros whenever kx is an integer multiple of π, so $f(x = \sin(kx)$ obeys both boundary conditions provided

$$k \times a \;=\; n \times \pi \quad \text{for integer } n \;=\; 1, 2, 3, \ldots.$$

Altogether, we have a discrete series of solutions

$$f(x) \;=\; \sin\frac{n\pi x}{a} \quad \text{for } n \;=\; 1, 2, 3, \ldots \tag{24}$$

while

$$C \;=\; +\left(\frac{n\pi}{a}\right)^2 \;>\; 0. \tag{25}$$

Next, eq. (20) for the $g(y)$ function. For a positive C, the general solution to the differential equation $g'' - Cg = 0$ is

$$g(y) \;=\; \alpha\exp(+\kappa x) \;+\; \beta\exp(-\kappa x) \quad \text{for} \quad \kappa \;=\; +\sqrt{C} \;=\; \frac{n\pi}{a}. \tag{26}$$

However, the asymptotic condition $g \to 0$ for $y \to +\infty$ eliminates the terms with the positive exponent, and we are left with

$$g(y) \;=\; \exp\left(-\frac{n\pi y}{a}\right) \tag{27}$$

(up to an overall constant factor). Thus altogether, we end up with an infinite but discrete series of potentials $V(x, y) = f(x) \times g(y)$ that obey the conditions [1], [2], and [3], namely

$$V(x, y) \;=\; \text{const} \times \sin\left(\frac{n\pi}{a}x\right) \times \exp\left(-\frac{n\pi}{a}y\right) \tag{28}$$

for all integer $n = 1, 2, 3, \ldots$.

Note that the Laplace equation [1] and the $V = 0$ or $V \to 0$ boundary conditions [2] and [3] are linear WRT the potential $V(x, y)$. Thus, any linear combination of solutions to [1,2,3] is also a solution; in particular, any linear combination of the separated-variables solutions (28),

$$V(x,y) \;=\; \sum_{n=1}^{\infty} A_n \times \sin\left(\frac{n\pi}{a}x\right) \times \exp\left(-\frac{n\pi}{a}y\right) \tag{29}$$

for arbitrary constant coefficients A_n is a solution to [1,2,3]. That is, although such linear combinations generally do not factorize into a product of $f(x)$ and $g(y)$, they do obey the Laplace equation [1] and the homogeneous boundary conditions [2] and [3]. Moreover, in the Hilbert space of all solutions to [1,2,3], the separate-variables solutions (28) form a complete basis, which means that any solution to [1,2,3] can be expanded into a series (29) for some coefficients A_n.

Proof for the interested students:

Any function of x on the interval $0 \le x \le a$ can be expanded into a Fourier series, and if the function obeys boundary conditions $f(0) = f(a) = 0$, then the Fourier series involves only the sines and not the cosines,

$$f(x) \;=\; \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a} \quad \text{for} \quad a_n \;=\; \frac{2}{a}\int_0^a f(x) \times \sin\frac{n\pi x}{a}\,dx. \tag{30}$$

Now look at the slice of the slot in question for any fixed y and do a similar Fourier transform of the potential as a function of x. Doing this for each y results in

$$V(x,y) \;=\; \sum_{n=1}^{\infty} g_n(y) \times \sin\frac{n\pi x}{a} \tag{31}$$

for some y-dependent Fourier coefficients $g_n(y)$, specifically

$$g_n(y) \;=\; \frac{2}{a}\int_0^a V(x,y) \times \sin\frac{n\pi x}{a}\,dx. \tag{32}$$

For each term in the series (31),

$$\begin{aligned} \triangle\left(g_n(y) \times \sin\frac{n\pi x}{a}\right) \;&=\; -(n\pi/a)^2 \sin\frac{n\pi x}{a} \times g_n(y) \;+\; \sin\frac{n\pi x}{a} \times g_n''(y) \\ &=\; \Big(g_n''(y) \;-\; (n\pi/a)^2 g_n(y)\Big) \times \sin\frac{n\pi x}{a}, \end{aligned} \tag{33}$$

hence

$$\triangle V(x,y) \;=\; \sum_{n=1}^{\infty} \Big(g_n''(y) \;-\; (n\pi/a)^2 g_n(y)\Big) \times \sin\frac{n\pi x}{a}. \tag{34}$$

This Laplacian must vanish for all x and y, so every term in its Fourier expansion must vanish for all

y, thus every $g_n(y)$ must obey

$$g_n''(y) \; - \; (n\pi/a)^2 \times g_n(y) \; = \; 0. \tag{35}$$

Solving this equation under asymptotic condition [3] (and hence $g_n(y) \to 0$ for $y \to +\infty$) gives us

$$g_n(y) \; = \; A_n \times \exp\left(-\frac{n\pi}{a}y\right) \tag{36}$$

for some constant overall coefficient A_n. Plugging this solution into the Fourier series (31), we arrive at

$$V(x,y) \; = \; \sum_{n=1}^{\infty} A_n \times \exp\left(-\frac{n\pi}{a}y\right) \times \sin\frac{n\pi x}{a}, \tag{37}$$

exactly as in eq. (29). *Quod erat demonstrandum.*

For general coefficients A_n, the series (29) obeys the conditions [1,2,3] — the Laplace equations, and the boundary conditions at the left wall, the right wall, and at $y \to +\infty$ — but not the boundary condition [4] at the $y = 0$ front of the slot. As we shall see in a moment, it is that condition [4] that determines the values of the coefficients A_n. Indeed, at $y = 0$ the potential given by a general series (29) becomes

$$V(x,y=0) \; = \; \sum_{n=1}^{\infty} A_n \times \exp\left(-\frac{n\pi}{a}(y=0)\right) \times \sin\frac{n\pi x}{a} \; = \; \sum_{n=1}^{\infty} A_n \times \sin\frac{n\pi x}{a} \tag{38}$$

since

$$\exp\left(-\frac{n\pi}{a} \times (y=0)\right) \; = \; 1. \tag{39}$$

On the other hand, at $y = 0$ we want to have

$$V(x,y=0) \; = \; \text{given } V_b(x), \tag{40}$$

which in terms of the A_n means

$$\sum_{n=1}^{\infty} A_n \times \sin\frac{n\pi x}{a} \; = \; \text{given } V_b(x). \tag{41}$$

In other words, the A_n are the coefficients of the Fourier expansion of the boundary potential into sine waves, so they obtain as the Fourier integrals

$$A_n \; = \; \frac{2}{a}\int_0^a V_b(x) \times \sin\frac{n\pi x}{a}\, dx. \tag{42}$$

Example#1: sine wave.

For a simple example, suppose the $V_b(x)$ measured across the $y = 0$ front of the slot is a sine wave, say

$$V_b(x) \;=\; V_0 \times \sin\frac{3\pi x}{a}\,. \tag{43}$$

In this case, we do not need to evaluate the integrals (42) to determine the coefficients A_n. Instead, we simply compare the front potential (43) to the Fourier series (41):

$$\sum_{n=1}^{\infty} A_n \times \sin\frac{n\pi x}{a} \;=\text{should be}=\; V_0 \times \sin\frac{3\pi x}{a}\,, \tag{44}$$

which immediately tells us that

$$A_3 \;=\; V_0, \quad \text{all other } A_n \;=\; 0. \tag{45}$$

Consequently, plugging these coefficients into the series (29), we find that the potential throughout the slot is

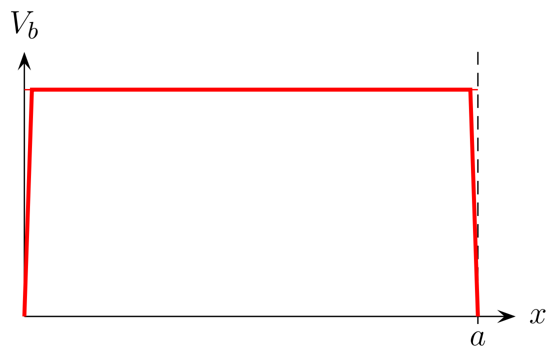
$$V(x,y) \;=\; V_0 \times \sin\frac{3\pi x}{a} \times \exp\left(-\frac{3\pi x}{a}\right). \tag{46}$$

Example#2: non-zero constant.

For another example, suppose the $y = 0$ front of the slot is covered by a separate conducting plate that *almost* touches the side walls at $x = 0$ and $x = a$ but is electrically insulated from them. This front plate is un-grounded and has non-zero constant potential V_0, thus

$$V_b(x) \;=\; V_0 \;=\; \text{const} \quad \text{for all } 0 < x < a. \tag{47}$$

In practice, the $V_b(x)$ would be constant for *almost* all x between 0 and a, but in the tiny gaps between the front plate and the side walls $V_b(x)$ would change very rapidly between V_0 and zero,



(48)

But to simplify our calculations, let's assume a flat non-zero potential (47) throughout the front of the slot. Consequently, performing the Fourier integrals (42), we obtain

$$\begin{aligned} A_n &= \frac{2}{a}\int_0^a V_0 \times \sin\frac{n\pi x}{a}\,dx \;=\; \frac{2}{a}\left(-\frac{V_0 a}{n\pi}\cos\frac{n\pi x}{a}\right)\Bigg|_0^a \\ &= \frac{2V_0}{n\pi}\Big(-\cos(n\pi)\,+\,\cos(0)\Big) \;=\; \frac{2V_0}{n\pi}\Big(-(-1)^n\,+\,1\Big) \\ &= \frac{2V_0}{n\pi}\times\begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n, \end{cases} \end{aligned} \tag{49}$$

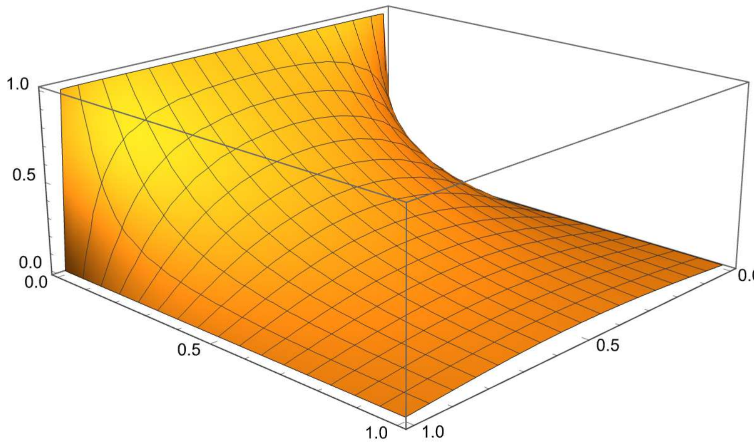
and therefore

$$V(x,y) \;=\; \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\text{odd}\,n}\frac{1}{n}\times\sin\frac{n\pi x}{a}\times\exp\left(-\frac{n\pi y}{a}\right). \tag{50}$$

There happens to be an analytic formula for this infinite sum, namely

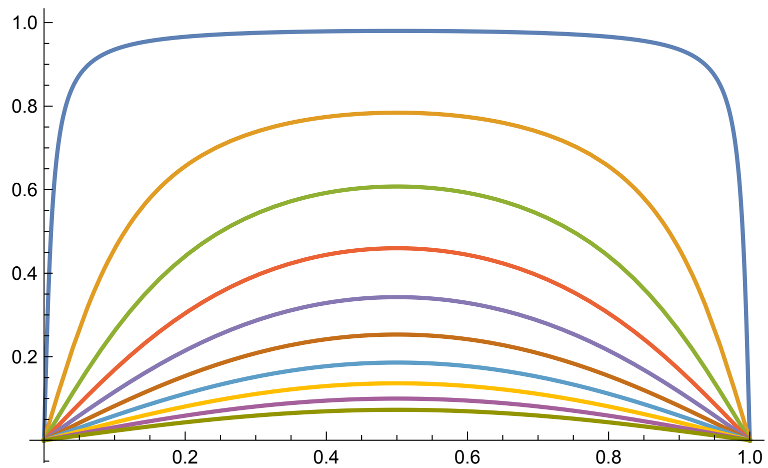
$$V(x,y) \;=\; \frac{2V_0}{\pi}\arctan\left(\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right), \tag{51}$$

but it's easier to understand the physical behavior of the potential (50) with a 3D plot:



(52)

And here are the cross-sectional profiles of $V(x)$ at specific fixed y's, namely $(y/a) = 0.01$, 0.11, 0.21, 0.31, 0.41, 0.51, 0.61, 0.71, 0.81, and 0.91:



(53)

As you can see, at small $y \ll a$ — near the front of the slot — the profile looks similar to the boundary $V_b(x)$, but with slower more rounded rises at x near zero or a, and a bit lower in the middle. As we go move into the slot — to larger y — we get lower and more rounded profiles with slower rises at the $x = 0$ and $x = a$ ends. And for larger y's — deeper and deeper into the slot — the profiles start looking just like the sine wave $\sin(\pi x/a)$ with smaller and smaller amplitudes.

The reason for this behavior becomes clear when we write the potential as the series (50): All the exponentials $\exp(-n\pi y/a)$ shrink with increasing y, but the exponentials with larger

n shrink faster that the exponentials with smaller n. Consequently, for large y/a the leading $n = 1$ terms completely dominates the potential, and we end up with

$$V \approx \frac{4V_0}{\pi} \times \exp(-\pi y/a) \times \sin(\pi x/a) \quad \text{for } y \gtrsim a, \tag{54}$$

a sine wave with a decreasing amplitude, exactly as we see on the plot (53).

Another 2D Example: Finite Slot Depth

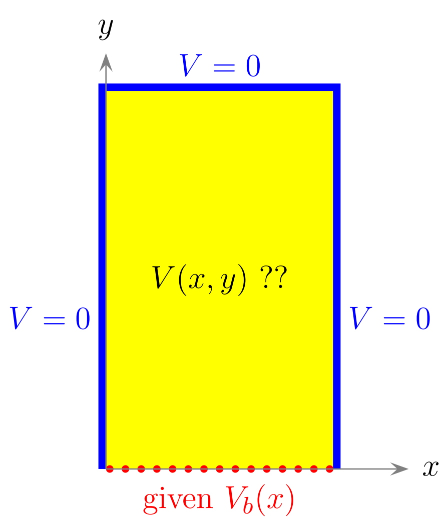
In this example, instead of a slot that's infinite in the $y \to +\infty$ direction we give it a finite depth b. In other words, the slot has a finite rectangular cross-section $a \times b$ (but infinite length in the $z \to \pm\infty$ directions). Three out of slot's four walls are made from a grounded metal — hence

$$V(x, y) = 0 \text{ for } x = 0 \text{ or } x = a \text{ or } y = b, \tag{55}$$

while the fourth wall at $y = 0$ is non-conducting. As before, there are no charges inside the slot but there are unknown charges outside it, and when we measure the potential along the $y = 0$ wall we find that it depends on x but not on z,

$$V(x, y = 0, z) = V_b(x \text{ only}). \tag{56}$$

Consequently, thanks to the translational symmetry in the z direction, we end up with a 2D problem: Finding $V(x, y \text{ only})$ inside the slot. Or rather, finding $V(x, y)$ inside the yellow rectangle $0 \leq x \leq a$, $0 \leq y \leq b$ on this diagram:



(57)

As before, we look for a $V(x,y)$ which obeys 4 conditions: [1] the Laplace equation; [2] $V=0$ at the left wall and the right wall of the pipe, $x=0$ or $x=a$; [3′] $V=0$ at the back wall $y=b$; and [4] at the front wall $y=0$, the potential agrees with the known boundary potential $V_b(x)$. Note that the conditions [1], [2], and [4] are exactly the same as for the infinite slot, and only the condition [3′] is different.

Using the separation-of-variables method, we proceed similarly to the infinite slot example. That is, we start by looking at the potentials of the form

$$V(x,y) \;=\; f(x)\times g(y) \tag{58}$$

which obey the Laplace equation [1] and the $V=0$ boundary conditions [2] and [3′]. As before, we find that Laplace equation requires

$$f''(x) \;+\; C\times f(x) \;=\; 0, \quad g''(y) \;-\; C\times g(y) \;=\; 0, \quad \text{for the same constant } C. \tag{59}$$

Moreover, the the boundary conditions for the $f(x)$ are the same as for the infinite slot, hence similar solutions

$$f(x) \;=\; \sin\frac{n\pi x}{a} \quad \text{for an integer } n=1,2,3,\ldots, \quad C \;=\; +(n\pi/a)^2 \;>\; 0. \tag{60}$$

As to the $g(y)$ function, the general solution to the $g''(y)-C\times g(y)=0$ equation is

$$g(y) \;=\; \alpha\times e^{+\kappa y} \;+\; \beta\times e^{-\kappa y} \tag{61}$$

for

$$\kappa \;=\; +\sqrt{C} \;=\; \frac{n\pi}{a}. \tag{62}$$

But now we have a different boundary condition for the $g(y)$: Having $V=0$ at the back wall $y=b$ calls for $g(y=b)=0$, and hence

$$\beta \;=\; -\alpha\times e^{2\kappa b}. \tag{63}$$

Equivalently, we may write

$$\alpha, \beta \;=\; \mp\frac{A}{2} \times e^{\mp\kappa b} \tag{64}$$

for some overall constant A, and therefore

$$g(y) \;=\; \frac{A}{2}\Big(-e^{-\kappa b+\kappa y} \,+\, e^{+\kappa b-\kappa y}\Big) \;=\; A \times \sinh(\kappa(b-y)). \tag{65}$$

Altogether, the variable-separated solution of the conditions $[1, 2, 3']$ for the finite-depth slot is

$$V(x,y) \;=\; A \times \sin\frac{n\pi x}{a} \times \sinh\frac{n\pi(b-y)}{a} \tag{66}$$

for an integer $n = 1, 2, 3, \ldots$.

Similar to the infinite slot, there is an infinite but a discrete series of such solutions, and they form a basis in the space of all solutions to the conditions $[1, 2, 3']$. Consequently, the most general solution to these conditions is a linear combination

$$V(x,y) \;=\; \sum_{n=1}^{\infty} A_n \times \sin\frac{n\pi x}{a} \times \sinh\frac{n\pi(b-y)}{a} \tag{67}$$

for some constant coefficients A_n. And to find the value of these coefficients, we need to match the potential (67) at the front wall $y = 0$ to the known boundary potential $V_b(y)$,

$$V(x, y=0) \;=\; \sum_{n=1}^{\infty}\left(A_n \times \sinh\frac{n\pi b}{a}\right) \times \sin\frac{n\pi x}{a} \;=\text{should be}=\; \text{given } V_b(x), \tag{68}$$

In terms of the Fourier transform of $V_b(x)$ into a series of the sine waves, this means that the $A_n \times \sinh(n\pi b/a)$ should match the coefficients of the n^{th} sine wave, thus

$$A_n \;=\; \frac{1}{\sinh(n\pi b/a)} \times \frac{2}{a}\int_0^a V_b(x) \times \sin\frac{n\pi x}{a}\,dx. \tag{69}$$

Or, if the given $V_b(x)$ happens to be a sine wave,

$$V_b(x) \;=\; V_0 \times \sin\frac{m\pi x}{a} \tag{70}$$

for an integer m, then

$$A_m = \frac{V_0}{\sinh(m\pi b/a)}, \quad \text{all other } A_n = 0, \tag{71}$$

and consequently

$$V(x,y) = V_0 \times \sin\frac{m\pi x}{a} \times \frac{\sinh(m\pi(b-y)/a)}{\sinh(m\pi b/a)} \tag{72}$$

inside the slot.

3D Example: Half-Infinite Rectangular Pipe

For a 3D example of the separation-of-variables method, consider a pipe with a rectangular $a \times b$ cross-section, but unlike a slot from the previous examle, all 4 sides the pipe are conducting and grounded. On the other hand, the length of this pipe is infinite in only one direction, so in the obvious (x,y,z) coordinates, the interior of the pipe is limited to

$$0 < x < a, \quad 0 < y < b, \quad 0 < z < +\infty, \tag{73}$$

and the pipe has a rectangular opening at $z = 0$. As usual, in this example there are no electric charges inside the pipe but there unknown charges outside it, and we are given the measured potential $V_b(x,y,z=0)$ across the pipe's opening; our task is to find the potential $V(x,y,z)$ throughout the pipe's interior.

Mathematically, we are looking at the potential $V(x,y,z)$ in the region (73) which obeys the following conditions:

[1] V obeys the 3D Laplace equation, $\triangle V(x,y,z) = 0$;

[2] V vanishes on the 4 grounded walls of the pipe,

$$V(x,y,z) = 0 \text{ when } x = 0, \text{ or } x = a, \text{ or } y = 0, \text{ or } y = b; \tag{74}$$

[3] deep inside the pipe, the potential asymptotes to zero, $V(x,y,z) \to 0$ for $z \to +\infty$;

[4] at the pipe's opening $z = 0$ the potential matches the given boundary potential,

$$V(x, y, z = 0) \;=\; \text{given } V_b(x, y). \tag{75}$$

Using the separation-of-variables method, we start by looking at the potentials of the form

$$V(x, y, z) \;=\; f(x) \times g(y) \times h(z) \tag{76}$$

which obeys the Laplace equation [1] and the homogeneous boundary conditions [2] and [3] — but don't worry about the inhomogeneous condition [4]. Eventually, we shall find an infinite series of such solutions, and then we shall look for a linear combination of these solutions that happens to obey the condition [4].

So let's start with the Laplace equation. In 3D,

$$\begin{aligned}\triangle V \;&=\; \frac{\partial^2 V}{\partial x^2} \;+\; \frac{\partial^2 V}{\partial y^2} \;+\; \frac{\partial^2 V}{\partial z^2} \\ &\qquad \langle\!\langle \text{ for } V \text{ as in eq. (76)} \rangle\!\rangle \\ &=\; f''(x) \times g(y) \times h(z) \;+\; f(x) \times g''(y) \times h(z) \;+\; f(x) \times g(y) \times h''(z),\end{aligned} \tag{77}$$

hence

$$\frac{\triangle V}{V} \;=\; \frac{f''(x)}{f(x)} \;+\; \frac{g''(y)}{g(y)} \;+\; \frac{h''(z)}{h(z)}\,, \tag{78}$$

and we want this expression to vanish for all x, y, z. But the first term here depends only on the x coordinate, the second — only on the y, and the third — only on the z, so the only way they can add up to zero for all independent x, y, z is if each one of these terms is a constant. Thus,

$$\begin{aligned}\frac{f''(x)}{f(x)} \;&=\; C_1 \;=\; \text{const}, \\ \frac{g''(y)}{g(y)} \;&=\; C_2 \;=\; \text{const}, \\ \frac{h''(z)}{h(z)} \;&=\; C_3 \;=\; \text{const}, \\ \text{and} \quad C_1 \;+\; C_2 \;&+\; C_3 \;=\; 0.\end{aligned} \tag{79}$$

Next, the homogeneous boundary conditions [2] and [3] for the potential translate to the boundary conditions of the f, g, and h functions as

$$\begin{aligned} f(x=0) &= f(x=a) = 0, \\ g(y=0) &= g(y=b) = 0, \\ h(z) &\to 0 \text{ for } z \to +\infty. \end{aligned} \tag{80}$$

Altogether, the $f(x)$ function obeys

$$f''(x) - C_1 \times f(x) = 0, \quad f(0) = f(a) = 0, \tag{81}$$

exactly as in the previous 2D examples, so it has the same solutions:

$$f(x) = \sin\frac{m\pi x}{a}, \quad C_1 = -(m\pi/a)^2 < 0, \tag{82}$$

for an integer $m = 1, 2, 3, \ldots$. Likewise, the $g(y)$ function obeys similar conditions

$$g''(y) - C_2 \times g(y) = 0, \quad g(0) = g(b) = 0, \tag{83}$$

so it also has similar solutions:

$$g(y) = \sin\frac{n\pi y}{b}, \quad C_2 = -(n\pi/b)^2, \tag{84}$$

for an integer $n = 1, 2, 3, \ldots$. Note: the two integers m and n in eqs. (82) and (84) are completely independent from each other.

Now let's pick any particular positive integers m and n. For any choice of these integers, we have

$$C_3 = -C_1 - C_2 = +(m\pi/a)^2 + (n\pi/b)^2 > 0, \tag{85}$$

so let's define

$$\kappa_{m,n} \stackrel{\text{def}}{=} +\sqrt{C_3} = +\sqrt{(m\pi/a)^2 + (n\pi/b)^2}. \tag{86}$$

In terms of this $\kappa_{m,n}$, the conditions for the $h(z)$ function become

$$h''(z) - \kappa_{m,n}^2 \times h(z) = 0, \quad h(z) \to 0 \text{ for } z \to +\infty, \tag{87}$$

with the only solution to these conditions being

$$h(z) \;=\; \exp\bigl(-\kappa_{m,n}\times z\bigr). \tag{88}$$

Altogether, we see that all the variable-separated solutions to the conditions [1,2,3] for the potential inside the pipe have form

$$V(x,y,z) \;=\; \text{const}\times\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b}\times\exp(-\kappa_{m,n}z) \tag{89}$$

for positive integers m and n. Similar to the previous 2D examples, we got an infinite but discrete set of solutions, although in the present 3D case we got a double series labeled by two independent integers m and n rather than a single series. Hence, a generic linear combinations of the solutions (89) is a double sum

$$V(x,y,z) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\times\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b}\times\exp(-\kappa_{m,n}z) \tag{90}$$

for some constant coefficients $A_{m,n}$. Note that the conditions [1,2,3] for the potential are all linear and homogeneous, so any linear combination of solutions is a solution, which means that any potential of the form (90) obeys these conditions. Moreover, the solutions (89) form a basis in the Hilbert space of all the solutions, so any solution can be expanded into a double sum (90) for some coefficients $A_{m,n}$.

In particular, the solution for the full problem — including the inhomogeneous boundary condition at the $z=0$ opening of the pipe — must have the form (90) for some coefficients $A_{m,n}$, and the values of such coefficients follow from the given boundary potential $V_b(x,y)$ at $z=0$. To find these coefficient, let's evaluate eq. (90) for $z=0$: Since $\exp(-\kappa_{m,n}z)=1$ for $z=0$ regardless of the value of $\kappa_{m,n}$, we get

$$V(x,y,z=0) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\times\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b} \;=\text{should be}=\; \text{given } V_b(x,y). \tag{91}$$

The double sum in this formula looks like a double Fourier expansion of $V(x,y,z=0)$ into sine waves of x and of y, so the coefficients $A_{m,n}$ obtain from the corresponding Fourier integrals as

$$A_{m,n} \;=\; \frac{4}{ab}\int\limits_0^a dx\int\limits_0^b dy\, V_b(x,y)\times\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b}\,. \tag{92}$$

Example:

Suppose the pipe has a square cross-section $a\times a$ (thus $b=a$), and the boundary potential at the pipe's opening is a double-sine wave

$$V_b(x,y) \;=\; V_0\times\sin\frac{3\pi x}{a}\times\sin\frac{4\pi y}{a}\,. \tag{93}$$

In this case, we do not need to perform the integrals (92) to find the Fourier coefficients $A_{m,n}$. Instead, we simply compare eq. (93) to the Fourier series (91):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\times\sin\frac{m\pi x}{a}\times\sin\frac{n\pi y}{b} \;=\; V_0\times\sin\frac{3\pi x}{a}\times\sin\frac{4\pi y}{a}\,, \tag{94}$$

which immediately tells us that

$$A_{3,4} \;=\; V_0 \quad \text{while all other } A_{m,n} \;=\; 0. \tag{95}$$

Consequently, the double sum (90) for the potential inside the pipe has only one non-zero term, thus

$$V(x,y,z) \;=\; V_0\times\sin\frac{3\pi x}{a}\times\sin\frac{4\pi y}{a}\times\exp(-\kappa_{3,4}z), \tag{96}$$

where

$$\kappa_{3,4} \;=\; \sqrt{(3\pi/a)^2\,+\,(4\pi/a)^2} \;=\; (\pi/a)\times\sqrt{3^2+4^2} \;=\; (\pi/a)\times 5. \tag{97}$$

Altogether,

$$V(x,y,z) \;=\; V_0\times\sin\frac{3\pi x}{a}\times\sin\frac{4\pi y}{a}\times\exp\left(\frac{-5\pi z}{a}\right). \tag{98}$$

Separation of Variables in Polar Coordinates

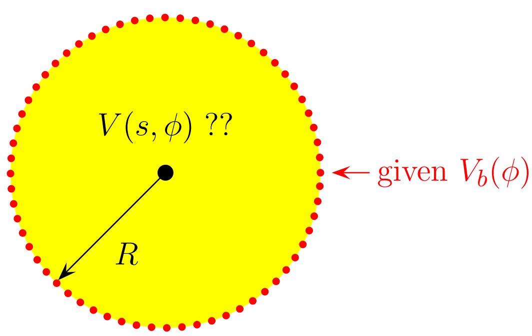
The separation-of-variables method can work in different coordinate systems, depending on the shape of the boundary in the problem in question. In this section, we shall see a couple of 2D examples with circular boundaries, for which the variables are best separated in the polar coordinates (s, ϕ) rather than the Cartesian coordinates (x, y).

Cylindrical Cavity

Consider an infinitely long cylindrical cavity of radius R — for example a round pipe with non-conducting walls. There are no electric charges inside the cavity, but there are some charges outside it, and maybe also on its surface. We do not know these charges, but we do know the potential V_b on the cavity's surface. Moreover, in the cylindrical coordinates (z, ϕ, z) — where the cavity's surface is at $s = R$, — the boundary potential $V_b(\phi, z)$ happens to be depend only on the ϕ coordinate but not on the z. Consequently, this system has a translational symmetry in z direction, so the potential inside the cavity should also be z-independent,

$$V_b(\phi, z) \;=\; V_b(\phi\text{ only}) \quad\Longrightarrow\quad V(s, \phi, z) \;=\; V(s, \phi\text{ only}). \tag{99}$$

Thus, we got ourselves a 2D problem: Given the potential $V_b(\phi)$ on a circle of radius R, find the potential everywhere inside that circle.



(100)

Given the shape of the boundary, it's best to solve this 2D problem in polar coordinates (s, ϕ). However, when working in polar coordinates, one should remember that the ϕ variable is an angle and $\phi + 2\pi$ direction in space is exactly the same as ϕ. Consequently, all physically measurable quantities — such as the electric potential $V(s, \phi)$ — must be periodic functions of

ϕ,

$$V(s, \phi + 2\pi) \;=\; V(s, \phi). \tag{101}$$

Thus, despite not having any boundaries in the ϕ direction, the periodicity condition (101) acts as a kind of a boundary condition. Also, the circle's center at $s = 0$ is the singular point of the polar coordinate system, but physically there it's just another point inside the circle, so the potential $V(s, \phi)$ should remain finite for $s \to 0$.

Altogether, this gives us the following mathematical problem: Find a function $V(s, \phi)$ for

$$0 \le s \le R, \qquad 0 \le \phi \le 2\pi \tag{102}$$

which obeys the following conditions:

[1] $V(s, \phi)$ obeys the Laplace equation in polar coordinates,

$$\triangle V(s, \phi) \;=\; \frac{\partial^2 V}{\partial s^2} \;+\; \frac{1}{s}\frac{\partial V}{\partial s} \;+\; \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} \;=\; 0; \tag{103}$$

[2] $V(s, \phi)$ is periodic in ϕ with period 2π; in terms of ϕ limited to the $0 \le \phi \le 2\pi$ interval, this means

$$V(s, \phi = 2\pi) \;=\; V(s, \phi = 0) \quad \text{and} \quad \frac{\partial V}{\partial \phi}(s, \phi = 2\pi) \;=\; \frac{\partial V}{\partial \phi}(s, \phi = 0); \tag{104}$$

[3] At the center $s = 0$, the potential $V(s, \phi)$ should be finite and ϕ-independent;

[4] At the outer boundary $s = R$, the potential should have the given boundary value,

$$V(s = R, \phi) \;=\; \text{given } V_b(\phi). \tag{105}$$

In the separation-of-variables method, we start by looking at solutions to conditions [1,2,3] (but not [4]) in the form

$$V(s,\phi) \;=\; f(s)\times g(\phi). \tag{106}$$

We start with the Laplace equation (103), which for potentials of the form (106) yields

$$\triangle V \;=\; f''(s)\times g(\phi) \;+\; \frac{f'(s)}{s}\times g(\phi) \;+\; \frac{f(s)}{s^2}\times g''(\phi) \;=\text{should be}=\; 0. \tag{107}$$

Let's multiply this equation by s^2 and divide by $V = fg$, which gives us

$$\frac{s^2}{V}\times\triangle V \;=\; \frac{s^2 f''(s)}{f(s)} \;+\; \frac{sf'(s)}{f(s)} \;+\; \frac{g''(\phi)}{g(\phi)} \;=\text{should be}=\; 0. \tag{108}$$

In this formula, the first two terms depend only on s while the third term depends only on ϕ, so the only way they can add up to zero for all s and all ϕ is if both combinations are constants:

$$\begin{aligned} \frac{s^2 f''(s)}{f(s)} \;+\; \frac{sf'(s)}{f(s)} \;&=\; +C \;=\; \text{const},\\ \frac{g''(\phi)}{g(\phi)} \;&=\; -C \;=\; \text{const},\\ &\text{for the same constant } C. \end{aligned} \tag{109}$$

Next, consider the g equation $g''(\phi)+Cg(\phi)=0$ for a constant C. In general, the solutions to this equation are

$$\text{for } C=+m^2\geq 0,\quad g(\phi) \;=\; A\cos(m\phi) \;+\; B\sin(m\phi), \tag{110}$$

$$\text{for } C=-\mu^2\leq 0,\quad g(\phi) \;=\; A\cosh(\mu\phi) \;+\; B\sinh(\mu\phi). \tag{111}$$

However, we want not just any solution but a *periodic solution* $g(\phi+2\pi)=g(\phi)$, which requires trigonometric rather than hyperbolic sine and cosine, hence $C=+m^2>0$. Moreover, *a period compatible with 2π requires integer $m=0,1,2,3,4,\ldots$.* Thus,

$$C \;=\; +m^2 \text{ for } m=0,1,2,3,\ldots \quad\text{and}\quad g(\phi) \;=\; A\cos(m\phi) \;+\; B\sin(m\phi). \tag{112}$$

Now consider the f equation for $C=+m^2$,

$$s^2\times f''(s) \;+\; s\times f'(s) \;-\; m^2\times f(s) \;=\; 0. \tag{113}$$

This equation is linear in f and *homogeneous* in s, so let's look for solutions of the form

$f(s) = s^\alpha$ for some power α. Indeed, plugging such f into the equation yields

$$0 \;=\; s^2 \times \alpha(\alpha-1)s^{\alpha-2} \;+\; s \times \alpha s^{\alpha-1} \;-\; m^2 \times s^\alpha \;=\; s^\alpha \times \bigl(\alpha^2 - m^2\bigr), \tag{114}$$

which is satisfied whenever

$$\alpha^2 - m^2 \;=\; 0 \quad\Longrightarrow\quad \alpha \;=\; \pm m. \tag{115}$$

For $m \neq 0$ there are two distinct roots, hence two independent solutions to eq. (113), and since this equation is linear, its general solution is a linear combination

$$f(s) \;=\; D \times s^{+m} \;+\; E \times s^{-m} \tag{116}$$

for some constants D and E.

For $m = 0$ the roots (115) coincide so we get only one solution rather than two. However, for $m = 0$, eq.(113) reduces to

$$sf''(s) \;+\; f'(s) \;=\; 0, \tag{117}$$

which is a first-order equation for the $f'(s)$, and its general solution is

$$f'(s) \;=\; \frac{E}{s} \quad\Longrightarrow\quad f(s) \;=\; D \;+\; E \times \ln(s) \tag{118}$$

for some constants D and E.

In any case, we want more than a general solution to the equation (113), we want the solution which obeys condition [3], namely no singularity at the cylinder's axis $s = 0$. This condition rules out negative powers of s for $m \neq 0$ or the logarithm for $m = 0$, which leaves us with

$$f(s) \;=\; \text{const} \times s^{+m} \;=\; \text{const}' \times \left(\frac{s}{R}\right)^m. \tag{119}$$

Altogether, we have an infinite series of solutions to conditions [1,2,3], namely

$$\begin{aligned} V_0(s,\phi) \;&=\; A_0 \;=\; \text{const for } m=0, \text{ and} \\ V_m(s,\phi) \;&=\; A\cos(m\phi) \times (s/R)^m \;+\; B\sin(m\phi) \times (s/R)^m \quad \text{for integer } m = 1,2,3,\ldots. \end{aligned} \tag{120}$$

Consequently, a general solution to [1,2,3] is a linear combination

$$V(s,\phi) \;=\; A_0 \;+\; \sum_{m=1}^{\infty}\Bigl(A_m\cos(m\phi) \;+\; B_m\sin(m\phi)\Bigr) \times \left(\frac{s}{R}\right)^m \tag{121}$$

for some constant coefficients A_m and B_m. Or in terms of complex exponentials $e^{\pm im\phi}$ with complex coefficients,

$$\begin{aligned} V(s,\phi) \;&=\; A_0 \;+\; \sum_{m=1}^{\infty}\Bigl(\tfrac{1}{2}(A_m+iB_m)e^{+im\phi} \;+\; \tfrac{1}{2}(A_m-iB_m)e^{-im\phi}\Bigr) \times \left(\frac{s}{R}\right)^m \\ &=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \times \left(\frac{s}{R}\right)^{|m|}, \end{aligned} \tag{122}$$

$$\text{where } C_0 \;=\; A_0, \quad C_{+m} \;=\; \tfrac{1}{2}(A_m+iB_m), \quad C_{-m} \;=\; \tfrac{1}{2}(A_m-iB_m) \;=\; C^*_{+m}. \tag{123}$$

Finally, the coefficients C_m follows from the boundary condition [4] on the surface of the cylinder:

$$@s=R, \quad V(R,\phi) \;=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \;=\; \text{given } V_b(\phi), \tag{124}$$

so the C_m obtain from expanding the periodic $V_b(\phi)$ into the Fourier series. Hence, the reverse Fourier transform gives

$$C_m \;=\; \frac{1}{2\pi}\int\limits_0^{2\pi} V_b(\phi) \times e^{-im\phi}\,d\phi. \tag{125}$$

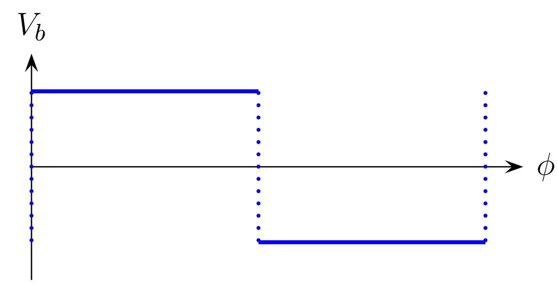
Or if you prefer the expansion (121) into real sine and cosine waves,

$$\begin{aligned} B_m \;&=\; \frac{2}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\sin(m\phi)\,d\phi, \\ A_m \;&=\; \frac{2}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\cos(m\phi)\,d\phi, \\ \text{except } A_0 \;&=\; \frac{1}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\,d\phi. \end{aligned} \tag{126}$$

Example: Square Wave.

Suppose the cylinder's surface is spit into 2 conducting halves (but insulated from each other) with potentials $\pm V_0$. Thus, as a function of ϕ, the boundary potential looks like a square wave,

$$V_b(\phi) \;=\; \begin{cases} +V_0 & \text{for } 0 < \phi < \pi, \\ -V_0 & \text{for } \pi < \phi < 2\pi. \end{cases} \tag{127}$$



This boundary potential is antisymmetric: $V_b(-\phi) = -V_b(\phi)$, or rather

$$V_b(2\pi - \phi) \;=\; -V_b(\phi), \tag{128}$$

so its Fourier transform has no cosine waves but only sine waves. Thus all $A_m = 0$, while

$$\begin{aligned} B_m \;&=\; \frac{V_0}{\pi}\int\limits_0^{\pi} \sin(m\phi)\,d\phi \;-\; \frac{V_0}{\pi}\int\limits_{\pi}^{2\pi} \sin(m\phi)\,d\phi \\ &=\; \frac{V_0}{m\pi}\big[\cos(0) \;-\; 2\cos(m\pi) \;+\; \cos(2m\pi)\big] \;=\; \frac{V_0}{m\pi} \times \begin{cases} 4 & \text{for odd } m, \\ 0 & \text{for even } m. \end{cases} \end{aligned} \tag{129}$$

Consequently, the potential inside the cylinder is given by the series

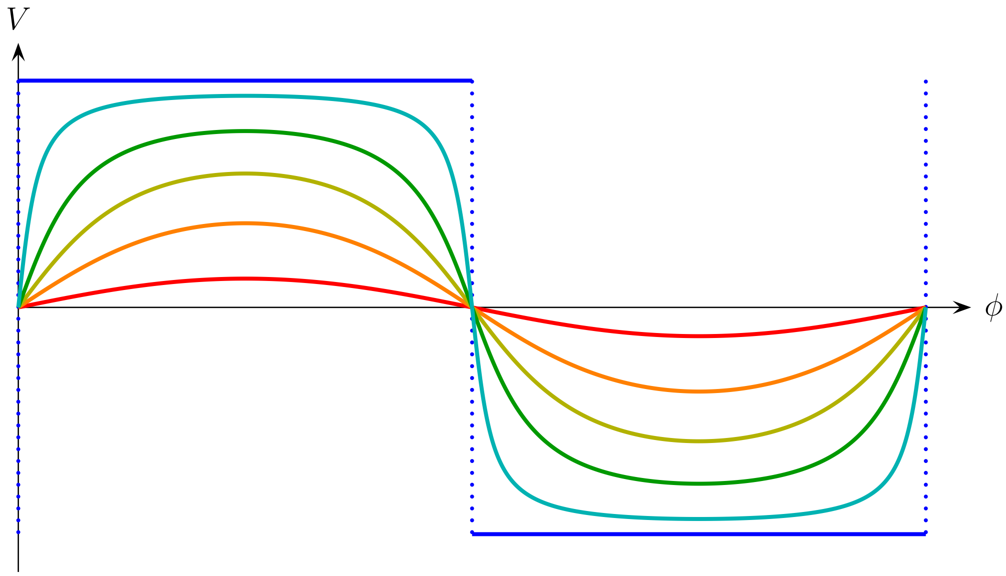
$$V(s,\phi) \;=\; \frac{4V_0}{\pi} \sum_{m=1,3,5,\ldots}^{\text{odd}\, m} \frac{\sin(m\phi)}{m} \times \left(\frac{s}{R}\right)^m, \tag{130}$$

which can be analytically summed up to

$$V(r,s) \;=\; \frac{2V_0}{\pi} \times \arctan\left(\frac{2Rs}{R^2 - s^2} \times \sin\phi\right). \tag{131}$$

To illustrate this potential graphically, let me plot it as a function of ϕ for $s = 0.1R$, $0.3R$,

$s = 0.5R$, $s = 0.7R$, $s = 0.9R$, and $s = R$:



(132)

Note: the closer we are to the axis, the smaller is the amplitude of the $V(\phi)$ curve, and the curve looks more and more like the sine wave. Mathematically, this happens because the larger–m terms in the series (130) carry larger powers of (s/R), so for small s/R ratios they become small compared to the leading $m = 1$ term. Consequently, close to the axis where $(s/R) \ll 1$ we may approximate the whole series by its leading term $(s/R)\sin(\phi)$.

Outside A Cylinder

Now consider a slightly different problem: instead of a cylindrical cavity, we have a charged cylinder surrounded by empty space. We don't know the charges inside the cylinder or on its surface, all we know is the boundary potential $V_b(\phi)$ which happens to be independent of z coordinate, and we need to find out the potential $V(s, \phi)$ outside the cylinder (which we presume to be also z independent).

Proceeding similarly to the previous example, we start by looking for $V(s, \phi) = f(s) \times g(\phi)$ which obeys the Laplace equation and is periodic in ϕ. This leads us to

$$C = +m^2 \text{ for } m = 0, 1, 2, 3, \ldots \quad \text{and} \quad g(\phi) = A\cos(m\phi) + B\sin(m\phi). \tag{112}$$

and hence

$$f(s) = \begin{cases} D + E \times \ln(s) & \text{for } m = 0, \\ D \times s^{+|m|} + E \times s^{-|m|} & \text{for } m \neq 0. \end{cases} \tag{133}$$

However, this time we are concerned with the asymptotic behavior for $s \to \infty$ rather than the axis of the cylinder at $s = 0$. Specifically, we want the potential to go to zero — or at least to stay finite — for $s \to \infty$, and this rules out the positive powers of s as well as $\ln(s)$. Consequently, outside of the cylinder

$$f(s) \;=\; \text{const} \times s^{-|m|} \tag{134}$$

instead of $f(s) \propto s^{+|m|}$ inside the cylinder.

Combining the s and ϕ dependence, we find

$$V(s,\phi) \;=\; A_0 \;+\; \sum_{m=1}^{\infty} \Bigl(A_m \cos(m\phi) \;+\; B_m \sin(m\phi) \Bigr) \times \left(\frac{R}{s} \right)^m \tag{135}$$

for some constants A_m and B_m, or in terms of complex exponentials $e^{\pm im\phi}$,

$$V(s,\phi) \;=\; \sum_{m=-\infty}^{+\infty} C_m \times e^{im\phi} \times \left(\frac{R}{s} \right)^{|m|}. \tag{136}$$

Finally, the complex coefficients $C_m = C_{-m}^*$ here — or if you prefer, the real coefficients A_m and B_m, — obtain from expanding the boundary potential into the Fourier series, precisely as in eqs. (125) or (126).

Separation of Variables in Spherical Coordinates

Problems with Axial Symmetry

Now consider a 3D problem: Find the potential $V(r,\theta,\phi)$ inside a spherical cavity — or outside a sphere — when we are given the potential $V_b(\theta,\phi)$ on the spherical surface. For simplicity, let's focus on potentials with axial symmetry:

$$V_b(\theta,\phi) \;=\; V_b(\theta \text{ only}) \quad\Longrightarrow\quad V(r,\theta,\phi) \;=\; V(r,\theta \text{ only}). \tag{137}$$

Mathematically, we seek the potential which:
[1] Obeys the 3D Laplace equation.
[2] Is single-valued, non-singular, and smooth as a function of θ.
[3] Is well behaved at the center $r \to 0$ if we work inside the sphere, or asymptotes to zero for $r \to \infty$ if we work outside the sphere.
[4] Has given boundary values at the sphere's surface, $V(r=R,\theta) = V_b(\theta)$.

Using the separation of variables method, we first seek to satisfy the conditions [1,2,3] for a potential of the form

$$V(r,\theta) \;=\; f(r) \times g(\theta), \tag{138}$$

find an infinite series of solutions, then look for a linear combination which satisfies the condition [4].

Let's start with the Laplace equation in the spherical coordinates:

$$\begin{aligned} \triangle V(r,\theta,\phi) \;=\; & \frac{\partial^2 V}{\partial r^2} \;+\; \frac{2}{r} \times \frac{\partial V}{\partial r} \\ & +\; \frac{1}{r^2} \times \frac{\partial^2 V}{\partial \theta^2} \;+\; \frac{1}{r^2\,\tan\theta} \times \frac{\partial V}{\partial \theta} \\ & +\; \frac{1}{r^2\,\sin^2\theta} \times \frac{\partial^2 V}{\partial \phi^2}\,. \end{aligned} \tag{139}$$

For the potential of the form (138), the Laplacian becomes

$$\triangle V \;=\; \left(f''(r) \,+\, \frac{2f'(r)}{r}\right) \times g(\theta) \;+\; \frac{f(r)}{r^2} \times \left(g''(\theta) \,+\, \frac{g'(\theta)}{\tan\theta}\right), \tag{140}$$

hence

$$\frac{r^2}{V} \times \triangle V \;=\; \left(\frac{r^2 f''(r)}{f(r)} \;+\; \frac{2rf(r)'}{f(r)} \right) + \left(\frac{g''(\theta)}{g(\theta)} \;+\; \frac{g'(\theta)}{g(\theta)\,\tan\theta} \right), \tag{141}$$

where the two terms inside the first () depend only on radius r while the two terms inside the second () depend only on the latitude θ. Consequently, the Laplace equation $\triangle V \equiv 0$ for all r, θ requires

$$r^2 \times \frac{f''(r)}{f(r)} \;+\; 2r \times \frac{f'(r)}{f(r)} \;=\; +C, \tag{142}$$

$$\frac{g''(\theta)}{g(\theta)} \;+\; \frac{1}{\tan\theta} \times \frac{g'(\theta)}{g(\theta)} \;=\; -C, \tag{143}$$

$$\text{for the same constant } C. \tag{144}$$

Next, consider the g equation (143), or equivalently

$$g''(\theta) \;+\; \frac{g'(\theta)}{\tan\theta} \;+\; C \times g(\theta) \;=\; 0. \tag{145}$$

Let's change the independent variable here from θ to $x = \cos\theta$, thus

$$g(\theta) \;=\; P(\cos\theta) \tag{146}$$

for some function $P(x)$. Consequently, by the chain rule for derivatives,

$$\frac{dg}{d\theta} \;=\; -\sin\theta \times \left.\frac{dP}{dx}\right|_{x=\cos\theta} \tag{147}$$

and hence

$$\frac{d^2g}{d\theta^2} \;=\; -\cos\theta \times \left.\frac{dP}{dx}\right|_{x=\cos\theta} \;+\; \sin^2\theta \times \left.\frac{d^2P}{dx^2}\right|_{x=\cos\theta}, \tag{148}$$

so plugging these derivatives into eq. (145) we arrive at

$$\begin{aligned} 0 \;&=\; -\cos\theta \times \frac{dP}{dx} \;+\; \sin^2\theta \times \frac{d^2P}{dx^2} \;+\; \frac{-\sin\theta}{\tan\theta} \times \frac{dP}{dx} \;+\; C \times P \\ &=\; (1 - \cos^2\theta) \times \frac{d^2P}{dx^2} \;-\; (\cos\theta + \cos\theta) \times \frac{dP}{dx} \;+\; C \times P. \end{aligned} \tag{149}$$

In terms of $x = \cos\theta$, this is the *Legendre equation* for the $P(x)$,

$$(1-x^2)\times P''(x) \;-\; 2x\times P'(x) \;+\; C\times P(x) \;=\; 0. \tag{150}$$

Without explaining how to solve the Legendre equation, let me briefly describe its solutions. Because the coefficient $(1-x^2)$ of the leading derivative term in this equation vanishes for $x = \pm 1$ — which correspond to the two poles of the sphere, $\theta = 0$ and $\theta = \pi$, — a general solution of the Legendre equation is singular at these two points. Moreover, for generic values of the constant C, all solutions are singular at $x = +1$, or at $x = -1$, or at both points. However, for special values of C — specifically

$$C \;=\; \ell(\ell+1), \quad \text{for integer } \ell = 0, 1, 2, 3, \ldots, \tag{151}$$

— there is a solution which is regular at all x, namely the *Legendre polynomial* of degree ℓ,

$$P_\ell(x) \;=\; \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell}\,(x^2-1)^\ell. \tag{152}$$

The overall coefficient here is chosen such that at $x = +1$ all these polynomials become $P_\ell(1) = 1$, while for $x = -1$ $P_\ell(-1) = (-1)^\ell$. Here are a few explicit Legendre polynomials for small ℓ:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= x,\\
P_2(x) &= \tfrac{3}{2}x^2 \;-\; \tfrac{1}{2},\\
P_3(x) &= \tfrac{5}{2}x^3 \;-\; \tfrac{3}{2}x,\\
P_4(x) &= \tfrac{35}{8}x^4 \;-\; \tfrac{15}{4}x^2 \;+\; \tfrac{3}{8},\\
P_5(x) &= \tfrac{63}{8}x^5 \;-\; \tfrac{35}{4}x^3 \;+\; \tfrac{15}{8}x,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{153}$$

The Legendre polynomial are 'orthogonal' to each other when we use $\int_{-1}^{+1} dx$ as the measure,

$$\int\limits_{-1}^{+1} P_\ell(x)\times P_{\ell'}(x) \;=\; \begin{cases} 0 & \text{for any } \ell' \neq \ell,\\[4pt] \dfrac{2}{2\ell+1} & \text{for } \ell' = \ell. \end{cases} \tag{154}$$

Consequently, any analytic function of x ranging from -1 to $+1$ may be expanded in a series of Legendre polynomials,

$$\text{any } H(x) \;=\; \sum_{\ell=0}^{\infty} H_\ell \times P_\ell(x) \quad \text{for} \quad H_\ell \;=\; \frac{2\ell+1}{2} \int\limits_{-1}^{+1} H(x) \times P_\ell(x)\, dx. \tag{155}$$

Going back to eq. (145) for the $g(\theta)$, we see that it has solutions that are regular at both poles $\theta = 0$ and $\theta = \pi$ **only for** $C = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$, namely

$$g(\theta) \;=\; P_\ell(\cos\theta). \tag{156}$$

For these values of the constant C, eq. (142) for the $f(r)$ becomes

$$r^2 \times f''(r) \;+\; 2r \times f'(r) \;-\; \ell(\ell+1) \times f(r) \;=\; 0. \tag{157}$$

This equation is linear in f and is *homogeneous* in r, so let's look for the solutions of the form $f(r) = r^\alpha$ for some constant power α. Indeed plugging such an f into the equation (157) yields

$$\begin{aligned} 0 \;&=\; r^2 \times \alpha(\alpha-1) r^{\alpha-2} \;+\; 2r \times \alpha r^{\alpha-1} \;-\; \ell(\ell+1) \times r^\alpha \\ &=\; r^\alpha \times \big(\alpha(\alpha-1) + 2\alpha - \ell(\ell+1)\big) \end{aligned} \tag{158}$$

so the differential equation is satisfied whenever

$$\alpha(\alpha-1) \;+\; 2\alpha \;=\; \alpha(\alpha+1) \;=\; \ell(\ell+1) \quad\Longrightarrow\quad \alpha = \ell \;\textbf{or}\; \alpha = -(\ell+1). \tag{159}$$

Thus, the general solution to eq. (142) has form

$$f(r) \;=\; A \times r^\ell \;+\; \frac{B}{r^{\ell+1}} \tag{160}$$

for some constant coefficients A and B.

The specific solution we need depends on whether we are looking for the potential inside the sphere or outside the sphere.

- For the inside-the-sphere solution we want the potential to be non-singular at the center, which rules out negative powers of the radius r. In terms of eq. (160) this means $B = 0$ and hence

$$f(r) \;=\; \text{const} \times r^{\ell} \;=\; \text{const}' \times \left(\frac{r}{R}\right)^{\ell}. \tag{161}$$

- For the outside-the-sphere solution, we want the potential to asymptote to zero for $r \to \infty$, which rules out positive powers of the radius. In terms of eq. (160) this means $A = 0$ and hence

$$f(r) \;=\; \frac{\text{const}}{r^{\ell+1}} \;=\; \text{const}' \times \left(\frac{R}{r}\right)^{\ell+1}. \tag{162}$$

Altogether, the general solution to the conditions [1,2,3] is given by the series:

Inside the sphere,

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \left(\frac{r}{R}\right)^{\ell}. \tag{163}$$

Outside the sphere,

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \left(\frac{R}{r}\right)^{\ell+1}. \tag{164}$$

In both cases the coefficients C_ℓ are constants, whose values are determined by the remaining condition [4], namely the boundary condition at the sphere's surface:

$$V(r=R,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times 1 = \text{ given } V_b(\theta). \tag{165}$$

To solve this condition for the C_ℓ, we use the orthogonality of the Legendre polynomials and hence eq. (155): Treat the given boundary potential $V_b(\theta)$ as a function of $x = \cos\theta$, then

$$V_b(x) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(x) \quad \text{for} \quad C_\ell \;=\; \frac{2\ell+1}{2} \int_{-1}^{+1} V_b(x) \times P_\ell(x)\, dx. \tag{166}$$

Or in terms of θ rather than $x = \cos\theta$,

$$C_\ell \;=\; \frac{2\ell+1}{2} \int_{0}^{\pi} V_b(\theta) \times P_\ell(\cos\theta) \times \sin\theta\, d\theta. \tag{167}$$

Example: $V_b(\theta) = V_0 \times \cos(3\theta)$.

For boundary potentials which are manifest polynomials of $\cos\theta$ — or can be brought to such form using simple trigonometry, such as in our example

$$V_b \;=\; V_0 \cos(3\theta) \;=\; 4V_0 \cos^3\theta \;-\; 3V_0 \cos\theta, \tag{168}$$

— we do not need to evaluate the integrals (167) to find the coefficients C_ℓ. Instead, we may simply expand the polynomial $V_b(\cos\theta)$ as a finite sum — rather than an infinite series — of Legendre polynomials using their explicit forms (153). Indeed, power by power in $x = \cos\theta$ we have

$$\begin{gathered} x \;=\; P_1(x), \quad x^2 \;=\; \tfrac{1}{3}\big(2P_2(x) + P_0(x)\big), \quad x^3 \;=\; \tfrac{1}{5}\big(2P_3(x) + 3P_1(x)\big), \\ x^4 \;=\; \tfrac{1}{35}\big(8P_4(x) + 20P_2(x) + 7P_0(x)\big), \quad \ldots \end{gathered} \tag{169}$$

Consequently, for our example we have

$$V_b(\cos\theta) \;=\; 4V_0 \times \frac{2P_3(\cos\theta) + 3P_1(\cos\theta)}{5} - 3V_0 \times P_1(\cos\theta) \;=\; \tfrac{8}{5}V_0 \times P_3(\cos\theta) - \tfrac{3}{5}V_0 \times P_1(\cos\theta), \tag{170}$$

hence

$$C_1 \;=\; -\tfrac{3}{5}V_0, \quad C_3 \;=\; +\tfrac{8}{5}V_0, \quad \text{all other } C_\ell \;=\; 0. \tag{171}$$

Therefore, inside the sphere the potential is

$$V(r,\theta) \;=\; -\tfrac{3}{5}V_0 \times P_1(\cos\theta) \times \left(\frac{r}{R}\right) \;+\; \tfrac{8}{5}V_0 \times P_3(\cos\theta) \times \left(\frac{r}{R}\right)^3, \tag{172}$$

while outside the sphere

$$V(r,\theta) \;=\; -\tfrac{3}{5}V_0 \times P_1(\cos\theta) \times \left(\frac{R}{r}\right)^2 \;+\; \tfrac{8}{5}V_0 \times P_3(\cos\theta) \times \left(\frac{R}{r}\right)^4. \tag{173}$$

Charges on the Spherical Surface

Consider a thin spherical shell with some surface charge density $\sigma(\theta,\phi)$ — and no other charges inside or outside the shell. For simplicity, assume axial symmetry, thus $\sigma(\theta$ only). Let's find out the potential both inside and outside the spherical shell due to this charge density.

Surface charge densities make for discontinuous electric fields, but the potential V is continuous across the charged surface. Thus, while in the present situation we do not know the boundary potential $V_b(\theta)$ on the spherical surface, we do know its the same potential both immediately inside and immediately outside the surface. Consequently, the potential $V(r,\theta)$ inside and outside the sphere is given by the equations (163) and (164) for the same coefficients C_ℓ, whatever they are. In other words,

$$\forall\, r,\theta: \quad V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} \left(\dfrac{r}{R}\right)^{\ell} & \text{for } r<R, \\[2ex] \left(\dfrac{R}{r}\right)^{\ell+1} & \text{for } r>R. \end{cases} \tag{174}$$

Next, consider the radial component of the electric field:

$$E_r \;=\; -\frac{\partial V(r,\theta)}{\partial r} \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} -\ell\,\dfrac{r^{\ell-1}}{R^\ell} & \text{for } r<R, \\[2ex] +(\ell+1)\,\dfrac{R^{\ell+1}}{r^{\ell+2}} & \text{for } r>R. \end{cases} \tag{175}$$

Unlike the potential, this radial electric field is discontinuous across the sphere. Indeed, near the sphere

$$E_r(r\approx R) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \begin{cases} \dfrac{-\ell}{R} & \text{just inside the sphere,} \\[2ex] \dfrac{+(\ell+1)}{R} & \text{just outside the sphere,} \end{cases} \tag{176}$$

with discontinuity

$$\mathrm{disc}(E_r) \;=\; E_r(r=R+\epsilon) \,-\, E_r(r=R-\epsilon) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times P_\ell(\cos\theta) \times \frac{2\ell+1}{R}\,. \tag{177}$$

Physically, this discontinuity is caused by the surface charge density on the sphere,

$$\mathrm{disc}(E_r) \;=\; \frac{\sigma}{\epsilon_0}\,. \tag{178}$$

Consequently, the charge density as a function of θ is related to the coefficients C_ℓ of the potential (174) according to

$$\sigma(\theta) \;=\; \epsilon_0\ \mathrm{disc}(E_r(\theta)) \;=\; \frac{\epsilon_0}{R} \times \sum_{\ell=0}^{\infty}(2\ell+1)\times C_\ell \times P_\ell(\cos\theta). \tag{179}$$

We may also reverse this relation according to eq. (155) to get the coefficients C_ℓ from the $\sigma(\theta)$,

$$C_\ell \;=\; \frac{R}{2\epsilon_0} \times \int\limits_0^\pi \sigma(\theta)\times P_\ell(\cos\theta)\times\sin\theta\,d\theta. \tag{180}$$

For example, suppose the sphere is neutral on the whole, but has a quadrupole charge density

$$\sigma(\theta) \;=\; \sigma_0 \times \frac{3\cos^2\theta - 1}{2} \;=\; \sigma_0 \times P_2(\cos\theta). \tag{181}$$

Comparing this angular dependence with eq. (179), we immediately see that the only non-zero coefficient C_ℓ is the C_2, specifically

$$C_2 \;=\; \frac{R\sigma_0}{5\epsilon_0}\,. \tag{182}$$

Consequently, inside the sphere the potential is

$$V(r,\theta) \;=\; \frac{\sigma_0}{5\epsilon_0} \times \frac{r^2}{R} \times P_2(\cos\theta), \tag{183}$$

while outside the sphere

$$V(r,\theta) \;=\; \frac{\sigma_0}{5\epsilon_0} \times \frac{R^4}{r^3} \times P_2(\cos\theta). \tag{184}$$

Metal Sphere in External Electric Field

Now consider another example: a metal sphere in uniform external electric field. That is, far away from the sphere the electric field asymptotes to the uniform $\mathbf{E} = E\hat{\mathbf{z}}$, hence

$$\text{for } r \to \infty, \quad V \;\to\; -Ez \;=\; -Er \times \cos\theta \;=\; -Er \times P_1(\cos\theta). \tag{185}$$

The sphere itself is neutral, so without loss of generality we may assume it has zero potential.

Let's find the potential outside the sphere for these boundary conditions. Since we no longer have $V \to 0$ at infinity, the radial function $f_\ell(r)$ could be a general combination of two solutions,

$$f_\ell(r) \;=\; A_\ell \times r^\ell \;+\; \frac{B_\ell}{r^{\ell+1}} \tag{186}$$

with $A_\ell \neq 0$. On the other hand, asking for $V = 0$ all over the sphere requires $f_\ell(r = R) = 0$ and hence

$$B_\ell = \; -R^{2\ell+1} \times A_\ell\,. \tag{187}$$

Consequently, the general form of the potential outside the sphere looks like

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} A_\ell \times P_\ell(\cos\theta) \times \left(r^\ell \;-\; \frac{R^{2\ell+1}}{r^{\ell+1}}\right) \tag{188}$$

for some coefficients A_ℓ.

To find these coefficients, we compare the asymptotic behavior of the potential (188) for large r,

$$V \;\longrightarrow\; \sum_{\ell=0}^{\infty} A_\ell \times P_\ell(\cos\theta) \times r^\ell \tag{189}$$

to the desired asymptotics (185). This comparison immediately tells us that

$$A_1 \;=\; -E, \quad \text{all other } A_\ell \;=\; 0, \tag{190}$$

hence

$$V(r,\theta) \;=\; -E\left(r \;-\; \frac{R^3}{r^2}\right) \times \cos\theta, \tag{191}$$

or in Cartesian coordinates

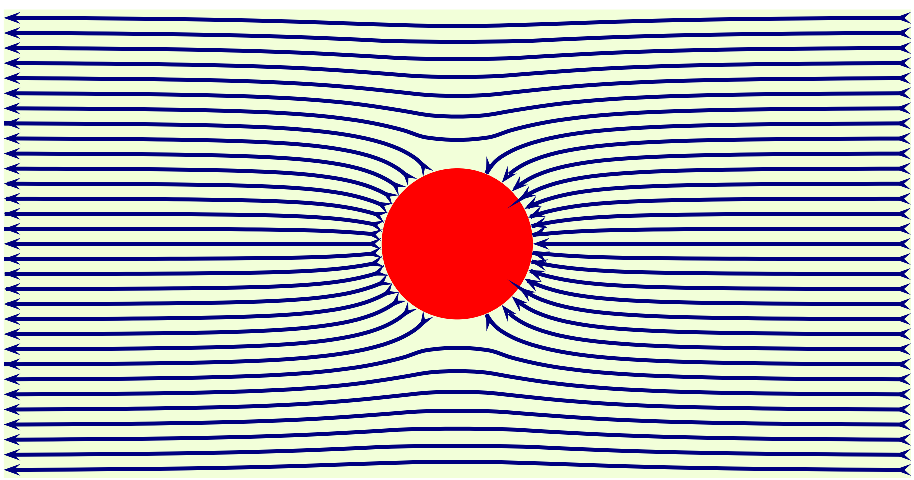
$$V(x,y,z) \;=\; -Ez \;+\; ER^3 \times \frac{z}{(x^2+y^2+z^2)^{3/2}} \tag{192}$$

The first term here is due to the external electric field, while the second term is due to induced charges on the sphere's surface.

Taking the gradient of the potential (192), we obtain the net electric field,

$$\mathbf{E}(x,y,z) \;=\; E\hat{\mathbf{z}} \;+\; ER^3\left(\frac{3z}{r^4}\,\hat{\mathbf{r}} \;-\; \frac{1}{r^3}\,\hat{\mathbf{z}}\right) \;=\; E\hat{\mathbf{z}} \;+\; \frac{ER^3}{r^3}\left(2\frac{z}{r}\,\hat{\mathbf{z}} \;-\; \frac{x}{r}\,\hat{\mathbf{x}} \;-\; \frac{y}{r}\,\hat{\mathbf{y}}\right). \tag{193}$$

Here is the picture of the field lines for this electric field:



Finally, the surface charge density $\sigma(\theta)$ on the metal sphere follows from the radial electric field immediately outside the metal:

$$\begin{aligned} E_r(\theta) \;&=\; -\frac{\partial V}{\partial r} \;=\; +E\cos\theta \times \frac{\partial}{\partial r}\left(r \;-\; \frac{R^3}{r^2}\right) \\ &=\; E\cos\theta \times \left(1 \;+\; \frac{2R^3}{r^3}\right) \;\xrightarrow[r\to R]{}\; E\cos\theta \times 3, \end{aligned} \tag{194}$$

hence

$$\sigma(\theta) \;=\; \epsilon_0 E_r(r\to R) \;=\; 3\epsilon_0 E\cos\theta. \tag{195}$$

Spherical Harmonics for Problems Without Axial Symmetry

Finally, consider a more general 3D problem with a spherical boundary, but with a given boundary potential $V_b(\theta,\phi)$ (or a given boundary charge $\sigma(\theta,\phi)$) which is not axially symmetric but depends on both angular coordinates θ and ϕ. In this case, instead of the Legendre polynomials $P_\ell(\cos\theta)$ we should use the *spherical harmonics* $Y_{\ell,m}(\theta,\phi)$. You will study these spherical harmonics in some detail in the Quantum Mechanics class in the context of angular momentum quantization, hydrogen atom wavefunctions, *etc.*, *etc.* For the moment, let me skip the details and simply summarize a few key properties of the spherical harmonics.

- The spherical harmonics are solutions to the partial differential equation

$$\frac{\partial^2 Y}{\partial\theta^2} \;+\; \frac{1}{\tan\theta}\,\frac{\partial Y}{\partial\theta} \;+\; \frac{1}{\sin^2\theta}\,\frac{\partial^2 Y}{\partial\phi^2} \;=\; -\ell(\ell+1)Y \tag{196}$$

  subject to the conditions of single-valuedness and no singularities anywhere on the sphere. In terms of the θ and ϕ coordinates this means periodicity in ϕ and no singularities at the poles $\theta=0$ and $\theta=\pi$.

- The solutions exist only for integer $\ell=0,1,2,3,\ldots$. For each such ℓ, there are $2\ell+1$ independent solutions $Y_{\ell,m}(\theta,\phi)$ labeled by another integer m running from $-\ell$ to $+\ell$.

- The $Y_{\ell,m}$ have form $Y_{\ell,m}(\theta,\phi)=(\text{const})\times P_{\ell(m)}(\cos\theta)\times\exp(im\phi)$ where the $P_{\ell(m)}(x)$ are called the *associate Legendre polynomials*, even though some of them are not really polynomials. Instead, $P_{\ell(m)}(\cos\theta)=(\sin\theta)^{|m|}\times$ degree $(\ell-|m|)$ polynomial of $\cos\theta$.

- For $m\neq 0$ the spherical harmonics are complex; by convention, $Y^*_{\ell,m}=(-1)^m Y_{\ell,-m}$. Also, all the harmonics with $m\neq 0$ vanish at the poles $\theta=0$ and $\theta=\pi$.

- The only harmonics which do not vanish at the poles are the $Y_{\ell,0}$. These harmonics are independent of ϕ and are proportional to $P_\ell(\cos\theta)$, but have different normalization: $Y_{\ell,0}(\theta,\not\phi)=\sqrt{(2\ell+1)/4\pi}\times P_\ell(\cos\theta)$.

- The spherical harmonics are orthogonal to each other and normalized to 1. That is

$$\iint Y^*_{\ell,m}(\theta,\phi)\,Y_{\ell',m'}(\theta,\phi)\,d^2\Omega(\theta,\phi) \;=\; \delta_{\ell,\ell'}\delta_{m,m'}\,. \tag{197}$$

- Any smooth, single-valued function $g(\theta,\phi)$ can be decomposed into a series of spherical harmonics,

$$g(\theta,\phi) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} C_{\ell,m} Y_{\ell,m}(\theta,\phi) \quad \text{for} \quad C_{\ell,m} \;=\; \iint g(\theta,\phi)\, Y^*_{\ell,m}(\theta,\phi)\, d^2\Omega(\theta,\phi). \quad (198)$$

- Let $F(r,\theta,\phi) = r^\ell \times Y_{\ell,m}(\theta,\phi)$. Then in Cartesian coordinates, $F(x,y,z)$ is a homogeneous polynomial in x,y,z of degree ℓ. Moreover, $F(x,y,z)$ obeys the Laplace equation.

Now let's apply the spherical harmonics to the electrostatic potential problems with spherical boundaries but with ϕ-dependent boundary conditions. Mathematically, we look for a function $V(r,\theta,\phi)$ which:
[1] Obeys the Laplace equation inside or outside some sphere of radius R.
[2] Is smooth and single-valued everywhere in the volume in question; in particular, V is periodic in ϕ and has no singularities at $\theta = 0$ or $\theta = \pi$.
[3] For the inside of a spherical cavity, V is smooth at $r \to 0$; for the outside or a sphere, V asymptotes to zero for $r \to \infty$.
[4] On the spherical boundary the potential has given form, $V(R,\theta,\phi) = V_b(\theta,\phi)$.

Using the separation of variables method, we start by looking for solutions to conditions [1,2,3] (but not [4]) of the form

$$V(r,\theta,\phi) \;=\; f(r) \times g(\theta,\phi); \quad (199)$$

note incomplete separation of variables at this stage. In light of eq. (139) for the Laplace operator in spherical coordinates,

$$\frac{r^2}{V} \times \Delta V \;=\; \frac{r^2 f''}{f} \;+\; \frac{2rf'}{f} \;+\; \frac{1}{g}\left(\frac{\partial^2 g}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial g}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\phi^2}\right), \quad (200)$$

so to get a solution to the Laplace equation $\Delta V = 0$ we need

$$\frac{\partial^2 g}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial g}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\phi^2} + C \times g \;=\; 0, \quad (201)$$

$$r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - C \times f \;=\; 0 \quad (202)$$

for the same constant C. By inspection, eq. (201) is the same as eq. (196), so we know that the solutions exist only for $C = \ell(\ell+1)$ for integer $\ell = 0, 1, 2, 3, \ldots$, and the solutions are the spherical harmonics $g(\theta,\phi) = Y_{\ell,m}(\theta,\phi)$ or their linear combinations. Thus,

$$V(r,\theta.\phi) \;=\; f(r) \times Y_{\ell,m}(\theta,\phi) \tag{203}$$

where the radial function $f(r)$ obeys

$$r^2 f''(r) \;+\; 2rf'(r) \;-\; \ell(\ell+1)f(r) \;=\; 0. \tag{204}$$

As we saw earlier in these notes, the solutions to this equation have form

$$f(r) \;=\; A \times r^\ell \;+\; \frac{B}{r^{\ell+1}} \tag{205}$$

for some constants A and B. For a spherical cavity, regularity of the solution at the center requires $B = 0$ while for an outside of a sphere the asymptotic condition at ∞ requires $A = 0$. However, for a space between two spherical boundaries, we may have both $A \neq 0$ and $B \neq 0$.

Altogether, the general solution to conditions [1,2,3] for the inside of a spherical cavity has form

$$V(r,\theta,\phi) \;= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times \left(\frac{r}{R}\right)^{\ell} \times Y_{\ell,m}(\theta,\phi), \tag{206}$$

while the general solution for the outside of a sphere looks like

$$V(r,\theta,\phi) \;= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times \left(\frac{R}{r}\right)^{\ell+1} \times Y_{\ell,m}(\theta,\phi). \tag{207}$$

In both cases, the constant coefficients $C_{\ell,m}$ follow from the boundary condition [4] at the spherical surface:

$$V(R,\theta,\phi) \;= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times Y_{\ell,m}(\theta,\phi) \;=\; \text{given } V_b(\theta,\phi). \tag{208}$$

Since the spherical harmonics form a complete orthonormal basis for the functions of the spherical angles (θ,ϕ), we may use eq. (198) to obtain the coefficients $C_{\ell,m}$ for any given boundary potential $V_b(\theta,\phi)$ on the spherical surface, namely

$$C_{\ell,m} = \iint V_b(\theta,\phi)\times Y^*_{\ell,m}(\theta,\phi)\, d^2\Omega(\theta,\phi). \tag{209}$$

A Few Words About Cylindrical Coordinates

There are many 3D boundary problems that can be solved by separating the variables in the cylindrical coordinates (s,ϕ,z). For example, consider a round pipe of length that's infinite in only one direction $z\to+\infty$. The pipe has grounded conducting surface at $s=R$, and a circular opening at $z=0$. There are no charges inside the pipe, we know the boundary potential $V_n(s,\phi)$ across the opening, and we want to find the potential inside the pipe.

Unfortunately, for the separated-variables potentials of the form

$$V(s,\phi,z) = f(s)\times g(\phi)\times h(z) \tag{210}$$

with non-constant $h(z)$, the differential equation for the radial function $f(s)$ is the Bessel equation and its solutions are Bessel functions. While there is nothing wrong with the Bessel functions as such, they are unfamiliar to many undergraduate students, and explaining them is beyond the scope of this *Classical Electrodynamics* class.

Thus, I am going to skip over separation of variables in cylindrical coordinates. The students who are interested in this subject can read about in in J. D. Jackson's graduate-level textbook "Classical Electrodynamics", §3.7–8 (of the third edition).