Problem **2.15**:

By spherical symmetry, the electric field has the radial direction and its magnitude depends only on the radial coordinate,

$$\mathbf{E}(\mathbf{r}) \;=\; E(r \text{ only})\,\hat{\mathbf{r}}. \tag{1}$$

Moreover, the magnitude $E(r)$ follows from the Gauss Law:

$$E(r) \;=\; \frac{Q[\text{inside radius } r]}{4\pi\epsilon_0\, r^2}\,. \tag{2}$$

Now let's focus on the thick spherical shell in question (*cf.* figure 2.25). In the region (i) — inside the cavity, $r < a$, — there are no charges inside the Gaussian sphere of radius r, hence $Q[r] = 0$, and the electric field vanishes, $E = 0$.

In the region (ii) — in the thickness of the shell, $a < r < b$, — the Gaussian sphere comprises the part of the shell between radia a and r. The net charge inside such a Gaussian sphere is

$$Q[r] \;=\; \int_a^r \rho(r') \times dV(r') \tag{3}$$

where $dV(r') = 4\pi r'^2 \times dr'$ while $\rho(r') = k/r'^2$ for a constant k. Consequently

$$Q[r] \;=\; \int_a^r \frac{k}{r'^2} \times 4\pi r'^2\, dr' \;=\; 4\pi k \int_a^r dr' \;=\; 4\pi k(r-a), \tag{4}$$

hence by the Gauss Law,

$$E(r) \;=\; \frac{4\pi k(r-a)}{4\pi\epsilon_0 r^2}\,. \tag{5}$$

Finally, in the region (iii) — outside the shell, $r > b$, — the net charge inside the

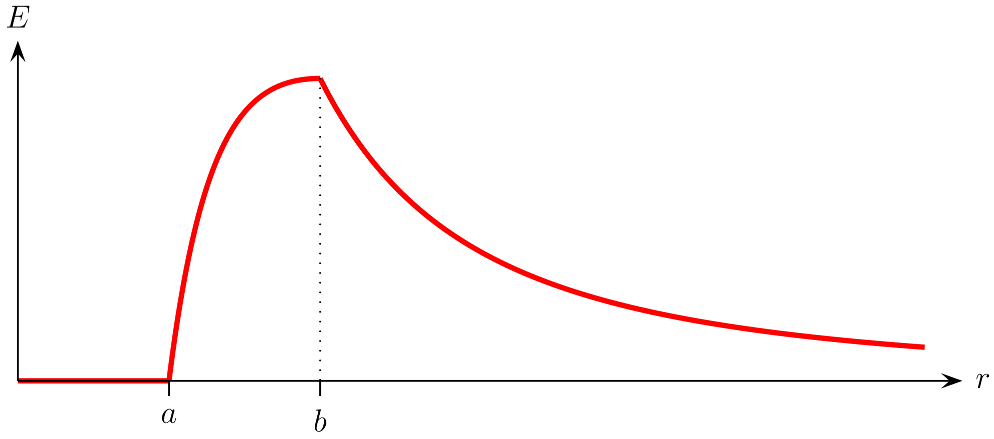
Gaussian sphere of radius r is the total charge of the shell,

$$Q[r] \;=\; Q_{\rm tot} \;=\; \int\limits_a^b \rho(r') \times dV(r') \;=\; 4\pi k \times (b-a). \tag{6}$$

Consequently, by the Gauss Law,

$$E(r) \;=\; \frac{4\pi k(b-a)}{4\pi\epsilon_0\, r^2}\,. \tag{7}$$

Finally, here is the plot of the electric field as a function of the radius for $b = 2a$:



<u>Problem **2.16**</u>:

By the axial symmetry of the system, the electric field is $\perp$ to the axis and points in the radial direction directly away from or towards the axis. Also, its magnitude depends only on the radial distance from that axis. In the cylindrical coordinates (s, ϕ, z),

$$\mathbf{E}(s,\phi,z) \;=\; E(s\text{ only})\,\hat{\mathbf{s}}. \tag{8}$$

Also, the magnitude $E(s)$ follows from the Gauss Law where the Gaussian surface is a coaxial cylinder of radius s and length L:

$$E(s) \;=\; \frac{Q[\text{inside } s \text{ and } L]}{2\pi s L\epsilon_0}\,. \tag{9}$$

Moreover, since all the charges are uniform WRT to the z axis, the charge inside the Gaussian cylinder is always proportional to its length, so if we define the *linear charge density* $\lambda(s)$

inside radius s as

$$\lambda(s) \;=\; \frac{1}{L}\times Q[\text{inside } s\times L \text{ cylinder}], \tag{10}$$

then

$$E(s) \;=\; \frac{\lambda(s)}{2\pi\epsilon_0\, s}\,. \tag{11}$$

Now consider the cable in question. Inside the inner cylinder (region (i), $s < a$) we have uniform volume charge density $\rho = \text{const}$, hence the net linear density inside a Gaussian cylinder of radius $s < a$ is

$$\lambda(s) \;=\; \int\limits_{s'<s} \rho\, d\text{Area} \;=\; \int\limits_0^s \rho\times 2\pi s'\, ds' \;=\; \rho\times\pi s^2. \tag{12}$$

Consequently, by the Gauss Law (11), the electric field in this region is

$$E(s) \;=\; \frac{\lambda(s)=\pi s^2\rho}{2\pi\epsilon_0\, s} \;=\; \frac{\rho}{2\epsilon_0}\times s. \tag{13}$$

Note: the field increases linear with the radius insider the inner cable.

Inside the neutral volume of the outer shell (region (ii), $a < s < b$), the net charge inside a Gaussian cylinder of radius s is simply the net charge of the inner cable. In terms of the linear charge density,

$$\lambda(s) \;=\; \rho\times\pi a^2, \tag{14}$$

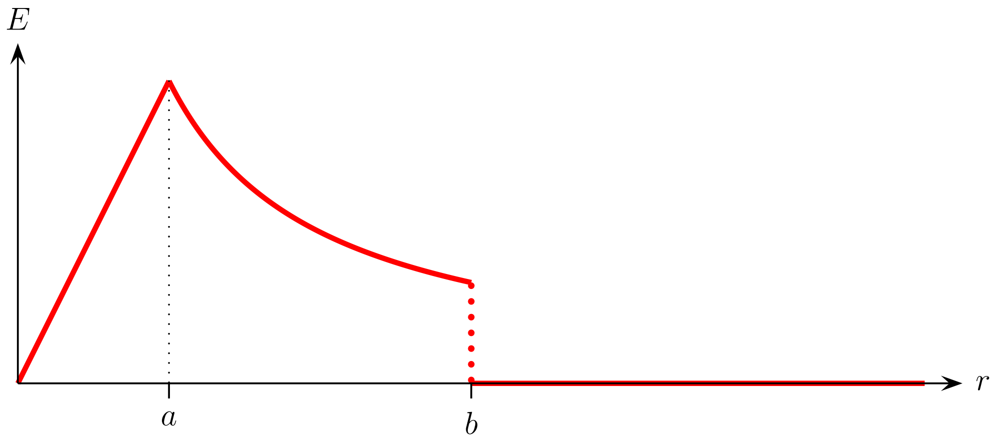
hence the electric field

$$E(s) \;=\; \frac{\lambda(s)=\pi a^2\rho}{2\pi\epsilon_0\, s} \;=\; \frac{\rho}{2\epsilon_0}\times\frac{a^2}{s}\,. \tag{15}$$

In this region, the electric field decreases as $1/s$.

Finally, outside the outer shell (region (iii), $s > b$), the Gaussian surface encloses both the inner cable and the outer shell, and their electric charges cancel each other — the cable as a whole is electrically neutral. Consequently, outside the outer shell $\lambda(s) = 0$ and the electric field vanishes, $E(s) = 0$.

Here is the plot of the electric field as a function of s for $b = 3a$:



Problem **2.18**:

Note: to be precise, the electric charge density (by volume) is

$$\rho = \begin{cases} +\rho & \text{inside the first sphere but outside the second sphere,} \\ -\rho & \text{inside the second sphere but outside the first sphere,} \\ 0 & \text{in the overlapping region.} \end{cases} \tag{16}$$

This way, the entire system is a *superposition* of two uniformly charged balls — one of density $+\rho$, the other of density $-\rho$ — so in the overlapping region the net charge density is $+\rho - \rho = 0$.

Consequently, the electric field of this charged system is the *superposition* $\mathbf{E}_{\text{net}} = \mathbf{E}_1 + \mathbf{E}_2$ where $\mathbf{E}_1$ is the field due to the first charged ball only and the $\mathbf{E}_2$ is the field due to the second ball only. In particular, inside the overlap of the two balls,

$$\mathbf{E}_{\text{net}}(\mathbf{r}) \;=\; \mathbf{E}_1(\mathbf{r}\text{ inside ball\#1}) \;+\; \mathbf{E}_2(\mathbf{r}\text{ inside ball\#2}). \tag{17}$$

The electric field inside a ball of uniform charge density follows from the Gauss Law. As explained in my notes on Gauss Law applications,

$$\begin{aligned} E(r) &= \frac{Q[\text{inside the Gaussian sphere of radius } r]}{4\pi\epsilon_0\, r^2} \\ &= \frac{\frac{4\pi}{3}r^3\times\rho}{4\pi\epsilon_0\, r^2} \;=\; \frac{\rho}{3\epsilon_0}\times r \end{aligned} \tag{18}$$

Moreover, the direction of this electric field is radial, so in vector notations

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{\rho}{3\epsilon_0}\times\mathbf{r} \qquad \text{(inside the ball).} \tag{19}$$

To be precise, this formula applies to the ball centered at the origin of the coordinate system. For a ball centered at some point $\mathbf{R}_{\text{center}}$, the field inside the ball is

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{\rho}{3\epsilon_0}\times(\mathbf{r}-\mathbf{R}_{\text{center}}). \tag{20}$$

Now let's plug this formula into eq. (17) for the superposition of electric fields in the overlap of the two balls. Let the first ball's center be at $\mathbf{R}_1$ while the second ball's center is at $\mathbf{R}_2$. Than in the overlapping region,

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}) &= \frac{+\rho}{3\epsilon_0}\times(\mathbf{r}-\mathbf{R}_1), \\ \mathbf{E}_2(\mathbf{r}) &= \frac{-\rho}{3\epsilon_0}\times(\mathbf{r}-\mathbf{R}_2), \\ \mathbf{E}_{\text{net}}(\mathbf{r}) &= \mathbf{E}_1(\mathbf{r}) \;+\; \mathbf{E}_2(\mathbf{r}) \\ &= \frac{\rho}{3\epsilon_0}\times\Big((\mathbf{r}-\mathbf{R}_1) \;-\; (\mathbf{r}-\mathbf{R}_2)\Big) \\ &= \frac{\rho}{3\epsilon_0}\times(\mathbf{R}_2-\mathbf{R}_1). \end{aligned} \tag{21}$$

Note that $\mathbf{r}$ — the point where we measure the electric field — cancels out from the formula on the last line here! This means that *the electric field in the overlapping region is uniform and parallel, i.e.*, it's constant as a vector! The direction of this field if from the center of the positive ball to the center of the negative ball, and its magnitude is

$$E \;=\; \frac{\rho}{3\epsilon_0}\times d. \tag{22}$$

Problem **1.15**:

$$\begin{aligned} \mathbf{V} &= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}, && \text{(a)} \\ \nabla\cdot\mathbf{V} &= \frac{\partial}{\partial x}(V_x = x^2) + \frac{\partial}{\partial y}(V_y = 3xz^2) + \frac{\partial}{\partial z}(V_z = -2xz) \\ &= 2x + 0 + (-2x) = 0. && (23) \end{aligned}$$

$$\begin{aligned} \mathbf{V} &= xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}, && \text{(b)} \\ \nabla\cdot\mathbf{V} &= \frac{\partial}{\partial x}(V_x = xy) + \frac{\partial}{\partial y}(V_y = 2yz) + \frac{\partial}{\partial z}(V_z = 3zx) \\ &= y + 2z + 3x. && (24) \end{aligned}$$

$$\begin{aligned} \mathbf{V} &= y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}, && \text{(c)} \\ \nabla\cdot\mathbf{V} &= \frac{\partial}{\partial x}(V_x = y^2) + \frac{\partial}{\partial y}(V_y = 2xy + z^2) + \frac{\partial}{\partial z}(V_z = 2yz) \\ &= 0 + 2x + 2y. && (25) \end{aligned}$$

Problem **1.18**:

In general

$$\nabla\times\mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{\mathbf{z}}. \tag{26}$$

$$\begin{aligned} \mathbf{V} &= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}, && \text{(a)} \\ \nabla\times\mathbf{V} &= \left(\frac{\partial(-2xz)}{\partial y} - \frac{\partial(3xz^2)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(x^2)}{\partial z} - \frac{\partial(-2xz)}{\partial x}\right)\hat{\mathbf{y}} \\ &\quad + \left(\frac{\partial(3xz^2)}{\partial x} - \frac{\partial(x^2)}{\partial y}\right)\hat{\mathbf{z}} \\ &= \big(0 - 6xz\big)\hat{\mathbf{x}} + \big(0 - (-2z)\big)\hat{\mathbf{y}} + \big(3z^2 - 0\big)\hat{\mathbf{z}} \\ &= -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}. && (27) \end{aligned}$$

$$
\begin{aligned}
\mathbf{V} &= xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}, & \text{(b)}\\
\nabla\times\mathbf{V} &= \left(\frac{\partial(3zx)}{\partial y} - \frac{\partial(2yz)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(xy)}{\partial z} - \frac{\partial(3zx)}{\partial x}\right)\hat{\mathbf{y}}\\
&\quad + \left(\frac{\partial(2yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)\hat{\mathbf{z}}\\
&= \big(0-2y\big)\hat{\mathbf{x}} + \big(0-3z\big)\hat{\mathbf{y}} + \big(0-x\big)\hat{\mathbf{z}}\\
&= -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}. & (28)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V} &= y^2\hat{\mathbf{x}} + (2xy+z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}, & \text{(c)}\\
\nabla\times\mathbf{V} &= \left(\frac{\partial(2yz)}{\partial y} - \frac{\partial(2xy+z^2)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(y^2)}{\partial z} - \frac{\partial(2yz)}{\partial x}\right)\hat{\mathbf{y}}\\
&\quad + \left(\frac{\partial(2xy+z^2)}{\partial x} - \frac{\partial(y^2)}{\partial y}\right)\hat{\mathbf{z}}\\
&= \big(2z-2z\big)\hat{\mathbf{x}} + \big(0-0\big)\hat{\mathbf{y}} + \big(2y-2y\big)\hat{\mathbf{z}}\\
&= 0. & (29)
\end{aligned}
$$

<u>Problem **1.20**</u>:

For example,

$$
\begin{aligned}
\mathbf{V} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}, & (30)\\
\nabla\cdot\mathbf{V} &= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(-2z)}{\partial z}\\
&= 1 + 1 - 2 = 0. & (31)\\
\nabla\times\mathbf{V} &= \left(\frac{\partial(-2z)}{\partial y} - \frac{\partial(y)}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial(x)}{\partial z} - \frac{\partial(-2z)}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial(y)}{\partial x} - \frac{\partial(x)}{\partial y}\right)\hat{\mathbf{z}}\\
&= (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (0-0)\hat{\mathbf{z}} = \mathbf{0}. & (32)
\end{aligned}
$$

<u>Problem **1.26**</u>:

The Laplacian is defined as

$$
\triangle = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,, \tag{33}
$$

and it may act on both scalar and vector fields. Let's apply this operator to the 4 fields in question:

$$
\begin{aligned}
T_a &= x^2 + 2xy + 3z + 4, && \text{(a)}\\
\frac{\partial^2}{\partial x^2} T_a &= 2 + 0 + 0 + 0 = 2,\\
\frac{\partial^2}{\partial y^2} T_a &= 0 + 0 + 0 + 0 = 0,\\
\frac{\partial^2}{\partial z^2} T_a &= 0 + 0 + 0 + 0 = 0,\\
\triangle T_a &= 2 + 0 + 0 = 2. && (34)
\end{aligned}
$$

$$
\begin{aligned}
T_b &= \sin x \times \sin y \times \sin z, && \text{(b)}\\
\frac{\partial^2}{\partial x^2} T_b &= (-\sin x) \times \sin y \times \sin z,\\
\frac{\partial^2}{\partial y^2} T_b &= \sin x \times (-\sin y) \times \sin z,\\
\frac{\partial^2}{\partial z^2} T_b &= \sin x \times \sin y \times (-\sin z),\\
\triangle T_b &= -3 \times \sin x \times \sin y \times \sin z. && (35)
\end{aligned}
$$

$$
\begin{aligned}
T_c &= \exp(-5x) \times \sin(4y) \times \cos(3z), && \text{(c)}\\
\frac{\partial^2}{\partial x^2} T_c &= \big(+25\exp(-5x)\big) \times \sin(4y) \times \cos(3z),\\
\frac{\partial^2}{\partial y^2} T_c &= \exp(-5x) \times \big(-16\sin(4x)\big) \times \cos(3z),\\
\frac{\partial^2}{\partial z^2} T_c &= \exp(-5x) \times \sin(4y) \times \big(-9\cos(3z)\big),\\
\triangle T_z &= \exp(-5x) \times \sin(4y) \times \cos(3z) \times (+25 - 16 - 9 = 0) = 0. && (36)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V} &= x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}, && \text{(d)}\\
\frac{\partial^2}{\partial x^2}\mathbf{V} &= 2\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} = 2\hat{\mathbf{x}},\\
\frac{\partial^2}{\partial y^2}\mathbf{V} &= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{x}} = \mathbf{0},\\
\frac{\partial^2}{\partial z^2}\mathbf{V} &= 0\hat{\mathbf{x}} + 6x\hat{\mathbf{y}} + 0\hat{\mathbf{z}} = 6x\hat{\mathbf{y}},\\
\triangle\mathbf{V} &= 2\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 6x\hat{\mathbf{y}}, && (37)
\end{aligned}
$$

or in components, $V_x = 2$, $V_y = 6x$, $V_z = 0$.

Problem **1.61**:

(a) Given a scalar field $T(x,y,z)$, let's make the vector field $\mathbf{V}(x,y,z) = T(x,y,z)\mathbf{c}$ for some constant vector $\mathbf{c}$. The divergence of this vector field follows from the Leibniz rule,

$$\nabla\cdot(T\mathbf{c}) \;=\; (\nabla T)\cdot\mathbf{c} \;+\; T(\nabla\cdot\mathbf{c}) \;=\; \mathbf{c}\cdot\nabla T \;+\; 0. \tag{38}$$

Hence, the volume integral of this divergence over some compact volume $\mathcal{V}$ is

$$\iiint\limits_{\mathcal{V}} \Big(\nabla\cdot(T\mathbf{c})\Big)\,d^3\text{Vol} \;=\; \iiint\limits_{\mathcal{V}} (\mathbf{c}\cdot\nabla T)\,d^3\text{Vol} \;=\; \mathbf{c}\cdot\iiint\limits_{\mathcal{V}} (\nabla T)\,d^3\text{Vol}. \tag{39}$$

By the Gauss Theorem, this integral is equal to the flux of $T\mathbf{c}$ through the complete surface $\mathcal{S}$ of the volume $\mathcal{V}$, thus

$$\iiint\limits_{\mathcal{V}} \Big(\nabla\cdot(T\mathbf{c})\Big)\,d^3\text{Vol} \;=\; \iint\limits_{\mathcal{S}} (T\mathbf{c})\cdot d^2\mathbf{A} \;=\; \mathbf{c}\cdot\iint\limits_{\mathcal{S}} T\,d^2\mathbf{A} \tag{40}$$

where the second equality follows from $\mathbf{c}$ being a *constant* vector. Altogether, we have

$$\mathbf{c}\cdot\iiint\limits_{\mathcal{V}} (\nabla T)\,d^3\text{Vol} \;=\; \mathbf{c}\cdot\iint\limits_{\mathcal{S}} T\,d^2\mathbf{A}. \tag{41}$$

Note that this equation must hold true for any constant vector $\mathbf{c}$, which can only happen if

$$\iiint\limits_{\mathcal{V}} (\nabla T)\,d^3\text{Vol} \;=\; \iint\limits_{\mathcal{S}} T\,d^2\mathbf{A}. \tag{42}$$

Quod erat demonstrandum.

(b) Now let's start with a vector field $\mathbf{V}(x,y,z)$ and cross it with a constant vector $\mathbf{c}$, thus $\mathbf{V}'(x,y,z) = \mathbf{V}(x,y,z)\times\mathbf{c}$. The divergence of this cross product follows from the Leibniz rule

$$\nabla\cdot(\mathbf{V}\times\mathbf{c}) \;=\; \mathbf{c}\cdot(\nabla\times\mathbf{V}) \;-\; \mathbf{V}\cdot(\nabla\times\mathbf{c}) \;=\; \mathbf{c}\cdot(\nabla\times\mathbf{V}) \;+\; \mathbf{0} \tag{43}$$

where the second equality follows the vector $\mathbf{c}$ being constant (and thus having zero curl). Consequently, taking the volume integral of this divergence over some compact volume $\mathcal{V}$, we get

$$\iiint_{\mathcal{V}} \Big(\nabla\cdot(\mathbf{V}\times\mathbf{c})\Big)\, d^3\mathrm{Vol} \;=\; \iiint_{\mathcal{V}} \Big(\mathbf{c}\cdot(\nabla\times\mathbf{V})\Big)\, d^3\mathrm{Vol} \;=\; \mathbf{c}\cdot \iiint_{\mathcal{V}} (\nabla\times\mathbf{V})\, d^3\mathrm{Vol}. \tag{44}$$

On the other hand, by the Gauss Theorem

$$\iiint_{\mathcal{V}} \Big(\nabla\cdot(\mathbf{V}\times\mathbf{c})\Big)\, d^3\mathrm{Vol} \;=\; \iint_{\mathcal{S}} (\mathbf{V}\times\mathbf{c})\cdot d^2\mathbf{A} \tag{45}$$

where $\mathcal{S}$ is the complete surface of the volume $\mathcal{V}$. Moreover,

$$(\mathbf{V}\times\mathbf{c})\cdot d^2\mathbf{A} \;=\; (d^2\mathbf{A}\times\mathbf{V})\cdot\mathbf{c} \;=\; -\mathbf{c}\cdot(\mathbf{V}\times d^2\mathbf{A}), \tag{46}$$

and since the vector $\mathbf{c}$ is constant,

$$\iint_{\mathcal{S}} (\mathbf{V}\times\mathbf{c})\cdot d^2\mathbf{A} \;=\; -\mathbf{c}\cdot \iint_{\mathcal{S}} \mathbf{V}\times d^2\mathbf{A}. \tag{47}$$

Altogether, we have

$$\mathbf{c}\cdot \iiint_{\mathcal{V}} (\nabla\times\mathbf{V})\, d^3\mathrm{Vol} \;=\; -\mathbf{c}\cdot \iint_{\mathcal{S}} \mathbf{V}\times d^2\mathbf{A}. \tag{48}$$

And since this equation must hold true for any vector $\mathbf{V}$, this means

$$\iiint_{\mathcal{V}} (\nabla\times\mathbf{V})\, d^3\mathrm{Vol} \;=\; -\iint_{\mathcal{S}} \mathbf{V}\times d^2\mathbf{A}. \tag{49}$$

quod erat demonstrandum.

(c) This time, we start with two scalar fields $T(x,y,z)$ and $U(x,y,z)$ and form a vector field as

$$\mathbf{V}(x,y,z) \;=\; T(\nabla U). \tag{50}$$

The divergence of this vector field is

$$\nabla\cdot\mathbf{V} \;=\; \nabla\cdot\big(T\nabla U\big) \;=\; (\nabla T)\cdot(\nabla U) \;+\; T(\nabla^2 U) \tag{51}$$

where $\nabla^2 = \triangle$ is the Laplacian operator (33).

Applying the Gauss Theorem to this divergence, we find that for any compact volume $\mathcal{V}$ and its complete surface $\mathcal{S}$,

$$\iiint\limits_{\mathcal{V}} \Big(T(\triangle U) \;+\; (\nabla T)\cdot(\nabla U)\Big) \;=\; \iint\limits_{\mathcal{S}} T(\nabla U)\cdot d^2\mathbf{A}. \tag{52}$$

Quod erat demonstrandum.

(d) Let's proceed similar to the part (c), but anti-symmetrize $\mathbf{V}$ with respect to exchanging $T \leftrightarrow U$. That is, let's take

$$\mathbf{V}(x,y,z) \;=\; T(\nabla U) \;-\; U\nabla(T). \tag{53}$$

Then by the Leibniz rule for the divergences,

$$\begin{aligned}
\nabla\cdot\big(T\nabla(U)\big) \;&=\; (\nabla T)\cdot(\nabla U) \;+\; T(\nabla^2 U),\\
\nabla\cdot\big(U\nabla(T)\big) \;&=\; (\nabla U)\cdot(\nabla T) \;+\; U(\nabla^2 T),\\
\nabla\cdot\mathbf{V} \;&=\; \xcancel{(\nabla T)\cdot(\nabla U)} \;+\; T(\triangle U) \;-\; \xcancel{(\nabla U)\cdot(\nabla T)} \;+\; U(\triangle T).
\end{aligned} \tag{54}$$

Consequently, applying the Gauss Theorem to the $\mathbf{V}$ field and its divergence, we have for any compact volume $\mathcal{V}$ and its complete surface $\mathcal{S}$,

$$\iiint\limits_{\mathcal{V}} \Big(T(\triangle U) \;-\; U(\triangle T)\Big) d^3\mathrm{Vol} \;=\; \iint\limits_{\mathcal{S}} \Big(T(\nabla U) \;-\; U(\nabla T)\Big)\cdot d^2\mathbf{A}. \tag{55}$$

Quod erat demonstrandum.

(e) This time we go back to a single scalar field $T(x,y,z)$, which we turn to a vector field $\mathbf{V}(x,y,z)$ by multiplying T by a constant vector $\mathbf{c}$, $\mathbf{V} = T\mathbf{c}$. The curl of such product is

$$\nabla\times(\mathbf{V}=T\mathbf{c}) \;=\; (\nabla T)\times\mathbf{c} \,+\, T(\nabla\times\mathbf{c}) \;=\; -\mathbf{c}\times(\nabla T) \,+\, \mathbf{0} \tag{56}$$

where the second equality follows from the constant vector $\mathbf{c}$ having zero curl. Consequently, when we integrate the curl (56) over the area of some surface $\mathcal{S}$, the integrand becomes

$$\Big(\nabla\times(T\mathbf{c})\Big)\cdot d^2\mathbf{A} \;=\; -\Big(\mathbf{c}\times(\nabla T)\Big)\cdot d^2\mathbf{A} \;=\; -\Big((\nabla T)\times d^2\mathbf{A}\Big)\cdot\mathbf{c}, \tag{57}$$

and since $\mathbf{c}$ is a constant vector,

$$\iint\limits_{\mathcal{S}}\Big(\nabla\times(T\mathbf{c})\Big)\cdot d^2\mathbf{A} \;=\; -\mathbf{c}\cdot\iint\limits_{\mathcal{S}}(\nabla T)\times d^2\mathbf{A}. \tag{58}$$

Now let $\mathcal{S}$ be a surface spanning some loop $\mathcal{C}$; that is, $\mathcal{S}$ has a single boundary and $\mathcal{C}$ is that boundary. Then by the Stokes' theorem

$$\iint\limits_{\mathcal{S}}\Big(\nabla\times(T\mathbf{c})\Big)\cdot d^2\mathbf{A} \;=\; \oint\limits_{\mathcal{C}}(T\mathbf{c})\cdot d\vec{\ell} \;=\; \mathbf{c}\cdot\oint\limits_{\mathcal{C}}T\,d\vec{\ell}. \tag{59}$$

Altogether, we have

$$-\mathbf{c}\cdot\iint\limits_{\mathcal{S}}(\nabla T)\times d^2\mathbf{A} \;=\; \mathbf{c}\cdot\oint\limits_{\mathcal{C}}T\,d\vec{\ell}, \tag{60}$$

and since this equation must hold for any vector $\mathbf{c}$, it follows that

$$\iint\limits_{\mathcal{S}}(\nabla T)\times d^2\mathbf{A} \;=\; -\oint\limits_{\mathcal{C}}T\,d\vec{\ell}. \tag{61}$$

Quod erat demonstrandum.

Problem **1.62**:

Let me first solve the five parts of the problem in the order they are given. And then I'll give an alternative solution to parts (a–d) in reverse order.

(a) Let's identify the hemispherical bowl in question with the Northern hemisphere of some sphere; in spherical coordinates (r, θ, ϕ) the radius is fixed $r \equiv R$, the latitude θ runs from 0 (the North pole) to $\pi/2$ (the equator), while the longitude ϕ runs from 0 to 2π.

By the axial symmetry of the hemisphere, the area vector $\mathbf{A}$ has to point along the symmetry axis, *i.e.*, in the z direction. So the only component of the area vector we need to calculate is the z component A_z.

For any infinitesimal part of the sphere — and thus of the hemisphere in question — the $d^2\mathbf{A}$ vector is $\perp$ to the surface and therefore points radially out from the center of the sphere, $d^2\mathbf{A} = d^2A\hat{\mathbf{r}}$. At latitude θ, this vector makes angle θ with the z axis, hence

$$(d^2\mathbf{A})_z \;=\; d^2A \times \cos\theta. \tag{62}$$

In spherical coordinates $d^2A = R^2 \sin\theta \, d\theta \, d\phi$, hence

$$(d^2\mathbf{A})_z \;=\; \cos\theta \times R^2 \sin\theta \times d\theta \, d\phi. \tag{63}$$

Therefore, the net area vector of the hemisphere (or rather its z component) is the integral

$$\begin{aligned}
A_z &= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \, R^2 \sin\theta \cos\theta \\
&= 2\pi R^2 \times \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \\
&= 2\pi R^2 \times \int_0^{\pi/2} d\big(\tfrac{1}{2}\sin^2\theta\big) \\
&= 2\pi R^2 \times \Big(\tfrac{1}{2}\sin^2\theta\Big)\Big|_{\theta=0}^{\theta=\pi/2} \\
&= 2\pi R^2 \times \Big(\tfrac{1}{2} - 0\Big) \\
&= \pi R^2,
\end{aligned} \tag{64}$$

same as the area of the disk in the equatorial plane.

In vector notations

$$\mathbf{A} \;=\; \pi R^2\,\hat{\mathbf{z}}. \tag{65}$$

(b) Suppose $\mathcal{S}$ is a closed surface: it has no boundary, and $\mathcal{S}$ itself is the complete boundary of some volume $\mathcal{V}$. Then as we saw in problem **1.161(a)**,

$$\iiint\limits_{\mathcal{V}} (\nabla T)\, d^3\mathrm{Vol} \;=\; \iint\limits_{\mathcal{S}} T\, d^2\mathbf{A} \tag{66}$$

for any scalar field $T(x,y,z)$. For our purposes, we take $T \equiv 1$, so the integral on the RHS of (66) becomes simply the vector area of $\mathcal{S}$. At the same time, on the LHS $\nabla T \equiv 0$, so the integral vanishes. Thus, the vector area of a closed surface $\mathcal{S}$ is zero.

(c) Let two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same boundary $\mathcal{C}$. Note: $\mathcal{C}$ must be the *complete* boundary of each surface, they cannot have other boundaries besides $\mathcal{C}$.

Let us glue the two surfaces together along their common boundary. Let us also reverse the orientation of one of the surfaces — say, the $\mathcal{S}_2$ — which means reversing the direction of the infinitesimal area vectors $d^2\mathbf{A}$. This way, for both surfaces $+\mathcal{S}_1$ and $-\mathcal{S}_2$, the $d^2\mathbf{A}$ vector points from the inside of the volume trapped between the $\mathcal{S}_1$ and $\mathcal{S}_2$ to the outside of that volume. Consequently, the combined surface $\mathcal{S} = +\mathcal{S}_1 - \mathcal{S}_2$ acts as a complete surface of some volume, so it's closed and has no boundaries. Therefore, as we have seen in part (b), the combined surface has zero vector area.

On the other hand,

$$\mathbf{A}(\mathcal{S}) \;=\; \iint\limits_{S} d^2\mathbf{A} \;=\; \iint\limits_{\mathcal{S}_1} d^2\mathbf{A} - \iint\limits_{\mathcal{S}_2} d^2\mathbf{A} \;=\; \mathbf{A}(\mathcal{S}_1) - \mathbf{A}(\mathcal{S}_2)\,, \tag{67}$$

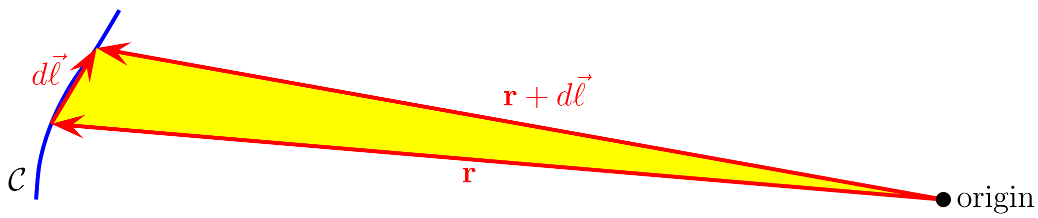
where the minus sign comes from the orientation reversal. So the fact that the combined surface $\mathcal{S}$ has zero vector area means that the area vectors of the $\mathcal{S}_1$ and the $\mathcal{S}_2$ surfaces must be equal,

$$\mathbf{A}(\mathcal{S}_1) \;=\; \mathbf{A}(\mathcal{S}_2)\,, \tag{68}$$

quod erat demonstrandum.

(d) First, the solution suggested in the textbook. By part (c), the area vectors of all surfaces spanning the same loop $\mathcal{C}$ are equal, so instead of calculating the area vector of the original surface $\mathcal{S}$, let's calculate the area vector of the cone with base at the loop $\mathcal{C}$ and the vertex at the coordinate origin $(x = 0, y = 0, z = 0)$. That is, the cone *spans* the finite straight lines from the origin to all points of the loop $\mathcal{C}$.

If we partition the loop $\mathcal{C}$ into infinitesimal intervals $d\vec{\ell}$, then the cone is partitioned into infinitesimal triangles with sides $\mathbf{r}$, $d\vec{\ell}$, and $\mathbf{r} + d\vec{\ell}$ as shown on the figure below:



The vector area of this triangle is

$$d^2\mathbf{A} = \tfrac{1}{2}\mathbf{r} \times d\vec{\ell}, \tag{69}$$

so the vector are of the whole cone is the integral

$$\mathbf{A}(\text{cone}) = \frac{1}{2}\oint_{\mathcal{C}} \mathbf{r} \times d\vec{\ell}. \tag{70}$$

And since the original surface $\mathcal{S}$ has the same boundary $\mathcal{C}$ as this cone, it also has the same vector area.

(e) Let's apply the result of the problem **1.61(e)** to the scalar field $T(x, y, z) = \mathbf{c} \cdot \mathbf{r}$ where $\mathbf{c}$ is a constant vector. The gradient of this field is simply $\mathbf{c}$:

$$\begin{aligned} T(x,y,z) &= c_x \times x + c_y \times y + c_z \times z, \\ \frac{\partial T}{\partial x} &= c_x\,, \quad \frac{\partial T}{\partial y} = c_y\,, \quad \frac{\partial T}{\partial z} = c_z\,, \\ \nabla T &= c_x\hat{\mathbf{x}} + c_y\hat{\mathbf{y}} + c_z\hat{\mathbf{z}} = \mathbf{c}. \end{aligned} \tag{71}$$

Consequently,

$$\iint\limits_{\mathcal{S}} (\nabla T) \times d^2\mathbf{A} \;=\; \iint\limits_{\mathcal{S}} \mathbf{c} \times d^2\mathbf{A} \;=\; \mathbf{c} \times \iint\limits_{\mathcal{S}} d^2\mathbf{A} \;=\; \mathbf{c} \times \mathbf{A}(\mathcal{S}). \tag{72}$$

At the same time, according to problem **1.61(e)**,

$$\iint\limits_{\mathcal{S}} (\nabla T) \times d^2\mathbf{A} \;=\; -\oint\limits_{\mathcal{C}} T\, d\vec{\ell} \;=\; -\oint\limits_{\mathcal{C}} (\mathbf{c} \cdot \mathbf{r})\, d\vec{\ell}.$$

Therefore,

$$\oint\limits_{\mathcal{C}} (\mathbf{c} \cdot \mathbf{r})\, d\vec{\ell} \;=\; -\mathbf{c} \times \mathbf{A}(\mathcal{S}) \;=\; \mathbf{A}(\mathcal{S}) \times \mathbf{c}, \tag{73}$$

quod erat demonstrandum.

Alternative solution to parts (a–d).

This time, I start by proving part (d) without using parts (a–c); and then parts (c), (b), and (a) will follow trivially from part (d).

My key to part (d) is the vector field $\mathbf{V}(x, y, z) = \mathbf{c} \times \mathbf{r}$ (where $\mathbf{c}$ is a constant vector) whose curl is $\nabla \times \mathbf{V} = 2\mathbf{c}$. Indeed, by inspection in components:

$$\begin{aligned}
\big(\nabla \times \mathbf{V}\big)_x \;&=\; \frac{\partial}{\partial y}\Big(V_z \;=\; c_x \times y \;-\; c_y \times x\Big) \;-\; \frac{\partial}{\partial z}\Big(V_y \;=\; c_z \times x \;-\; c_x \times z\Big) \\
&=\; (+c_x) \;-\; (-c_x) \;=\; 2c_x\,, \\
\big(\nabla \times \mathbf{V}\big)_y \;&=\; \frac{\partial}{\partial z}\Big(V_x \;=\; c_y \times z \;-\; c_z \times y\Big) \;-\; \frac{\partial}{\partial x}\Big(V_z \;=\; c_x \times y \;-\; c_y \times x\Big) \\
&=\; (+c_y) \;-\; (-c_y) \;=\; 2c_y\,, \\
\big(\nabla \times \mathbf{V}\big)_z \;&=\; \frac{\partial}{\partial x}\Big(V_y \;=\; c_z \times x \;-\; c_x \times z\Big) \;-\; \frac{\partial}{\partial y}\Big(V_x \;=\; c_y \times z \;-\; c_z \times y\Big) \\
&=\; (+c_z) \;-\; (-c_z) \;=\; 2c_z\,.
\end{aligned} \tag{74}$$

Now let apply the Stokes' theorem to this vector field:

$$\oint\limits_{\mathcal{C}} (\mathbf{c} \times \mathbf{r}) \cdot d\vec{\ell} \;=\; \iint\limits_{\mathcal{S}} \Big(\nabla \times (\mathbf{c} \times \mathbf{r})\Big) \cdot d^2\mathbf{A} \;=\; \iint\limits_{\mathcal{S}} (2\mathbf{c}) \cdot d^2\mathbf{A} \;=\; 2\mathbf{c} \cdot \mathbf{A}. \tag{75}$$

At the same time, in the integrand on the LHS here

$$(\mathbf{c}\times\mathbf{r})\cdot d\vec{\ell}\ =\ (\mathbf{r}\times d\vec{\ell})\cdot\mathbf{c} \tag{76}$$

and hence

$$\oint\limits_{\mathcal{C}}(\mathbf{c}\times\mathbf{r})\cdot d\vec{\ell}\ =\ \mathbf{c}\cdot\oint\limits_{\mathcal{C}}\mathbf{r}\times d\vec{\ell}. \tag{77}$$

Altogether, we get

$$2\mathbf{c}\cdot\mathbf{A}\ =\ \mathbf{c}\cdot\oint\limits_{\mathcal{C}}\mathbf{r}\times d\vec{\ell}, \tag{78}$$

and this equality must hold for any constant vector $\mathbf{c}$. Therefore, the vector area of the surface $\mathcal{S}$ must be

$$\mathbf{A}(\mathcal{S})\ =\ \frac{1}{2}\oint\limits_{\mathcal{C}}\mathbf{r}\times d\vec{\ell}, \tag{79}$$

which completes the proof of part (d).

Part (c) follows trivially from the formula (79) for the vector area of a surface. Indeed, when any two surfaces have the same boundary $\mathcal{C}$, then the loop integral on the RHS of eq. (79) is exactly the same for both surfaces, hence the same vector are for both surfaces.

Given part (c), part (b) is just a special case: if a surface has no boundary, then the integral on the RHS of eq. (79) is zero, so the vector area of the surface is zero.

Finally, part (a) is also a special case of part (c). The boundary of the hemisphere is the equatorial circle, and the flat disk in the equatorial plane has the same boundary. By part (c) the hemisphere and the disk have the same vector areas, and since the disk is flat, its vector area is simply its ordinary area πR^2 times the unit vector $\perp$ to the disk. Hence, the vector area of the hemisphere has magnitude πR^2 (unlike its ordinary area $2\pi R^2$), and its direction is along the symmetry axis of the hemisphere.

Problem **2.20**:

A *static* electric field $\mathbf{E}(x, y, z)$ must have zero curl. By inspection, the curl of the (a) field is

$$\begin{aligned}
\mathbf{E}_a &= kxy\hat{\mathbf{x}} \;+\; 2kyz\hat{\mathbf{y}} \;+\; 3kxz\hat{\mathbf{z}}, \\
\nabla \times \mathbf{E}_a &= \left(\frac{\partial(E_z = 3kxz)}{\partial y} - \frac{\partial(E_y = 2kyz)}{\partial z}\right)\hat{\mathbf{x}} \\
&\quad + \left(\frac{\partial(E_x = kxy)}{\partial z} - \frac{\partial(E_z = 3kxz)}{\partial x}\right)\hat{\mathbf{y}} \\
&\quad + \left(\frac{\partial(E_y = 2kyz)}{\partial x} - \frac{\partial(E_x = kxy)}{\partial y}\right)\hat{\mathbf{z}} \\
&= (0 - 2ky)\hat{\mathbf{x}} \;+\; (0 - 3kz)\hat{\mathbf{y}} \;+\; (0 - kx)\hat{\mathbf{z}} \;=\; -k(2y\hat{\mathbf{x}} + 3z\hat{\mathbf{y}} + x\hat{\mathbf{z}}) \\
&\neq 0
\end{aligned} \tag{80}$$

So the (a) field is impossible in electrostatics.

As to the (b) field,

$$\begin{aligned}
\mathbf{E}_b &= ky^2\hat{\mathbf{x}} \;+\; k(2xy + z^2)\hat{\mathbf{y}} \;+\; 2kyz\hat{\mathbf{z}}, \\
\nabla \times \mathbf{E}_b &= \left(\frac{\partial(E_z = 2kyz)}{\partial y} - \frac{\partial(E_y = k(2xy + z^2))}{\partial z}\right)\hat{\mathbf{x}} \\
&\quad + \left(\frac{\partial(E_x = ky^2)}{\partial z} - \frac{\partial(E_z = 2kyz)}{\partial x}\right)\hat{\mathbf{y}} \\
&\quad + \left(\frac{\partial(E_y = k(2xy + z^2))}{\partial x} - \frac{\partial(E_x = 2kyz)}{\partial y}\right)\hat{\mathbf{z}} \\
&= (2kz - 2kz)\hat{\mathbf{x}} \;+\; (0 - 0)\hat{\mathbf{y}} \;+\; (2ky - 2ky)\hat{\mathbf{z}} \\
&= 0\hat{\mathbf{x}} \;+\; 0\hat{\mathbf{y}} \;+\; 0\hat{\mathbf{z}} \;=\; \mathbf{0},
\end{aligned} \tag{81}$$

So the (b) field is possible.

Now let's find the electric potential $V(x, y, z)$ for the (b) field by integrating

$$V(\mathcal{O}) \;-\; V(x, y, z) \;=\; \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\vec{\ell}, \tag{82}$$

where $\mathcal{O}$ is some reference point, and we may integrate along any path from $\mathcal{O}$ to the $\mathbf{r}$. For the sake of definiteness, let the reference point $\mathcal{O}$ be the coordinate origin, and let's us the following 3-leg path from $\mathcal{O} = (0,0,0)$ to the point (x,y,z) where we calculate the potential:

1. From $(0,0,0)$ to $(x,0,0)$ along the x axis.
2. From $(x,0,0)$ to $(x,y,0)$ parallel to the y axis.
3. From $(x,y,0)$ to (x,y,z) parallel to the z axis.

Along the first leg of the path, we have

$$\int\limits_{(0,0,0)}^{(x,0,0)} \mathbf{E}\cdot d\vec{\ell} \;=\; \int\limits_0^x dx'\,E_x(x',0,0) \;=\; \int\limits_0^x dx'\,(ky^2\;=\;0) \;=\; 0. \tag{83}$$

Along the second leg,

$$\begin{aligned}\int\limits_{(x,0,0)}^{(x,y,0)} \mathbf{E}\cdot d\vec{\ell} \;&=\; \int\limits_0^y dy'\,\Bigl(E_y(x,y',0) \;=\; k(2xy'+2z^2) \;=\; 2kx\times y'\Bigr)\\ &=\; \Bigl(kx\times y'^2\Bigr)\Big|_{y'=0}^{y'=y} \;=\; kxy^2.\end{aligned} \tag{84}$$

Finally, along the third leg

$$\begin{aligned}\int\limits_{(x,y,z)}^{(x,y,z)} \mathbf{E}\cdot d\vec{\ell} \;&=\; \int\limits_0^z dz'\,\Bigl(E_z(x,y,z') \;=\; 2ky\times z'\Bigr)\\ &=\; \Bigl(ky\times z'^2\Bigr)\Big|_{z'=0}^{z'=z} \;=\; kyz^2.\end{aligned} \tag{85}$$

Altogether,

$$V(0,0,0) \;-\; V(x,y,z) \;=\; \int\limits_{(0,0,0)}^{(x,y,z)} \mathbf{E}\cdot d\vec{\ell} \;=\; 0 \;+\; kxy^2 \;+\; kyz^2. \tag{86}$$

So is we set the potential at the reference point $(0,0,0)$ to zero, then the electric potential everywhere else is

$$V(x, y, z) \;=\; -k(xy^2 + yz^2). \tag{87}$$

To check our calculation, let's take (minus) the gradient of this potential and compare to the electric field $\mathbf{E}_(b)$:

$$-\frac{\partial V}{\partial x} \;=\; +ky^2, \quad -\frac{\partial V}{\partial y} \;=\; +k(2xy + z^2), \quad -\frac{\partial V}{\partial z} \;=\; +2kyz, \tag{88}$$

thus

$$-\nabla V(x, y, z) \;=\; ky^2\hat{\mathbf{x}} \;+\; k(2xy + z^2)\hat{\mathbf{y}} \;+\; 2kyz\hat{\mathbf{z}} \;= \text{indeed} =\; \mathbf{E}(x, y, z). \tag{89}$$

<u>Problem **2.21**</u>:

There are two ways to calculate the potential due to a uniformly charged solid ball: I can do it directly from the Coulomb law for the potential,

$$V(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0} \iint\limits_{\text{ball}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dx'\, dy'\, dz', \tag{90}$$

similarly to how I have done the potential of a spherical shell in my notes on gradient, divergence, curl, and related issues (pages 19–21). Alternatively, I can start with the electric field obtained from the symmetry and the Gauss Law, and get the potential from the field. Let me follow the second route, since it's clearly what the textbook expects — and it's also much easier.

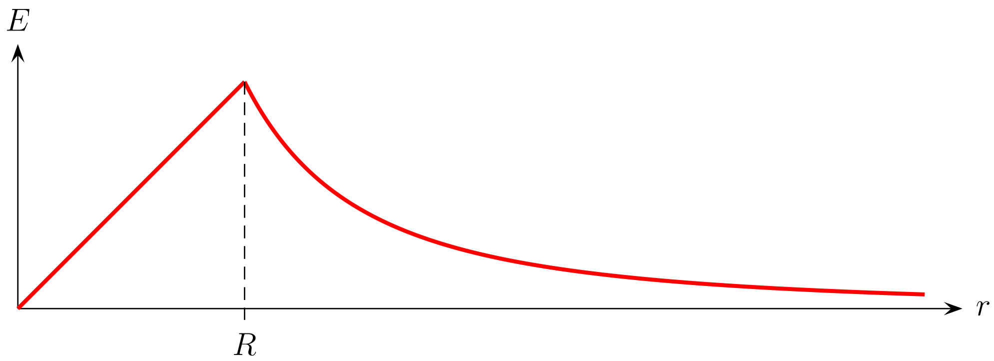
The electric field of a uniformly charged solid ball is worked out in detail in my notes on applications of the Gauss Law (page 4), and also in the textbook example 2.3 (pages 71–72). Outside the ball, the electric field is the same as if the entire charge was concentrated at the ball's center,

$$\text{for } r > R, \quad \mathbf{E}(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\hat{\mathbf{r}}}{r^2}\,, \tag{91}$$

while inside the ball, the electric field grows linearly with the distance from the center,

$$\text{for } r < R, \quad \mathbf{E}(r) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{r}{R^3}\,\hat{\mathbf{r}}. \tag{92}$$

Graphically,



(93)

To verify these formulae, let's take the divergence of the electric field and compare to the electric charge density ρ: Outside the ball,

$$\nabla\cdot\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\times\nabla\left(\frac{\hat{\mathbf{r}}}{r^2}\right) \;=\; \frac{Q}{4\pi\epsilon_0}\times\left(\left(\frac{d}{dr}+\frac{2}{r}\right)\frac{1}{r^2}\right) \;=\; 0 \tag{94}$$

— which agrees with $\rho = 0$ outside the ball,— while inside the ball,

$$\nabla\cdot\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0 R^3}\times\big(\nabla\cdot(r\hat{\mathbf{r}}=\mathbf{r})\big) \;=\; \frac{Q}{4\pi\epsilon_0 R^3}\times 3 \;=\; \frac{1}{\epsilon_0}\times\frac{Q}{(4\pi/3)R^3} \;=\; \frac{\rho(\text{inside})}{\epsilon_0}\,. \tag{95}$$

Anyhow, given the electric field $\mathbf{E}(x,y,z)$, the potential $V(x,y,z)$ obtains from integrating

$$V(x,y,z) \;=\; \int\limits_{(x,y,z)}^{\mathcal{O}} \mathbf{E}\cdot d\vec{\ell} \tag{96}$$

over any path we like from (x,y,z) to the reference point $\mathcal{O}$ (where we take $V(\mathcal{O})=0$), For a spherically symmetric electric field $\mathbf{E} = E(r\,\text{only})\hat{\mathbf{r}}$,

$$\mathbf{E}\cdot d\vec{\ell} \;=\; E(r)\,\hat{\mathbf{r}}\cdot d\vec{\ell} \;=\; E(r)\,dr \tag{97}$$

so the integral (96) over some path is simply the radial integral

$$V(r\,\text{only}) \;=\; \int_r^{R_{\mathcal{O}}} dr'\,E(r'). \tag{98}$$

Specifically, for the reference point at infinity,

$$V(r\,\text{only}) \;=\; \int_r^{\infty} dr'\,E(r'). \tag{99}$$

In particular, for $r > R$ outside the ball, the potential is

$$V(r) \;=\int_r^\infty \frac{Q}{4\pi\epsilon_0}\,\frac{1}{r'^2}\,dr' \;=\; \frac{Q}{4\pi\epsilon_0}\int_r^\infty d\left(\frac{-1}{r'}\right) \;=\; \frac{Q}{4\pi\epsilon_0}\left(\frac{-1}{\infty}-\frac{1}{r}\right) \;=\; \frac{Q}{4\pi\epsilon_0}\times\frac{1}{r}. \tag{100}$$

the same as the Coulomb potential due to a point charge Q at the center of the ball.

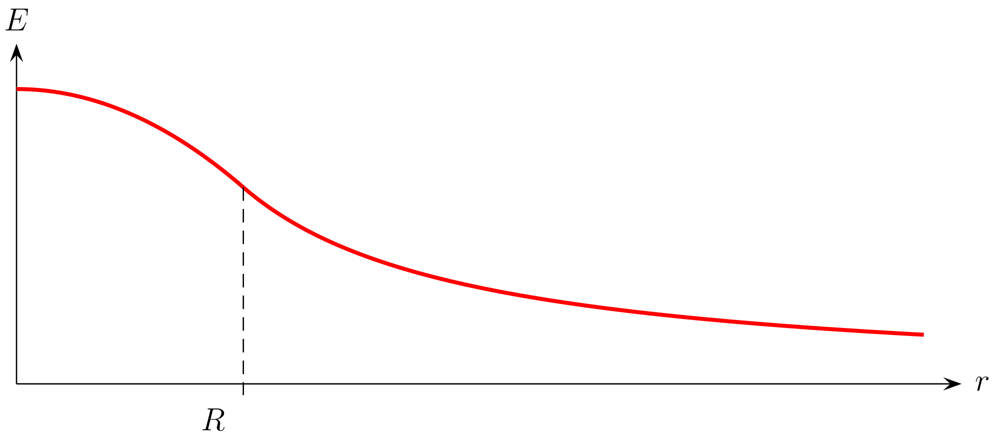
Now let's calculate the potential inside the ball, at $r < R$. This time, we should iintegrate the electric field from r to ∞ and use different formulae for the field inside and outside the ball. Thus,

$$\begin{aligned}
v(r) \;&=\int_r^\infty dr'\left(E(r') \;=\; \frac{Q}{4\pi\epsilon_0}\times\begin{cases}\frac{r'}{R^3} & \text{for } r' < R\\ \frac{1}{r'^2} & \text{for } r' > R\end{cases}\right)\\
&=\;\frac{Q}{4\pi\epsilon_0}\times\left\{\int_r^R dr'\,\frac{r'}{R^3} \;+\int_R^\infty dr'\,\frac{1}{r'^2}\right\}\\
&=\;\frac{Q}{4\pi\epsilon_0}\times\left\{\frac{R^2-r^2}{2R^3} \;+\; \frac{1}{R}\right\}\\
&=\;\frac{Q}{4\pi\epsilon_0}\times\frac{3R^2-r^2}{2R^3}\,.
\end{aligned}\tag{101}$$

Altogether, the electric potential due to a uniformly charged solid ball is

$$V(r) = \frac{Q}{4\pi\epsilon_0} \times \begin{cases} \dfrac{1}{r} & \text{for } r > R \text{ (outside the ball),} \\[2ex] \dfrac{3R^2 - r^2}{2R^3} & \text{for } r < R \text{ (inside the ball).} \end{cases} \tag{102}$$

Graphically,



(103)

Finally, to verify our calculation, let's take (minus) the gradient of the potential (102) and compare to the electric field of the solid ball:

$$-\nabla\frac{1}{r} = \frac{\hat{\mathbf{r}}}{r^2}, \qquad -\nabla\frac{3R^2 - r^2}{2R^3} = \frac{r\hat{\mathbf{r}}}{R^3}, \tag{104}$$

hence

$$-\nabla V(r) = \frac{Q}{4\pi\epsilon_0} \left\{ \begin{matrix} \dfrac{1}{r^2} & \text{outside the ball} \\ \dfrac{r}{R^3} & \text{inside the ball} \end{matrix} \right\} \hat{\mathbf{r}} = \text{indeed} = \mathbf{E}. \tag{105}$$

<u>Problem **2.22**</u>:

Similarly to the previous problem, let me start from the electric field of the wire,

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{\hat{\mathbf{s}}}{s} \tag{106}$$

where s is the distance from the wire in the plane $\perp$ to the wire and $\hat{\mathbf{s}}$ is the unit vector radially away from the wire in the $\perp$ plane. In the cylindrical coordinates (s, ϕ, z) where the wire runs along the z axis, s is the radial coordinate and $\hat{\mathbf{s}}$ is the unit vector in s direction.

The electric field (105) of the wire obtains from the axial symmetry and the Gauss Law, see my notes on applications of the Gauss Law (page 7), so I am not going to repeat the calculation here. Instead, let me simply derive the potential from the electric field (105).

As usual, the potential obtains from the integration

$$V(x,y,z) \;=\; \int\limits_{(x,y,z)}^{\mathcal{O}} \mathbf{E}\cdot d\vec{\ell} \tag{107}$$

along an arbitrary path from the point (x,y,z) where we calculate the potential to some fixed reference point $\mathcal{O}$ where we set $V=0$. Due to axial symmetry of the electric field of the wire, in cylindrical coordinates

$$\mathbf{E}(s,\phi,z) \;=\; E(s\,\text{only})\hat{\mathbf{s}} \quad\Longrightarrow\quad \mathbf{E}\cdot d\vec{\ell} \;=\; E(s)\,ds\,, \tag{108}$$

hence the potential V depends only on the s coordinate — the distance from the wire. Specifically,

$$V(s) \;=\int\limits_{s}^{S_{\mathcal{O}}} E(s')\,ds' \;=\int\limits_{s}^{S_{\mathcal{O}}} \frac{\lambda}{2\pi\epsilon_0}\times\frac{ds'}{s'} \;=\; \frac{\lambda}{2\pi\epsilon_0}\times\ln\frac{S(\mathcal{O})}{s}, \tag{109}$$

or in Cartesian coordinates (where the wire runs along the z axis),

$$V(x,y,z) \;=\; \frac{\lambda}{2\pi\epsilon_0}\times\ln\frac{S(\mathcal{O})}{\sqrt{x^2+y^2}}\,. \tag{110}$$

Note that we cannot get rid of the explicit $S(\mathcal{O})$ in this formula by sending the reference point to infinity since that would add an *infinite* constant to the potential. Likewise, we cannot put the reference point in the middle of the wire (if its infinitely thin), so we are stuck with an explicit $S(\mathcal{O})$ in the denominator inside the logarithm. We may set the $S(\mathcal{O})$ to any constant we like, but that constant has to be finite and non-zero.

It remains to check that (minus) gradient of the potential (109) agrees with the electric field (106) of the infinite thin wire. Indeed, by the chain rule

$$-\nabla V(s) \;=\; -\frac{dV}{ds}\nabla s \;=\; +\frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}\,\hat{\mathbf{s}} \;=\text{indeed}=\; \mathbf{E}(\text{wire}). \tag{111}$$