Problem **1.46**:

(a) Since the delta function $\delta(x)$ vanishes at all $x \neq 0$, its derivative also vanishes at all $x \neq 0$, hence

$$\text{for all } x \neq 0, \quad x \times \frac{d}{dx}\delta(x) \;=\; 0 \;=\; -\delta(x). \tag{1}$$

But at $x = 0$, both sides of the equation are badly singular, so to establish their equality we need to integrate. Specifically, we need to multiply both sides of the equation by $g(x)\,dx$ for some *smooth* function $g(x)$, then integrate over an interval from some $a < 0$ to some $b > 0$, and then check the equality

$$\int_a^b g(x) \times x \,\frac{d\delta(x)}{dx}\,dx \;\overset{??}{=}\; \int_a^b g(x) \times \big(-\delta(x)\big)\,dx. \tag{2}$$

Let's start by integrating the LHS by parts:

$$\begin{aligned}
\int_a^b g(x) \times x \,\frac{d\delta(x)}{dx}\,dx \;&=\; \int_a^b \; g(x) \times x \times d\big(\delta(x)\big) \\
&=\; \Big(g(x) \times x \times \delta(x)\Big)\Big|_a^b \;-\; \int_a^b \frac{d}{dx}\big(x \times g(x)\big) \times \delta(x)\,dx \\
&=\; 0 \quad \langle\!\langle \text{ because } \delta(a) = \delta(b) = 0 \,\rangle\!\rangle \\
&\qquad -\; \frac{d}{dx}\big(x \times g(x)\big)\Big|_{@x=0} \\
&=\; -\,\big(x \times g'(x) \;+\; g(x)\big)\big|_{@x=0} \\
&=\; -g(0).
\end{aligned} \tag{3}$$

But on the RHS of eq. (2) we get exactly the same result,

$$\int_a^b g(x) \times \big(-\delta(x)\big)\,dx = \;-\int_a^b g(x) \times \delta(x)\,dx \;=\; -g(0), \tag{4}$$

so both sides of eq. (2)are indeed equal to each other. And since this equality holds for any smooth function $g(x)$, this means that

$$x \times \frac{d\delta(x)}{dx} \times dx \;=\; -\delta(x) \times dx \quad \text{as measures,} \tag{5}$$

or in other words,

$$x \times \frac{d\delta(x)}{dx} \;=\; -\delta(x) \quad \text{as generalized functions.} \tag{6}$$

Quod erat demonstrandum.

(b) The step function $\Theta(x)$ is discontinuous at $x = 0$, but at all other $x \neq 0$ this function is *locally constant.* Consequently, its derivative vanishes at all $x \neq 0$, but becomes singular for $x = 0$,

$$\left.\frac{d\Theta}{dx}\right|_{@x=0} \;=\; \lim_{\Delta x \to 0} \frac{\Delta\Theta = +1}{\Delta x} \;=\; +\infty, \tag{7}$$

thus

$$\frac{d}{dx}\,\Theta(x) \;=\; \begin{cases} 0 & \text{for } x \neq 0, \\ +\infty & \text{for } x = 0. \end{cases} \tag{8}$$

To compare this singularity with the *delta* function, we multiply the derivative $\Theta'(x)$ by $g(x)\,dx$ (for some smooth $g(x)$) and integrate (from some $a < 0$ to some $b > 0$):

$$\begin{aligned}
&\int\limits_a^b g(x) \times \frac{d\Theta(x)}{dx} \times dx \qquad \langle\!\langle \text{ integrating by parts } \rangle\!\rangle \\
&\qquad = \Big(g(b) \times \Theta(b) \;-\; g(a) \times \Theta(a)\Big) \;-\; \int\limits_a^b g'(x) \times \Theta(x) \times dx \\
&\qquad\quad \langle\!\langle \text{ using } \Theta = \text{step function } \rangle\!\rangle \\
&\qquad = \Big(g(b) \;-\; 0\Big) \;-\; \int\limits_0^b g'(x)\,dx \quad \langle\!\langle \text{ note lower limit is 0 instead of } a \,\rangle\!\rangle \\
&\qquad = g(b) \;-\; \Big(g(b) \;-\; g(0)\Big) \;=\; g(0) \\
&\qquad = \int\limits_a^b g(x) \times \delta(x) \times dx.
\end{aligned} \tag{9}$$

And since this equality holds true for any smooth function $g(x)$, it follows that the *generalized functions* $d\Theta/dx$ and $\delta(x)$ are equal,

$$\frac{d\Theta(x)}{dx} = \delta(x). \tag{10}$$

Problem **1.48**:

(a)

$$\iiint\limits_{\text{space}} \left(r^2 + \mathbf{r}\cdot\mathbf{a} + a^2\right) \times \delta^{(3)}(\mathbf{r}-\mathbf{a}) \times d^3\mathrm{Vol} = \left(r^2 + \mathbf{r}\cdot\mathbf{a} + a^2\right)\Big|_{@\mathbf{r}=\mathbf{a}} = a^2 + a^2 + a^2 = 3a^2. \tag{11}$$

(b) First, let's work out the $\delta^{(3)}(5\mathbf{r})$. In one dimension, $\delta(5x) = \frac{1}{5}\delta(x)$, so in 3 dimensions

$$\delta^{(3)}(5\mathbf{r}) = \delta(5x) \times \delta(5y) \times \delta(5z) = \tfrac{1}{5}\delta(x) \times \tfrac{1}{5}\delta(y) \times \tfrac{1}{5}\delta(5z) = \frac{1}{5^3 = 125} \times \delta^{(3)}(\mathbf{r}). \tag{12}$$

Second, let's check that the integration volume $\mathcal{V}$ includes the point where the $\delta^{(3)}(5\mathbf{r})$ is infinite: Indeed, that point is the origin $\mathbf{r} = \mathbf{0}$, while the $\mathcal{V}$ is a cube centered on that origin, so that's OK. Consequently, for any smooth $f(\mathbf{r})$,

$$\iiint\limits_{\text{cube}} f(\mathbf{r}) \times \delta^{(3)}(5\mathbf{r}) \times d^3\mathrm{Vol} = \frac{1}{125} \int\limits_{\text{cube}} f(\mathbf{r}) \times \delta^{(3)}(\mathbf{r}) = \frac{1}{125} \times f(\mathbf{r} = \mathbf{0}). \tag{13}$$

In particular, for $f(\mathbf{r}) = |\mathbf{r} - \mathbf{b}|^2$, we have $f(\mathbf{0}) = |\mathbf{b}|^2$, hence

$$\text{the integral} = \frac{1}{125} \times |\mathbf{b}|^2 = \frac{b_x^2 + b_y^2 + b_z^2 = 0^2 + 4^2 + 3^2}{125} = \frac{25}{125} = \frac{1}{5}. \tag{14}$$

(c) In general,

$$\iiint\limits_{\mathcal{V}} f(\mathbf{r})\times\delta^{(3)}(\mathbf{r}-\mathbf{c})\times d^3\mathrm{Vol} \;=\; \begin{cases} f(\mathbf{c}) & \text{if } \mathbf{c} \text{ lies inside the volume } \mathcal{V}, \\ 0 & \text{if } \mathbf{c} \text{ lies outside the volume } \mathcal{V}. \end{cases} \tag{15}$$

So let's check if the point $\mathbf{c}$ in question lies inside the volume $\mathcal{V}$ or outside it, Since $\mathcal{V}$ happens to be the sphere — or rather the solid ball — of radius $R = 6$ centered on the origin, all we need is to check the distance of the point $\mathbf{c}$ from that origin, or in vector term, the magnitude $|\mathbf{c}|$. Specifically,

$$\begin{aligned} \mathbf{c}^2 \;&=\; c_x^2 + c_y^2 + c_z^2 \;=\; 5^2 + 3^2 + 2^2 \;=\; 38, \\ \text{hence } |\mathbf{c}| \;&=\; \sqrt{38} \;>\; 6 \;=\; R, \end{aligned} \tag{16}$$

thus the point $\mathbf{c}$ lies *outside* the volume $\mathcal{V}$. Consequently, *regardless of the details of the function* $f(\mathbf{r}) = [r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4]$, *the integral vanishes*

(d) First of all, $\delta^{(3)}(\mathbf{e}-\mathbf{r}) = +\delta^{(3)}(\mathbf{r}-\mathbf{e})$. Consequently, in light of eq. (15),

$$\text{the integral} \;=\; \iiint\limits_{\mathcal{V}} \big(\mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\big)\times\delta^{(3)}(\mathbf{r}-\mathbf{e})\times d^3\mathrm{Vol} \;=\; \big(\mathbf{e}\cdot(\mathbf{d}-\mathbf{e})\big)\times\begin{cases} 1 & \text{if } \mathbf{e} \text{ lies inside } \mathcal{V}, \\ 0 & \text{if } \mathbf{e} \text{ lies outside } \mathcal{V}. \end{cases} \tag{17}$$

This time, $\mathcal{V}$ is a sphere of radius $R = \frac{3}{2}$ centered at the point $\mathbf{f} = (2, 2, 2)$, so to find if the point $\mathbf{e}$ lies inside or outside that sphere, let's check the distance $|\mathbf{e}-\mathbf{f}|$:

$$\begin{aligned} |\mathbf{e}-\mathbf{f}|^2 \;&=\; (e_x - f_x \;=\; 3-2)^2 \;+\; (e_y - f_y \;=\; 2-2)^2 \;+\; (e_z - f_z \;=\; 1-2)^2 \\ &=\; 1 \;+\; 0 \;+\; 1 \;=\; 2, \\ |\mathbf{e}-\mathbf{f}| \;&=\; \sqrt{2} \;<\; \tfrac{3}{2} \;=\; R, \end{aligned} \tag{18}$$

thus $\mathbf{e}$ lies inside the integration volume $\mathcal{V}$. Hence, according to eq. (17),

$$\begin{aligned} \text{the integral} \;&=\; \big(\mathbf{e}\cdot(\mathbf{d}-\mathbf{e})\big)\times 1 \\ &=\; e_x(d_x - e_x) \;+\; e_y(d_y - e_y) \;+\; e_z(d_z - e_z) \\ &=\; 3\times(1-3) \;+\; 2\times(2-2) \;+\; 1\times(3-1) \\ &=\; -4. \end{aligned} \tag{19}$$

Problem **1.49**:

Method #1: As we have learned in class on Tuesday (2/2), the divergence of the Coulomb field of a point charge is the delta function,

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) \;=\; 4\pi\,\delta^{(3)}(\mathbf{r}). \tag{20}$$

Consequently, in light of eq. (15),

$$\begin{aligned}
\text{the integral } &= \iiint\limits_{\mathcal{V}} \exp(-r)\times\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right)\times d^3\mathrm{Vol}\\
&= 4\pi\times\iiint\limits_{\mathcal{V}} \exp(-r)\times\delta^{(3)}(\mathbf{r})\times d^3\mathrm{Vol}\\
&= 4\pi\exp(0)\times\begin{cases}1 & \text{if the origin } \mathbf{0} \text{ lies inside } \mathcal{V},\\ 0 & \text{if the origin } \mathbf{0} \text{ lies outside } \mathcal{V}.\end{cases}
\end{aligned} \tag{21}$$

The volume $\mathcal{V}$ happens to be a sphere centered at the origin, so the origin itself definitely lies inside the sphere. Consequently, regardless of the sphere's radius R,

$$\text{the integral } \;=\; 4\pi\exp(0) \;=\; 4\pi. \tag{22}$$

Method #2: Integration by parts using the Gauss Theorem. Indeed, for any scalar field $S(\mathbf{r})$ and any scalar field $\mathbf{V}(\mathbf{r})$.

$$S\big(\nabla\cdot\mathbf{V}\big) \;=\; \nabla\cdot(S\mathbf{V}) \;-\; \big(\nabla S\big)\cdot\mathbf{V}, \tag{23}$$

hence for any volume $\mathcal{V}$ and its complete surface $\mathcal{S}$,

$$\iiint\limits_{\mathcal{V}} S(\nabla\cdot\mathbf{V})\,d^3\mathrm{Vol} \;=\; \iint\limits_{\mathcal{S}} S\mathbf{V}\cdot\mathbf{d^2A} \;-\; \iiint\limits_{\mathcal{V}} \big(\mathbf{V}\cdot\nabla S\big)\,d^3\mathrm{Vol}. \tag{24}$$

For the problem at hand, $\mathcal{S}$ is the sphere of radius R centered at the origin, $\mathcal{V}$ is the solid ball filling that sphere, $S(\mathbf{r})=\exp(-r)$ and $\mathbf{V}(\mathbf{r})=(1/r^2)\hat{\mathbf{r}}$. Plugging all these into eq. (24), we obtain

$$\text{the integral} \;=\; \iint\limits_{\text{sphere}} \exp(-r)\,\frac{\hat{\mathbf{r}}}{r^2}\cdot\mathbf{d^2A} \;-\; \iiint\limits_{\text{ball}} \left(\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\exp(-r)\right) d^3\text{Vol}\,. \tag{25}$$

The surface integral on the RHS here is easy: the factors $\exp(-r)$ and $1/r^2$ are constant along the sphere, while the unit vector $\hat{\mathbf{r}}$ is $\perp$ to the spherical surface. Consequently,

$$\iint\limits_{\text{sphere}} \exp(-r)\,\frac{\hat{\mathbf{r}}}{r^2}\cdot\mathbf{d^2A} \;=\; \exp(-R)\times\frac{1}{R^2}\times\text{Area(sphere)} \;=\; \frac{\exp(-R)}{R^2}\times 4\pi R^2 \;=\; 4\pi\exp(-R). \tag{26}$$

As to the volume integral,

$$\nabla\exp(-r) \;=\; -\exp(-r)\,\hat{\mathbf{r}} \quad\Longrightarrow\quad \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\exp(-r) \;=\; -\frac{\exp(-r)}{r^2}\,, \tag{27}$$

which depends only on the radius. Consequently, we may use the spherical coordinates where

$$d^3\text{Vol} \;=\; r^2\,dr\times d^2\Omega \;\longrightarrow\; 4\pi r^2\,dr \quad \text{after integration over } \theta \text{ and } \phi. \tag{28}$$

Therefore,

$$\begin{aligned}\iiint\limits_{\text{ball}} \left(\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\exp(-r)\right) d^3\text{Vol} \;&=\; \int_0^R \frac{-\exp(-r)}{r^2}\times 4\pi r^2\times dr \\ &=\; -4\pi\times\int_0^R \exp(-r)\,dr \\ &=\; -4\pi\times\Bigl(1\,-\,\exp(-R)\Bigr).\end{aligned} \tag{29}$$

Finally, plugging the surface integral (26) and the volume integral (29) into eq. (25), we arrive at

$$\text{the integral} \;=\; 4\pi\exp(-R) + 4\pi\Bigl(1\,-\,\exp(-R)\Bigr) \;=\; 4\pi, \tag{30}$$

regardless of the sphere's radius R.

Nota bene: By inspection, both methods produce the same result.

Problem **2.48**:

The potential of a surface charge density is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iint \frac{\sigma\, d^2A'}{|\mathbf{r}-\mathbf{r}'|}, \tag{31}$$

so the potential difference between points $\mathbf{r}_1$ and $\mathbf{r}_2$ is

$$V(\mathbf{r}_1) - V(\mathbf{r}_2) = \frac{1}{4\pi\epsilon_0} \iint \left(\frac{1}{|\mathbf{r}_1-\mathbf{r}'|} - \frac{1}{|\mathbf{r}_2-\mathbf{r}'|}\right) \times \sigma\, d^2A \tag{32}$$

where $\mathbf{r}'$ runs over the charged surface.

Let's use the spherical coordinates (θ',ϕ') for the $\mathbf{r}'$ on the hemisphere. Then for $\mathbf{r}_1$ being the pole of the hemisphere

$$|\mathbf{r}_1-\mathbf{r}'| = 2R \times \sin\frac{\theta'}{2} \tag{33}$$

while for $\mathbf{r}'$ being the center of the sphere

$$|\mathbf{r}_2-\mathbf{r}'| \equiv R. \tag{34}$$

Consequently, for the uniform surface charge density σ, we obtain

$$\begin{aligned}
\Delta V &= V(\text{pole}) - V(\text{center}) \\
&= \frac{1}{4\pi\epsilon_0} \iint\limits_{\text{hemisphere}} \left(\frac{1}{2R\sin(\theta'/2)} - \frac{1}{R}\right) \times \sigma \times R^2 \sin\theta'\, d\theta'\, d\phi' \\
&= \frac{\sigma R^2}{4\pi\epsilon_0 R} \times \int_0^{\pi/2} d\theta' \sin\theta' \left(\frac{1}{2\sin(\theta'/2)} - 1\right) \times \left(\int_0^{2\pi} d\phi' = 2\pi\right) \\
&= \frac{\sigma R}{2\epsilon_0} \times \int_0^{\pi/2} d\theta' \sin\theta' \left(\frac{1}{2\sin(\theta'/2)} - 1\right).
\end{aligned} \tag{35}$$

To perform the remaining integral here, we use

$$\frac{\sin\theta'}{2\sin(\theta'/2)} = \cos(\theta'/2), \tag{36}$$

hence

$$\int_0^{\pi/2} \cos(\theta'/2) d\theta' = \int_0^{\pi/2} d\big(2\sin(\theta'/2)\big) = 2\sin\frac{\pi}{4} - 2\sin(0) = \sqrt{2}, \tag{37}$$

while

$$\int_0^{\pi/2} \sin\theta'\, d\theta' = \int_0^{\pi/2} d\big(-\cos\theta'\big) = -\cos(\pi/2) + \cos(0) = 1. \tag{38}$$

Consequently,

$$\int_0^{\pi/2} d\theta'\, \sin\theta' \left(\frac{1}{2\sin(\theta'/2)} - 1\right) = \sqrt{2} - 1 \tag{39}$$

and hence

$$\Delta V = \frac{\sigma R}{2\epsilon_0} \times (\sqrt{2} - 1). \tag{40}$$

Problem **2.59**:
(a) The would-be theorem is manifestly false. Here is the counterexample: Let $Q = 0$ (no net charge on the conductor), while the external field is due to a point charge q near the conductor. This external field induces charges on the conductor's surface, specifically on the near side of the conductor the induced charges have the opposite sign from q while the induced charges on the far side have the same sign as q. Consequently, the induced charges on the near side are attracted to q while the induced charges on the far side are repelled, but since the near side is closer to q, the attraction is stronger than the repulsion, and the net force is attractive.

Note: the net force on the conductor is towards the point charge q regardless of the sign of q! Consequently, reversing the external electric field by reversing the sign of q would not reverse the direction of the net force. Thus, the would-be theorem is manifestly false.

(b) Now suppose the *external* electric field $\mathbf{E}_e$ is uniform. The net electric field is the sum of this external field and the non-uniform field $\mathbf{E}_i$ of all the induced charges,

$$\mathbf{E}_{\text{net}}(\mathbf{r}) \;=\; \mathbf{E}_e \;+\; \mathbf{E}_i(\mathbf{r}), \tag{41}$$

so the net force on the conductor is

$$\mathbf{F}_{\text{net}} \;=\; \iint d^2A\,\sigma(\mathbf{r}')\mathbf{E}_{\text{net}}(\mathbf{r}') \;=\; \mathbf{E}_e \iint d^2A\,\sigma(\mathbf{r}') \;+\; \iint d^2A\,\sigma(\mathbf{r}')\mathbf{E}_i(\mathbf{r}') \;=\; \mathbf{E}_e Q_{\text{net}} \;+\; \mathbf{F}_{\text{int}}\,. \tag{42}$$

Note that thanks to uniformity of the external field, the first term here does not depend on how the charges are distributed over the conductor's surface but only on its net charge Q_{net}. As to the second term $\mathbf{F}_{\text{int}}$, it comprises the sum total of all the forces the charges on the conductor's surface exert on each other, and by the third law of Newton this sum must vanish, $\mathbf{F}_{\text{int}} = \mathbf{0}$.

Thus in a uniform external field $\mathbf{E}$ the net force on the conductor is simply

$$\mathbf{F}_{\text{net}} \;=\; Q_{\text{net}}\mathbf{E}_e\,. \tag{43}$$

Consequently, reversing the external field (while keeping the net charge of the conductor) also reverses the direction of the net force.

But such reversal would not work for a non-uniform external field.

<u>Problem **2.38**</u>:

(a) By spherical symmetry of the system, the electric charge density is uniform on each of the three surfaces in question, namely the inner sphere, the inner surface of the outer shell, and the outer surface of the outer shell. Let σ_R, σ_a, and σ_b denote these respective charge densities.

For the inner sphere, its whole charge q is uniformly distributed on its surface, hence

$$\sigma_R \;=\; \frac{q}{4\pi R^2}\,. \tag{44}$$

As to the outer shell, there are no volume charges within its thickness, but there are induced charges σ_a and σ_b on its inner and outer surfaces. To find them, we may use the Gauss Law for a spherical surface $\mathcal{S}$ of radius r between a and b (*i.e.*, $a < r < b$). The electric field within the conducting outer shell must vanish, so the electric flux through $\int$ is zero, which means that the net charge within $\mathcal{S}$ must vanish. But the charges within $\mathcal{S}$ comprise the charge q on the inner ball and the induced charge $Q_a = 4\pi a^2 \sigma_a$ on the inner surface of the shell; the induced charge $Q - b = 4\pi b^2 \sigma_b$ on the outer surface of the shell is outside $\mathcal{S}$, so it does not count. Consequently,

$$Q_a + q = 0 \quad \Longrightarrow \quad Q_a = -q. \tag{45}$$

Moreover, the net charge of the outer shell is zero, hence

$$Q_b + Q_a = 0 \quad \Longrightarrow \quad Q_b = -Q_a = +q. \tag{46}$$

Or in terms of the charge densities on the inner and outer surfaces of the shell,

$$\sigma_q = -\frac{q}{4\pi a^2}, \quad \sigma_b = +\frac{q}{4\pi b^2}. \tag{47}$$

(b) By the Gauss Law, the electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{q\,\hat{\mathbf{r}}}{4\pi\epsilon_0\, r^2} \tag{48}$$

between the inner ball and the shell (for $R < r < a$) and outside the shell (for $r > b$), but the field vanishes within the thickness of the shell (for $a < r < b$) or inside the inner ball (for $r < R$). Consequently, the potential at the outer surface of the shell is

$$V_b = \int_b^\infty dr\, \frac{q}{4\pi\epsilon_0\, r^2} = \frac{q}{4\pi\epsilon_0} \times \frac{1}{b}, \tag{49}$$

the potential on the inner surface of the shell is the same — the whole shell is equipotential,

$$V_a = V_b, \tag{50}$$

the potential on the surface of the inner ball is

$$V_R \;=\; V_a \;+\int\limits_R^b dr\,\frac{q}{4\pi\epsilon_0\,r^2} \;=\; \frac{q}{4\pi\epsilon_0}\times\frac{1}{b} \;+\; \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R}\;-\;\frac{1}{a}\right) \;=\; \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b}\;-\;\frac{1}{a}\;+\;\frac{1}{R}\right), \tag{51}$$

and the potential at the center of the system is the same as on the surface of the inner ball,

$$V_c \;=\; V_R \;=\; \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b}\;-\;\frac{1}{a}\;+\;\frac{1}{R}\right). \tag{52}$$

(c) When we ground the outer shell (by connecting it by a wire to the ground where $V=0$), some electric charge flows into it to change its potential to zero. But this extra charge goes to the outer surface only, the charge on the inner surface is fixed at $Q_a=-q$ to avoid the electric field within the thickness of the shell. Consequently, the net charge of the whole system is

$$Q_{\text{net}} \;=\; q \;+\; Q_a \;+\; Q_b \;=\; Q_b\,. \tag{53}$$

On the other hand, the net charge govern the potential of the outer surface (and hence the entire shell) relative to ∞ according to

$$\frac{Q_{\text{net}}}{4\pi\epsilon_0}\int\limits_b^\infty\frac{dr}{r^2} \;=\; \frac{Q_{\text{net}}}{4\pi\epsilon_0 b}\,, \tag{54}$$

so to set this potential to zero we need $Q_{\text{net}=0}$. Thus, in light of eq. (53), the *grounded* outer shell has

$$Q_a \;=\; -q \quad\text{but}\quad Q_b=0. \tag{55}$$

Or in terms of the surface charge densities,

$$\sigma_R \;=\; +\frac{q}{4\pi R^2}\,,\quad \sigma_a \;=\; -\frac{q}{4\pi a^2}\,,\quad \sigma_b \;=\; 0.$$

Now consider the electric field in different parts of the system. The net charge inside a

Gaussian sphere of radius r is

$$Q(r) \;=\; \begin{cases} 0 & \text{for } r > b, \\ 0 & \text{for } a < r < b, \\ q & \text{for } R < r < a, \\ 0 & \text{for } r < R, \end{cases} \tag{56}$$

so by the Gauss Law, the electric field between the ball and the inner surface of the shell is

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{q}{4\pi\epsilon_0}\,\frac{\hat{\mathbf{r}}}{r^2} \quad \text{for } R < r < a \text{ only,} \tag{57}$$

but it vanishes everywhere else. Consequently, the potential at the center of the system is

$$V_C \;=\; \int_0^\infty E_r(r)\,dr \;=\; \frac{q}{4\pi\epsilon_0}\int_R^a \frac{dr}{r^2} \;=\; \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R}-\frac{1}{a}\right). \tag{58}$$

Problem **2.39**:

In electrostatics — *i.e.*, no electric currents, just the static charges — the electric field $\mathbf{E}$ vanishes anywhere inside a conductor. Consequently, the volume charge density ρ vanishes throughout the bulk of a conductor, but at the conductor's surface there is generally a surface charge density σ.

For conductors with multiple surfaces — like the conductor depicted on textbook figure (2.49) — there is a very useful *rule of independent surfaces:* the charges on the surface of a cavity depend ONLY on the internal charges inside that cavity, while the charges on the outside surface depend only on the NET charge (of the conductor and everything enclosed within it) and on the external charges outside the outer surface. Specifically, the net electric field due to the charges inside a cavity and on the cavity's inner surface — and only due to these charges — must cancel everywhere outside that cavity. (Not just in the metal itself but also in the other cavities and in the outside world.) Likewise, the net electric field due to the outside charges (if any) and the charges on the outer surface — and only due to those charges — must cancel everywhere inside the outer surface (including the metal itself and all the cavities).

(a) Let's start with the cavity a. It is spherical in shape, with a charge q_a placed at the cavity's center. By the rule of independent surfaces, the charges on the cavity's surface do not care about any other charges. In fact, they do not care about anything outside the cavity, for all they know the cavity may be a spherical hole in an infinite conductor filling up the rest of the Universe. Consequently, as far as the cavity a and its surface are concerned, it has spherical symmetry around its center, and although the rest of the conductor does not have this symmetry, the cavity does not care! And thanks to this symmetry, the charge density σ on the cavity's surface is uniform!

Now let's combine the point charge q_a at the center of the cavity with the uniform surface charge σ_a at the cavity surface — the sphere of radius R_a. The net electric field of these charges must vanish outside the cavity. By Gauss Law, this calls for zero net charge on the surface plus inside the cavity, thus

$$q_a \;+\; 4\pi R_a^2 \times \sigma_a \;=\; 0, \tag{59}$$

and hence

$$\sigma_a \;=\; -\frac{q_q}{4\pi R_a^2}\,. \tag{60}$$

The same arguments apply to the cavity b and its surface. Again, there is effective spherical symmetry around the center of b despite the asymmetry of the rest of the conductor, hence uniform charge density σ_b on the cavity's surface. And by the Gauss Law,

$$q_b \;+\; 4\pi R_b^2 \times \sigma_b \;=\; 0, \tag{61}$$

and hence

$$\sigma_b \;=\; -\frac{q_b}{4\pi R_b^2}\,. \tag{62}$$

Now consider the outer surface of the metal. The whole metal is neutral, *i.e.* has zero net charge, although the three separate surfaces are charged. By charge conservation,

$$\begin{aligned} Q[\text{outer surface}] \;&=\; -Q[\text{surface of cavity } A] \;-\; Q[\text{surface of cavity } B] \\ &=\; +q_a \;+\; q_b \end{aligned} \tag{63}$$

where the second equality follows from eqs. (60) and (62).

As to how this charge is distributed over the surface, we again use the independent surfaces rule. The charges on the outer surface do not depend on cavities and anything inside them (except for the net charge); for all they care the conductor may be a solid metal ball — without any cavities or anything inside them. The only thing the outer surface cares about is the net charge (63) and that there are no extra charges *outside* the conductor. Consequently, the charges on the outer surface have spherical symmetry around the center of the outside sphere — despite the cavities not having that symmetry. Therefore, the outer surface has *uniform charge density*

$$\sigma_o = \frac{Q_o}{4\pi R^2} = \frac{q_a + q_b}{4\pi R^2}. \tag{64}$$

Indeed, we know from the Gauss Law that the uniformly charged spherical surface produces zero electric field inside the sphere, and that's precisely what we need for the charges on the outer surface.

This completes part (a) of the problem.

(b) Together, the charges on the surface of cavity (a) and the point charge q_a at the cavity's center cancel each other's electric fields anywhere outside that cavity — including the region outside the outer surface. Likewise, the charges on the surface of cavity (b) and the point charge q_b at that cavity's center cancel each other's electric fields anywhere outside cavity b — including the region outside the outer surface.

Thus, the net electric field outside the outer surface comes from the charges on that outer surface. By uniformity of these surface charges and by the Gauss Law, the net field outside the metal is simply the Coulomb field of the net charge as if it was at the center (of the outer surface), thus

$$\mathbf{E}(\text{outside}) = \frac{q_a + q_b}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \tag{65}$$

where the radius r is counted from the center of the outer sphere, and the unit vector $\hat{\mathbf{r}}$ point away from that center (to the point where you measure the field).

(c) As we already saw for part (b), the charges on the surface of cavity (b) and the point charge q_b at that cavity's center cancel each other's electric fields anywhere outside that cavity — in particular, anywhere inside the other cavity (a). Also, the charges on the outside surface of the conductor cancel each other's field inside the outer sphere — in particular, inside the cavity a. Thus, the only electric fields inside cavity (a) comes from the charges on its surface and the point charge q_a.

Moreover, by the spherical symmetry of the cavity a by itself, the charge density on its surface is uniform, so the electric field due to these charges vanishes inside the cavity. So the only non-zero field inside the cavity (a) is the Coulomb field of the point charge q_a inside the cavity center,

$$\mathbf{E}(\mathbf{r} \text{ inside cavity } a) \;=\; \frac{q_a}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_a}{|\mathbf{r}-\mathbf{r}_a|^3} \tag{66}$$

where $\mathbf{r}_a$ is the location of the cavity a's center.

Likewise, the electric field inside cavity b is the Coulomb field of the point charge q_b at that cavity's center, the electric fields due to all the other charges cancel out, thus

$$\mathbf{E}(\mathbf{r} \text{ inside cavity } b) \;=\; \frac{q_b}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_b}{|\mathbf{r}-\mathbf{r}_b|^3}\,. \tag{67}$$

(d) The electric force on the point charge q_a is precisely q_a times the electric field due to all the other charges — but not the field due to the q_a itself! But in part (c) we saw that anywhere inside cavity a, the electric fields due to all the charges other than the q_a — surface charges on the three surfaces plus the q_b — cancel each other! Thus, the net electric force on the charge q_a is zero!

Likewise, the net electric force on the charge q_b in the other cavity is also zero.

Note: The zero forces stem from the charges being located at the centers of the respective cavities. An off-center charge would feel a non-zero net force.

(e) Suppose we add a new charge q_c somewhere outside the outer surface of the conductor. By the rule of independent cavities, this would affect the distribution of electric charges on the outer surface — it wouldn't be uniform any more, although the net charge on the outer surface would still be the same $q_a + q_b$ — and the electric field outside the outer surface. But the new charge would not affect anything inside the outer surface, so the field inside the metal would remain exactly zero while the fields inside the cavities would remain exactly as in part (c). Also, the charge densities σ_a and σ_b on cavity walls would remain exactly as in part (a), and the forces on the charges q_a and q_b would remain exactly as in part (d).

Problem **2.42**:

According to the textbook equation (2.52), the net electric force on a surface of a conductor may be accounted by the *pressure*

$$P = \frac{\epsilon_0}{2} \times \mathbf{E}^2 \tag{2.52}$$

where $\mathbf{E}$ is the electric field just outside the conductor.

For a spherical conducting shell, the field just outside the shell has magnitude

$$E = \frac{Q}{4\pi\epsilon_0 R^2}, \tag{68}$$

so the pressure on the shell is

$$P = \frac{Q^2}{32\pi^2\epsilon_0 R^4}. \tag{69}$$

Note uniformity of this pressure, so the net force due to this pressure on some piece of the shell is

$$\mathbf{F} = P\mathbf{A} \tag{70}$$

where $\mathbf{A}$ is the *vector area* of that piece. As we have learned in the previous homework, the vector area of a hemisphere is πR^2 (as opposed to the ordinary area of $2\pi R^2$), so the net force on each hemisphere is

$$F = \frac{Q^2}{32\pi^2\epsilon_0 R^4} \times \pi R^2 = \frac{Q^2}{32\pi\epsilon_0 R^2}. \tag{71}$$

Physically, this is the net repulsive force between two halves of the spherical shell.

Problem **2.53**:

(a) I assume the electrodes — the negative cathode and the positive anode — are flat, parallel to each other, and much larger in size than the distance between them. This makes the problem essentially one-dimensional: the electric field's direction is always from the anode to the cathode ($\perp$ to both electrodes), the electrons fly in the opposite directions, and all the variables — the potential, the field, the charge density, the electrons' speed, *etc.*, — depends only on the x coordinate running $\perp$ to the electrodes.

In one dimension, the Laplacian of a potential $V(x, y, z) = V(x \text{ only})$ simplifies to the ordinary second derivative,

$$\triangle V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{d^2}{dx^2} V(x). \tag{72}$$

Consequently, the Poisson equation for the potential

$$\triangle V(x, y, z) = -\frac{1}{\epsilon_0} \rho(x, y, z) \tag{73}$$

becomes simply

$$\frac{d^2}{dx^2} V(x) = -\frac{1}{\epsilon_0} \rho(x). \tag{74}$$

Note: between the electrodes, the electric charge density of the electron cloud is negative, so the potential $V(x)$ should have positive second derivative.

(b) Let me make another big assumption: The electrons travel *ballistically* from the cathode to the anode without any collisions which may dissipate their energies. Collisions with other electrons do not count since they all move together in a cloud, but I assume there are no collisions with anything else — like the gas molecules in a poorly vacuumed tube.

Note: the ballistic assumption does not work for the electron gas permeating a metal where the electrons frequently collide with the phonons, the crystal defects, and with various impurities. Such collisions dissipate the electrons' velocities, and ultimately lead to the electric resistance and the Ohm's Law. Likewise, the ballistic assumption does not work for plasmas, electrolytes, semiconductors, *etc.*, *etc.* But it does work well for vacuum tubes, so it applies to the problem at hand.

Given the ballistic assumption, the net kinetic + potential energy of each electron is conserved as it flies from the cathode to the anode. Consequently, the electron's speed follows from its potential energy and hence from the electric potential $V(x)$.

Each electron emerges from the cathode with a rather small kinetic energy which we may neglect, $K \approx 0$. As the electron flies away from the cathode to $x > 0$, the potential $V(x)$ increases, so the potential energy of the negatively charged electron decreases by $\Delta U = -e \times \Delta V = -e \times V(x)$. (We set $V = 0$ at the cathode, so $\Delta V = V(x) - V(0)$ is simply $V(x)$.) By the energy conservation, the kinetic energy of the electron must increase by the same amount, $\Delta K = +e \times V(x)$, and since it had started with $K_0 \approx 0$, this means

$$\text{an electron at point } x \text{ has} \quad K = \frac{m_e v^2}{2} = +e \times V(x). \tag{75}$$

Note notations: the lower-case v is the electron's speed, while the upper-case V is the electric potential.

Solving eq. (75) for the election's speed, we find that

$$\text{an electron at point } x \text{ has speed} \quad v(x) = \sqrt{\frac{2e}{m_e} \times V(x)}\,. \tag{76}$$

(c) The *electric current density* $\mathbf{J}$ of the electron cloud follows from cloud's density n_e (in electrons per unit of volume) and its *drift velocity*

$$\mathbf{v}_d = \text{average velocity vector of an electron} = \frac{1}{N_e}\sum_{i=1}^{N_e} \mathbf{v}(\text{electron}\,\#\,i). \tag{77}$$

In metals, the drift velocity of the electron gas is much slower than the average speed of the random electron motion within the gas, but for electrons flying ballistically through the vacuum, the drift velocity is simply the velocity we found in part (b).

Anyhow, given the electron density n_e and the drift velocity $\mathbf{v}_d$, the current density is

$$\mathbf{J} \;=\; (-e)\, n_e\, \mathbf{v}_d\,, \tag{78}$$

or in terms of the electric charge density $\rho = (-e) \times n_e$,

$$\mathbf{J} \;=\; \rho\, \mathbf{v}_d\,. \tag{79}$$

For the problem at hand, the electrons fly in the positive x direction with drift velocity $\approx$ ballistic velocity $v(x)$, while the current flows in the opposite direction since the electrons are negatively charges. Also, the electron density and the velocity are uniform in the y and z directions, so the net current is simply

$$I \;=\; -J_x \times A \;=\; (-\rho) \times A \times v \tag{80}$$

where A is the electrodes' area. In a steady state, the overall current I must be the same all along the electron stream, thus

$$I \;=\; -\rho(x) \times A \times v(x) \quad \text{is independent of } x, \tag{81}$$

or in other words, the electrons cloud's charge density follows from its drift velocity as

$$-\rho(x) \;=\; \frac{\text{constant } I/A}{v(x)} \tag{82}$$

(d) For the electron cloud in a vacuum tube, the drift velocity of the electron cloud is simply the ballistic speed of an electron accelerated by the potential $V(x)$, namely

$$v(x) \;=\; \sqrt{\frac{2e}{m_e} \times V(x)}\,. \tag{76}$$

Plugging this velocity into eq. (82) relating the uniform current to the speed and charge density profiles, we obtain

$$-\rho(x) \;=\; \frac{I}{A} \times \sqrt{\frac{m_e}{2e}} \times \frac{1}{\sqrt{V(x)}}\,. \tag{83}$$

This formula gives us the charge density profile in terms of the electric potential $V(x)$.

On the other hand, the potential $V(x)$ obeys the Poisson equation, which in one dimension becomes simply

$$\frac{d^2}{dx^2}V(x) \;=\; \frac{-\rho(x)}{\epsilon_0}\,. \tag{74}$$

Combining this formula with eq. (83) for the $\rho(x)$, we end up with the differential equation

$$\frac{d^2}{dx^2}V(x) \;=\; \frac{I}{A}\times\frac{\sqrt{m_e/(2e)}}{\epsilon_0}\times\frac{1}{\sqrt{V(x)}} \tag{84}$$

for just the potential $V(x)$ — all the other variables have been eliminated.

(e) Now comes the hard part — solving the differential equation (84) for the potential. Its an ordinary differential equation rather than a PDE, but it's second-order and non-linear. Fortunately, it's not that hard to solve.

Let's start by combining all the constants in eq. (84) into a single constant

$$C \;=\; \frac{I}{A}\times\frac{\sqrt{m_e/(2e)}}{\epsilon_0}\,, \tag{85}$$

so the equation becomes simply

$$V''(x) \;=\; C\times V^{-1/2}(x). \tag{86}$$

Let's also write down the explicit boundary conditions for this equation. First, we set the potential at the cathode to zero, thus $V(x=0)=0$. Second, the problem tells us that the electric field of the electron cloud cancels the electric fields of the two electrodes *at the cathode*, thus $E(x=0)=0$, or in terms of the potential $V'(0)=0$. Altogether, we have two boundary conditions —

$$V(0) \;=\; 0,\quad V'(0) \;=\; 0, \tag{87}$$

— which is precisely the right number to uniquely determine the solution of the second-order equation (86).

Next, note the absence of any *explicit* dependence on x in the eq. (86), it simply relates the second derivative $V''(x)$ to the potential $V(x)$ at the same point x, whichever x it happens to be. For all such differential equations, there is a useful trick: write the first derivative dV/dx as some unknown function of the V itself,

$$\frac{dV}{dx} = g(V) \quad [\text{evaluated for } V(x)], \tag{88}$$

and then recast the original second-order equation (86) as a first-order equation for the $g(V)$. Indeed, by the chain rule

$$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}\, g\big(V(x)\big) = \frac{dg}{dV} \times \frac{dV}{dx} = \left(\frac{dg}{dV} \times g\right) @V(x)\,. \tag{89}$$

hence eq. (86) becomes

$$\left(\frac{dg}{dV} \times g\right) @V(x) = \frac{d^2V}{dx^2} = C \times V^{-1/2}(x), \tag{90}$$

or in terms of V and $g(V)$ regardless of x,

$$\frac{dg}{dV} \times g(V) = C \times V^{-1/2}. \tag{91}$$

To solve this first-order equation, we simply recast it in terms of the differentials dV and dg:

$$g \times dg = C \times V^{-1/2} \times dV, \tag{92}$$

$$d\Big(\tfrac{1}{2} g^2(V)\Big) = d\Big(2C \times \sqrt{V}\Big), \tag{93}$$

$$\text{hence } g^2(V) = 4C \times \sqrt{V} + \text{const.} \tag{94}$$

To determine the constant term in this generic solution, we need a boundary condition. According to the conditions (87), at the cathode $x = 0$ we have both $V = 0$ and $V' = 0$, or in terms of $g(V)$, both $V = 0$ and $g = 0$, thus

$$g(V = 0) \;=\; 0. \tag{95}$$

Plugging this condition into the general solution (94) sets the constant on the RHS to zero, thus

$$g(V) \;=\; \sqrt{4C} \times V^{1/4}. \tag{96}$$

Now let's plug this solution back into eq. (88) which relates $g(V)$ to the x-derivative of the potential:

$$\frac{dV(x)}{dx} \;=\; g(V) \;=\; \sqrt{4C} \times V^{1/4}(x). \tag{97}$$

This is a first-order equation for the $V(x)$, which we may solve by recasting it in terms of the differentials dV and dx:

$$\frac{dV}{V^{1/4}} \;=\; \sqrt{4C} \times dx, \tag{98}$$

$$d\Big(\tfrac{4}{3}\,V^{3/4}\Big) \;=\; d\Big(\sqrt{4C} \times x\Big), \tag{99}$$

$$V^{3/4}(x) \;=\; \tfrac{3}{4}\sqrt{4C} \times x \;+\; \text{const}. \tag{100}$$

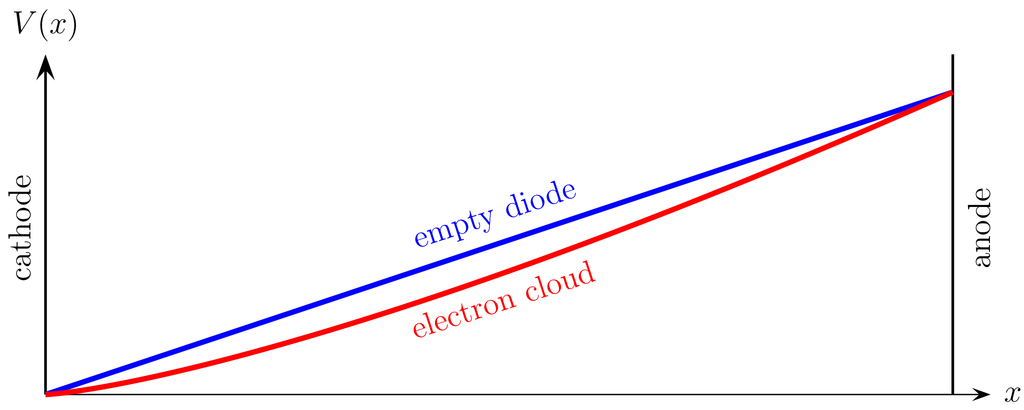
Finally, using the boundary condition $V = 0$ at $x = 0$, we set the constant in this general solution to zero and arrive at

$$\begin{aligned} V^{3/4}(x) \;&=\; \tfrac{3}{4}\sqrt{4C} \times x, \\ V(x) \;&=\; \big(\tfrac{9}{4}\,C\big)^{2/3} \times x^{4/3}. \end{aligned} \tag{101}$$

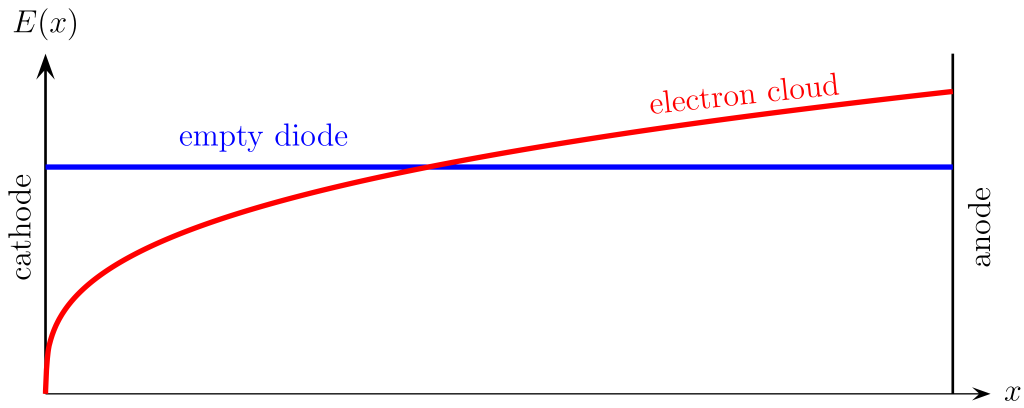
Note: Due to the space charge of the electron cloud, the potential inside the diode varies with the distance x from the cathode as $V \propto x^{4/3}$, so the electric field (in the negatives x direction) varies as

$$E(x) \;=\; -E_x(x) \;=\; +\frac{dV}{dx} \;\propto\; x^{1/3}. \tag{102}$$

By comparison, in an empty diode without the space charge, the electric field wold be constant while the potential would vary linearly with x. Here is the graphic illustration:



(103)



(104)

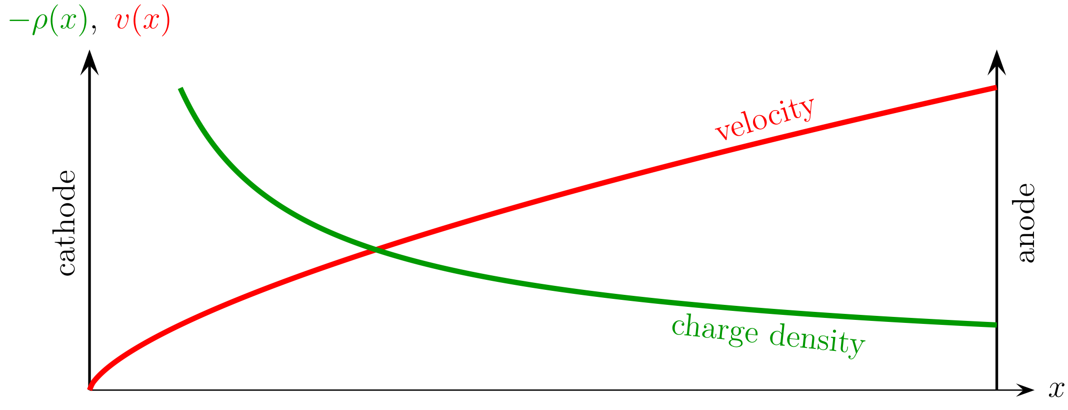
Finally, let me plot the velocity and the charge density of the electron cloud. Given $V(x) \propto x^{4/3}$, the velocity obtains from the energy equation (76):

$$v(x) \;\propto\; \sqrt{V(x)} \;\propto x^{2/3}, \tag{105}$$

and then the charge density follows from the constant current condition (82):

$$-\rho(x) \;\propto\; \frac{1}{v(x)} \;\propto x^{-2/3}. \tag{106}$$

Graphically,



(107)

(f) Given eq. (101) for the potential profile $V(x)$, let's plug in the anode point $x = d$ and demand that the potential there matches the anode potential V_0 — which is also the *voltage* between the anode at the cathode since we take $V(0) = 0$, —

$$@x = d, \quad V(d) \;=\; \left(\tfrac{9}{4}\,C\right)^{2/3} \times d^{4/3} \;=\; V_0 \tag{108}$$

Physically, this is the equation for the constant C in terms of the voltage V_0:

$$C \;=\; \frac{4V_0^{3/2}}{9d^2}\,. \tag{109}$$

At the same time, back in eq. (85) we have defined C in terms of the current I, the electrode area A an some fundamental constants as

$$C \;=\; \frac{I}{A} \times \frac{\sqrt{m_e/(2e)}}{\epsilon_0}\,, \tag{110}$$

Reconciling these two formulae for the C leads to a relation between the voltage on the diode and the current through it:

$$\frac{I}{A} \times \frac{\sqrt{m_e/(2e)}}{\epsilon_0} \;=\; \frac{4V_0^{3/2}}{9d^2}\,, \tag{111}$$

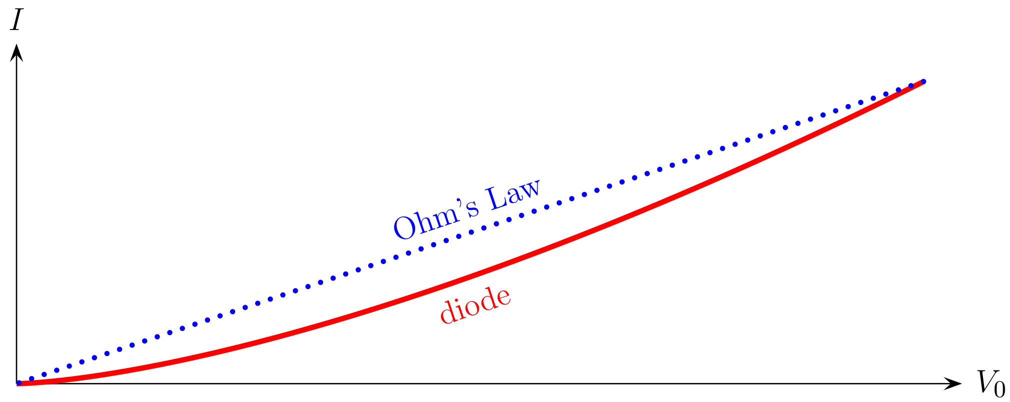
and hence

$$I \;=\; \mathcal{K} \times V_0^{3/2} \tag{2.56}$$

— the Child–Langmuir law — where the constant $\mathcal{K}$ is given by

$$\mathcal{K} \;=\; \frac{A}{d^2}\times\frac{4}{9}\,\epsilon_0\sqrt{\frac{2e}{m_e}} \;=\; \frac{A}{d^2}\times\left(2.33\cdot 10^{-6}\ \text{Ampere/Volt}^{3/2}\right). \tag{112}$$

Graphically, the current and the voltage are related as



(113)