Problem **2.2**:

First, let me write down the Coulomb law in the vector form. For a point charge Q located at $\mathbf{R}$, the electric field at some other point $\mathbf{r} = \mathbf{R} + \Delta\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0\,|\Delta\mathbf{r}|^2}\,\widehat{\Delta\mathbf{r}} = \frac{Q}{4\pi\epsilon_0}\,\frac{\Delta\mathbf{r}}{|\Delta\mathbf{r}|^3} = \frac{Q}{4\pi\epsilon_0}\,\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\,. \tag{1}$$

Note the third power of the distance in the denominator of the last expression here, since one factor of the distance cancels the length of the non-unit vector in the numerator.

For multiple charges Q_i located at $\mathbf{R}_i$, the net field at some probe point $\mathbf{r}$ is the vector sum

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_i Q_i\frac{\mathbf{r}-\mathbf{R}_i}{|\mathbf{r}-\mathbf{R}_i|^3}\,. \tag{2}$$

For the problem at hand, we have two charges: $Q_1 = +q$ at $R_1 = (-\frac{d}{2}, 0, 0)$ and $Q_2 = -q$ at $R_2 = (+\frac{d}{2}, 0, 0)$, while the point where we measure the electric field is at $\mathbf{r} = (0, 0, z)$. Thus, the $\Delta\mathbf{r}$ vectors for the two charges are

$$\Delta\mathbf{r}_1 = \mathbf{r} - \mathbf{R}_1 = (+\tfrac{d}{2}, 0, z) \quad\text{and}\quad \Delta\mathbf{r}_2 = \mathbf{r} - \mathbf{R}_2 = (-\tfrac{d}{2}, 0, z), \tag{3}$$

which have the same length

$$|\Delta\mathbf{r}_1| = |\Delta\mathbf{r}_2| = \mathcal{R} = \sqrt{(d/2)^2 + z^2}\,. \tag{4}$$

Consequently, the net electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{+q}{4\pi\epsilon_0}\,\frac{\Delta\mathbf{r}_1}{|\Delta\mathbf{r}_1|^3} + \frac{-q}{4\pi\epsilon_0}\,\frac{\Delta\mathbf{r}_2}{|\Delta\mathbf{r}_2|^3} = \frac{q}{4\pi\epsilon_0}\,\frac{1}{\mathcal{R}^3}\big(\Delta\mathbf{r}_1 - \Delta\mathbf{r}_2\big), \tag{5}$$

where

$$\Delta\mathbf{r}_1 - \Delta\mathbf{r}_2 = (\mathbf{r}-\mathbf{R}_1) - (\mathbf{r}-\mathbf{R}_2) = \mathbf{R}_2 - \mathbf{R}_1 = (d, 0, 0) \equiv d\,\hat{\mathbf{x}}. \tag{6}$$

Altogether, the electric field is

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{d}{\mathcal{R}^3}\,\hat{\mathbf{x}}. \tag{7}$$

Its magnitude is

$$E \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{d}{\mathcal{R}^3} \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{d}{\bigl(z^2+(d/2)^2\bigr)^{3/2}}\,, \tag{8}$$

while the direction is parallel to the x axis.

Problem **2.3**:

On figure 2.7, the charged rod lies along x axis, spanning points

$$\mathbf{R} \;=\; (x,0,0) \quad\text{for}\quad 0\le x\le L, \tag{9}$$

while we measure the electric field at point P with coordinates $\mathbf{r}_P=(0,0,z)$, hence the vector from a charged point to P is

$$\Delta\mathbf{r} \;=\; \mathbf{r}_P \;-\; \mathbf{R} \;=\; (-x,0,z) \quad\text{of length } |\Delta\mathbf{r}| \;=\; \sqrt{x^2+z^2}\,. \tag{10}$$

The rod has uniform charge density λ, thus $dQ=\lambda dx$. Hence, integrating over the rod's length, we obtain the electric field at point P as

$$\mathbf{E}(\mathbf{r}_P) \;=\; \frac{1}{4\pi\epsilon_0}\int_0^L dx\,\lambda\,\frac{\Delta\mathbf{r}}{|\Delta\mathbf{r}|^3} \;=\; \frac{\lambda}{4\pi\epsilon_0}\int_0^L dx\,\frac{(-x,0,z)}{(x^2+z^2)^{3/2}}\,, \tag{11}$$

or in components

$$\begin{aligned} E_x &= -\frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{x\,dx}{(x^2+z^2)^{3/2}}\,,\\ E_y &= 0,\\ E_z &= +\frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{z\,dx}{(x^2+z^2)^{3/2}}\,. \end{aligned} \tag{12}$$

Note that unlike the example I did in class on Tuesday, the x component of the electric field does not vanish since the probe is closer to the left end of the rod than to the right end.

Now let's do the integrals. For the E_x component, we have

$$\frac{x\,dx}{(x^2+z^2)^{3/2}} \;=\; \frac{d(x^2)}{2(x^2+z^2)^{3/2}} \;=\; \frac{d(x^2+z^2)}{2(x^2+z^2)^{3/2}} \;=\; d\left(\frac{-1}{\sqrt{x^2+z^2}}\right), \tag{13}$$

hence

$$\int\limits_0^L \frac{x\,dx}{(x^2+z^2)^{3/2}} \;=\; -\int\limits_{1/|z|}^{1/\sqrt{L^2+z^2}} d\left(\frac{1}{\sqrt{x^2+z^2}}\right) \;=\; \frac{1}{|z|} \;-\; \frac{1}{\sqrt{z^2+L^2}} \tag{14}$$

and therefore

$$E_x(P) \;=\; -\frac{\lambda}{4\pi\epsilon_0}\times\left(\frac{1}{|z|} \;-\; \frac{1}{\sqrt{z^2+L^2}}\right). \tag{15}$$

As to the z component, we change the integration variable from x to $\alpha = \arctan(x/|z|)$ $\implies x = |z|\tan\alpha$, thus

$$\frac{z\,dx}{(x^2+z^2)^{3/2}} \;=\; z\times\frac{|z|\,d\alpha}{\cos^2\alpha}\times\frac{\cos^3\alpha}{|z|^3} \;=\; \frac{1}{z}\times\cos\alpha\,d\alpha \;=\; \frac{1}{z}\times d\sin\alpha.$$

The integration range $0\le x\le L$ translates to α running from 0 to $\alpha_{\max} = \arctan(l/|z|)$ and hence $\sin\alpha$ running from 0 to

$$\sin\alpha_{\max} \;=\; \frac{\tan\alpha_{\max}}{\sqrt{1+\tan^2\alpha_{\max}}} \;=\; \frac{L/|z|}{\sqrt{1+(L/|z|)^2}} \;=\; \frac{L}{\sqrt{L^2+z^2}}\,. \tag{16}$$

Consequently,

$$\int\limits_0^L \frac{z\,dx}{(x^2+z^2)^{3/2}} \;=\; \frac{1}{z}\int\limits_0^{L/\sqrt{L^2+z^2}} d\sin\alpha \;=\; \frac{1}{z}\times\frac{L}{\sqrt{L^2+z^2}} \tag{17}$$

and therefore

$$E_z(P) \;=\; \frac{\lambda}{4\pi\epsilon_0}\times\frac{1}{z}\times\frac{L}{\sqrt{L^2+z^2}}\,. \tag{18}$$

Finally, let's check the $z\gg L$ limit. In that limit $\sqrt{L^2+z^2}\approx|z|$, hence

$$E_z(P) \;\approx\; \frac{\lambda}{4\pi\epsilon_0}\times\frac{L}{z\,|z|} \;=\; \frac{\lambda L = Q_{\rm net}}{4\pi\epsilon_0}\times\frac{\mathrm{sign}(z)}{|z|^2}\,, \tag{19}$$

which is basically the Coulomb field of a point-like charge at distance $|z|$ from the point P.

At the same time

$$\frac{1}{|z|} - \frac{1}{\sqrt{z^2+L^2}} \approx \frac{L^2}{2|z|^3}, \tag{20}$$

hence

$$E_x \approx -\frac{\lambda}{4\pi\epsilon_0} \times \frac{L^2}{|z|^3} \ll E_z,$$

which is appropriate for a distant almost-pointlike charge located directly above or below the point P.

Problem **2.5**:

The charged ring — which we take to lie in the (x, y) plane, with its center at the coordinate origin — is symmetric with respect to rotations around the z axis. Since the point P lies right on that symmetry axis, the electric field at that point cannot have any horizontal E_x or E_y components; only the E_z component is allowed by the symmetry.

To find the E_z component, we integrate over the charged ring:

$$E_z(P) = \frac{1}{4\pi\epsilon_0} \int\limits_{\text{ring}} \frac{\Delta z}{|\Delta\mathbf{r}|^3}\,\lambda\,d\ell, \tag{21}$$

where for every point of the ring

$$\Delta z = z \quad\text{and}\quad |\Delta r| = \sqrt{z^2+R_{\text{ring}}^2}. \tag{22}$$

Note that both the vertical distance Δz from P to the ring and the overall distance $|\Delta r|$ remain constant for all points of the ring. The charge density λ of the ring is also constant all along the ring, so all the factors under the integral (21) are constants! This makes evaluating the integral extremely easy:

$$\int\limits_{\text{ring}} \frac{\Delta z}{|\Delta\mathbf{r}|^3}\,\lambda\,d\ell = \frac{\Delta z}{|\Delta\mathbf{r}|^3} \times \lambda \times L_{\text{ring}} = \frac{z}{(z^2+R^2)^{3/2}} \times \lambda \times 2\pi R, \tag{23}$$

and hence

$$E_z(P) = \frac{\lambda \times 2\pi R}{4\pi\epsilon_0} \times \frac{z}{(z^2+R^2)^{3/2}}. \tag{24}$$

Problem **2.6**:

Similar to the previous problem, the charged disk is symmetric with respect to rotations of the z axis, while the point P lies on that axis. Consequently, the electric field at that point P does not have E_x or E_y components but only the E_z component, which obtains as

$$E_z(P) \;=\; \frac{1}{4\pi\epsilon_0} \iint\limits_{\text{disk}} \frac{\Delta z}{|\Delta\mathbf{r}|^3}\, dQ\,. \tag{25}$$

To perform this integral, let's use polar coordinates for the charged disk:

$$x \;=\; r\times\cos\phi, \quad y \;=\; r\times\sin\phi, \tag{26}$$

$$0 \;\le\; r \;\le\; R, \quad 0 \;\le\; \phi \;\le\; 2\pi, \tag{27}$$

$$d^2A \;=\; dr\times r\,d\phi \quad\Longrightarrow\quad dQ \;=\; \sigma\times r\,dr\times d\phi. \tag{28}$$

The vertical distance from P to any point of the disk is the same $\Delta z \equiv z$, but the overall distance depends on the radial coordinate of the disk:

$$|\Delta\mathbf{r}| \;=\; \sqrt{z^2+r^2}\,. \tag{29}$$

Finally, the surface charge density σ of the disk is uniform.

Plugging all these data into the integral (25), we obtain

$$E_z(P) \;=\; \frac{1}{4\pi\epsilon_0}\int\limits_0^R dr \int\limits_0^{2\pi} d\phi\, \frac{z\times\sigma\times r}{(r^2+z^2)^{3/2}} \;=\; \frac{z\times\sigma}{4\pi\epsilon_0}\times\int\limits_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}\times\int\limits_0^{2\pi} d\phi\,, \tag{30}$$

where in the second expression we have moved all constant factors in front of the integrals. In particular, nothing in the integrand depends on the polar angle ϕ, so the $\int d\phi$ integral is quite trivial,

$$\int\limits_0^{2\pi} d\phi \;=\; 2\pi. \tag{31}$$

This leaves us with

$$E_z(P) \;=\; \frac{2\pi \times z \times \sigma}{4\pi\epsilon_0} \times \int\limits_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}\,, \tag{32}$$

where

$$\frac{r\,dr}{(r^2+z^2)^{3/2}} \;=\; \frac{d(r^2)}{2(r^2+z^2)^{3/2}} \;=\; \frac{d(r^2+z^2)}{2(r^2+z^2)^{3/2}} \;=\; d\left(\frac{-1}{\sqrt{r^2+z^2}}\right). \tag{33}$$

Consequently

$$\int\limits_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} \;=\; \int\limits_{r=0}^{r=R} d\left(\frac{-1}{\sqrt{r^2+z^2}}\right) \;=\; \frac{-1}{\sqrt{R^2+z^2}} \,-\, \frac{-1}{\sqrt{0+z^2}} \;=\; \frac{1}{|z|} \,-\, \frac{1}{\sqrt{R^2+z^2}}\,, \tag{34}$$

and therefore

$$E_z(P) \;=\; \frac{z\times\sigma}{4\pi\epsilon_0} \times \left(\frac{1}{|z|} \,-\, \frac{1}{\sqrt{R^2+z^2}}\right) \;=\; \frac{\sigma\times\operatorname{sign}(z)}{2\epsilon_0} \times \left(1 \,-\, \frac{|z|}{\sqrt{R^2+z^2}}\right). \tag{35}$$

Or in vector notations,

$$\mathbf{E}(P) \;=\; \frac{\sigma}{2\epsilon_0} \times \left(1 \,-\, \frac{|z|}{\sqrt{R^2+z^2}}\right) \times \operatorname{sign}(z)\,\hat{\mathbf{z}}. \tag{36}$$

Finally, consider the extreme limits $R \gg z$ and $R \ll z$. In the $R \gg z$ limit,

$$\left(1 \,-\, \frac{|z|}{\sqrt{R^2+z^2}}\right) \;\approx\; 1, \tag{37}$$

and the electric field becomes

$$\mathbf{E}(P) \;\approx\; \frac{\sigma}{2\epsilon_0}\,\operatorname{sign}(z)\,\hat{\mathbf{z}}. \tag{38}$$

This is the electric field of the infinite charged sheet — as we have seen in Thursday class from the Gauss Law. Physically, for $R \gg z$ the disk viewed from the point P looks very large, so it makes sense to approximate it as an infinite charged sheet.

On the other hand, for $|z| \gg R$, the disk viewed from the point P looks tiny, almost point like, so its electric field should be approximated by the Coulomb field of a point charge

$$Q_{\rm net} \;=\; \sigma \times \pi R^2. \tag{39}$$

And indeed, in the $R \ll |z|$ limit

$$\left(1\,-\,\frac{|z|}{\sqrt{R^2+z^2}}\right) \;\approx\; \frac{R^2}{2z^2}\,, \tag{40}$$

hence

$$\mathbf{E}(P) \;\approx\; \frac{\sigma}{2\epsilon_0}\times\frac{R^2}{2z^2}\times{\rm sign}(z)\,\hat{\mathbf{z}} \;=\; \frac{\sigma\times\pi R^2}{4\pi\epsilon_0\times z^2}\times{\rm sign}(z)\,\hat{\mathbf{z}}\,, \tag{41}$$

which is precisely the Coulomb field of a point charge measured at a point P directly above or below the charge.

Problems **2.15, 2.16, 2.18** are postponed to the next homework set.

Problem **1.6**:
Note: the relation

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \;+\; \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) \;+\; \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) \;=\; 0 \quad \text{for any vectors } \mathbf{A},\mathbf{B},\mathbf{C} \tag{42}$$

is an example of a Jacobi identity. There are similar Jacobi identities for antisymmetric "products" of entities more complicated than vectors in 3 space dimensions. For example, the Poisson brackets of classical mechanics

$$\begin{gathered}\text{for } A(q_1,\ldots,q_n,p_1,\ldots,p_n) \quad \text{and} \quad B(q_1,\ldots,q_n,p_1,\ldots,p_n),\\ [A,B]_{\rm Poisson} \;\stackrel{\rm def}{=}\; \sum_{i=1}^{n}\left(\frac{\partial A}{\partial q_i}\times\frac{\partial B}{\partial p_i}\;-\;\frac{\partial A}{\partial p_i}\times\frac{\partial B}{\partial q_i}\right)\end{gathered} \tag{43}$$

and the commutator brackets of quantum mechanics

$$\text{for linear operators } \hat{A} \text{ and } \hat{B}, \quad [\hat{A},\hat{B}]_{\rm commutator} \;\stackrel{\rm def}{=}\; \hat{A}\hat{B}\;-\;\hat{B}\hat{A}$$

obey the Jacobi identitity

$$\text{for any } \hat{A},\hat{B},\hat{C},\quad [\hat{A},[\hat{B},\hat{C}]]\ +\ [\hat{B},[\hat{C},\hat{A}]]\ +\ [\hat{C},[\hat{A},\hat{B}]]\ =\ 0. \tag{44}$$

And there are many more examples, some of them quite useful for Physics.

But let's focus on the cross products of 3D vectors and prove the Jacobi identity (42). The simplest proof follows from the $\mathbf{BAC}-\mathbf{CAB}$ rule (textbook eq. (1.17) on page 8) for the double vector product:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\ =\ \mathbf{B}(\mathbf{A}\cdot\mathbf{C})\ -\ \mathbf{C}(\mathbf{A}\cdot\mathbf{B}). \tag{45}$$

Let's apply the same rule to the cyclically permuted vectors $\mathbf{A},\mathbf{B},\mathbf{C}$:

$$\mathbf{B}\times(\mathbf{C}\times\mathbf{A})\ =\ \mathbf{C}(\mathbf{B}\cdot\mathbf{A})\ -\ \mathbf{A}(\mathbf{B}\cdot\mathbf{C}), \tag{46}$$

$$\mathbf{C}\times(\mathbf{A}\times\mathbf{B})\ =\ \mathbf{A}(\mathbf{C}\cdot\mathbf{B})\ -\ \mathbf{B}(\mathbf{C}\cdot\mathbf{A}). \tag{47}$$

Now let's add up eqs. (45), (46), and (47):

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\ +\ \mathbf{B}\times(\mathbf{C}\times\mathbf{A})\ +\ \mathbf{C}\times(\mathbf{A}\times\mathbf{B})&\\ =\ \cancel{\mathbf{B}(\mathbf{A}\cdot\mathbf{C})}-\cancel{\mathbf{C}(\mathbf{A}\cdot\mathbf{B})}&\\ +\ \cancel{\mathbf{C}(\mathbf{B}\cdot\mathbf{A})}\ -\ \cancel{\mathbf{A}(\mathbf{B}\cdot\mathbf{C})}&\\ +\ \cancel{\mathbf{A}(\mathbf{C}\cdot\mathbf{B})}\ -\ \cancel{\mathbf{B}(\mathbf{C}\cdot\mathbf{A})}&\\ =\ 0.&\end{aligned} \tag{48}$$

On the left hand side here we have the LHS of the Jacobi identity (42), while on the right hand side we have a complete cancellation all the way to zero. This completes my proof, *quod erat demonstrandum.*

Besides the proof, let's see under what conditions can the double vector product appear to be associative. In general, it is NOT associative, so the relation

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\ \stackrel{??}{=}\ (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \tag{49}$$

does not work for generic vectors $\mathbf{A},\mathbf{B},\mathbf{C}$. But it might happen to hold true for some special vectors, so let's find out what does it take to 'accidentally' satisfy the relation (49).

Thanks to the antisymmetry of the cross product,

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} \;=\; -\mathbf{C}\times(\mathbf{A}\times\mathbf{B}). \tag{50}$$

Consequently, the difference between two sides of eq. (49) can be written as

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \;-\; (\mathbf{A}\times\mathbf{B})\times\mathbf{C} \;&=\; \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \;+\; \mathbf{C}\times(\mathbf{A}\times\mathbf{B})\\ &\qquad\langle\!\langle\text{ using the Jacobi identity (42)}\rangle\!\rangle\\ &=\; -\mathbf{B}\times(\mathbf{C}\times\mathbf{A}).\end{aligned} \tag{51}$$

To make the right hand side here vanish, we need $\mathbf{B}$ to be parallel to $\mathbf{C}\times\mathbf{A}$, which means that $\mathbf{B}$ should be $\perp$ to both $\mathbf{C}$ and $\mathbf{A}$. We may also have the $\mathbf{C}$ and $\mathbf{A}$ vectors be parallel to each other (which would give us $\mathbf{C}\times\mathbf{A}=0$), or any of the three vectors may vanish. These are the only possibilities — in all other cases, the RHS of eq. (51) does not vanish and the associativity relation (49) does not work.

<u>Problem **1.11**</u>:

(a) For $f(x,y,z)=x^2+y^3+z^4$, the partial derivatives are

$$\frac{\partial f}{\partial x} \;=\; 2x,\quad \frac{\partial f}{\partial y} \;=\; 3y^2,\quad \frac{\partial f}{\partial z} \;=\; 4z^3, \tag{52}$$

hence the gradient vector

$$\nabla f(x,y,z) \;=\; 2x\,\hat{\mathbf{x}} \;+\; 3y^2\,\hat{\mathbf{y}} \;+\; 4z^3\,\hat{\mathbf{z}}. \tag{53}$$

(b) For $f(x,y,z)=x^2y^3z^4$, the partial derivatives are

$$\frac{\partial f}{\partial x} \;=\; 2xy^3z^4,\quad \frac{\partial f}{\partial y} \;=\; 3x^2y^2z^4,\quad \frac{\partial f}{\partial z} \;=\; 4x^2y^3z^3, \tag{54}$$

hence the gradient vector

$$\nabla f(x,y,z) \;=\; 2xy^3z^4\,\hat{\mathbf{x}} \;+\; 3x^2y^2z^4\,\hat{\mathbf{y}} \;+\; 4x^2y^3z^3\,\hat{\mathbf{z}}. \tag{55}$$

(c) For $f(x,y,z) = e^x \sin(y) \ln(z)$, the partial derivatives are

$$\frac{\partial f}{\partial x} = e^x \times \sin(y)\ln(z), \quad \frac{\partial f}{\partial y} = \cos(y) \times e^x \ln(z), \quad \frac{\partial f}{\partial z} = \frac{1}{z} \times e^x \sin(y), \tag{56}$$

hence the gradient vector

$$\nabla f(x,y,z) = e^x \sin(y)\ln(z)\,\hat{\mathbf{x}} + e^x \cos(y)\ln(z)\,\hat{\mathbf{y}} + e^x \sin(y)\,\frac{1}{z}\,\hat{\mathbf{z}}. \tag{57}$$

<u>Problem **1.13**</u>:

In ordinary calculus, a derivative of a function of a function — for example the derivative of $g(f)$ for $f = f(x)$ with respect to x is given by the chain rule,

$$\frac{dg(f(x))}{dx} = \frac{dg}{df} \times \frac{df}{dx}. \tag{58}$$

The same rule applies for partial derivatives of function of functions. For example, let $f(s)$ be a function of s which is itself a function of the 3 coordinates (x,y,z). In this case, the partial derivatives of f with respect to the coordinates are given by the chain rule

$$\frac{\partial f(s(x,y,z))}{\partial x} = \frac{df}{ds} \times \frac{\partial s}{\partial x}, \quad \frac{\partial f(s(x,y,z))}{\partial y} = \frac{df}{ds} \times \frac{\partial s}{\partial y}, \quad \frac{\partial f(s(x,y,z))}{\partial z} = \frac{df}{ds} \times \frac{\partial s}{\partial z}. \tag{59}$$

In terms of the gradient vector, this means

$$\nabla f(s(x,y,z)) = \frac{df}{ds}\,\nabla s. \tag{60}$$

Now let's apply this rule to functions of the radius $r = |\mathbf{r}|$, or more generally, to functions of the distance $\mathcal{R} = |\mathbf{r} - \mathbf{R}_0|$ from some fixed point $\mathbf{R}_0$. For any such function $f(\mathcal{R})$, its gradient obtains as

$$\nabla f(\mathcal{R}) = \frac{df}{d\mathcal{R}} \nabla\mathcal{R}\,. \tag{61}$$

In particular, for powers of the distance $f(\mathcal{R}) = \mathcal{R}^n$,

$$\nabla\big(\mathcal{R}^n\big) = n\mathcal{R}^{n-1}\,\nabla\mathcal{R}\,, \tag{62}$$

for example

$$\nabla\mathcal{R}^2 = 2\mathcal{R}\,\nabla\mathcal{R}, \quad \nabla\frac{1}{\mathcal{R}} = \frac{-1}{\mathcal{R}^2}\nabla\mathcal{R}, \quad \textit{etc., etc.} \tag{63}$$

To complete the evaluation of all such gradients, all we need not is the $\nabla\mathcal{R}$. But actually, it is easier to start with $\nabla\mathcal{R}^2$. Indeed,

$$\mathcal{R}^2 = |\mathbf{r} - \mathbf{R}_0|^2 = (x - X_0)^2 + (y - Y_0)^2 + (z - Z_0)^2, \tag{64}$$

so its easy to evaluate its gradient as

$$\nabla\mathcal{R}^2 = 2(x - X_0)\hat{\mathbf{x}} + 2(y - Y_0)\hat{\mathbf{y}} + 2(z - Z_0)\hat{\mathbf{z}}, \tag{65}$$

which has a nice vector form:

$$\nabla\mathcal{R}^2 = 2(\mathbf{r} - \mathbf{R}_0). \tag{66}$$

At the same time, $\nabla\mathcal{R}^2 = 2\mathcal{R}\nabla\mathcal{R}$, hence the gradient of the distance $\mathcal{R}$ itself is

$$\nabla\mathcal{R} = \frac{1}{2\mathcal{R}}\nabla\mathcal{R}^2 = \frac{2(\mathbf{r} - \mathbf{R}_0)}{2|\mathbf{r} - \mathbf{R}_0|} = \widehat{\mathcal{R}}, \tag{67}$$

the unit vector in the radial direction from $\mathbf{R}_0$ to $\mathbf{r}$.

Consequently, the gradient of any function of $\mathcal{R}$ is the ordinary derivative of that function times a unit vector in the radial direction from the fixed point $\mathbf{R}_0$ to the point $\mathbf{r}$ where we take the gradient,

$$\nabla f(\mathcal{R}) \;=\; \frac{df}{d\mathcal{R}}\,\widehat{\mathcal{R}}\,. \tag{68}$$

In particular,

$$\nabla \mathcal{R}^2 \;=\; 2\mathcal{R}\widehat{\mathcal{R}} \;=\; 2\vec{\mathcal{R}}, \quad \nabla\frac{1}{\mathcal{R}} \;=\; \frac{-1}{\mathcal{R}^2}\,\widehat{\mathcal{R}}, \tag{69}$$

or for any other power $\mathcal{R}^n$,

$$\nabla \mathcal{R}^n \;=\; n\mathcal{R}^{n-1}\widehat{\mathcal{R}}. \tag{70}$$