Problem **2.34**:

(a) Back in homework #2 (problem **2.21**), we have found the electric potential of a uniformly charged solid ball of radius R:

$$\begin{aligned} \text{outside the ball,} \quad V(r) &= \frac{Q}{4\pi\epsilon_0} \times \frac{1}{r}\,, \\ \text{inside the ball,} \quad V(r) &= \frac{Q}{4\pi\epsilon_0} \times \frac{3R^2 - r^2}{2R^3}\,. \end{aligned} \tag{1}$$

The charge density inside the ball is

$$\rho = \frac{Q}{\frac{4\pi}{3}R^3} \tag{2}$$

while outside the ball $\rho = 0$. Consequently, the textbook equation (2.43) for the electrostatic potential energy of the charged ball yields

$$\begin{aligned} U &= \frac{1}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \rho(\mathbf{r}) \times V(\mathbf{r}) \times d^3\text{Vol} = \frac{1}{2} \iiint\limits_{\substack{\text{the}\\ \text{ball}}} \rho(\mathbf{r}) \times V(\mathbf{r}) \times d^3\text{Vol} \\ &= \frac{1}{2} \int\limits_0^R \frac{Q}{\frac{4\pi}{3}R^3} \times \frac{Q}{4\pi\epsilon_0} \frac{3R^2 - r^2}{2R^3} \times 4\pi r^2\, dr \\ &= \frac{3Q^2}{16\pi\epsilon_0 R^6} \times \int\limits_0^R (3R^2 r^2 - r^4)\, dr \\ &= \frac{3Q^2}{16\pi\epsilon_0 R^6} \times \left(3R^2 \times \frac{R^3}{3} - \frac{R^5}{5} = \frac{4R^5}{5} \right) \\ &= \frac{3}{20\pi} \times \frac{Q^2}{\epsilon_0 R}\,. \end{aligned} \tag{3}$$

(b) Another way to obtain the electrostatic potential energy of any continuous charge system is to integrate the square of the electric field over the whole space: According to the textbook equation (2.45),

$$U = \frac{\epsilon_0}{2} \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{E}^2(\mathbf{r})\, d^3\text{Vol}\,. \tag{4}$$

For the solid ball in question, the electric field is

$$\begin{aligned} \text{outside the ball,} \quad \mathbf{E}(\mathbf{r}) &= \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}\, \hat{\mathbf{r}}\,, \\ \text{inside the ball,} \quad \mathbf{E}(\mathbf{r}) &= \frac{Q}{4\pi\epsilon_0} \times \frac{r}{R^3}\, \hat{\mathbf{r}}\,. \end{aligned} \tag{5}$$

Consequently,

$$\begin{aligned} \iiint\limits_{\substack{\text{inside}\\\text{the ball}}} \mathbf{E}^2(\mathbf{r})\, d^3\text{Vol} &= \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \times \int\limits_0^R \frac{r^2}{R^6} \times 4\pi r^2\, dr \\ &= \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \times \frac{4\pi}{R^6} \times \int\limits_0^R r^4\, dr \\ &= \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \times \frac{4\pi}{R^6} \times \frac{R^5}{5} \\ &= \frac{1}{20\pi} \times \frac{Q^2}{\epsilon_0^2\, R}\,, \end{aligned} \tag{6}$$

while

$$\begin{aligned} \iiint\limits_{\substack{\text{outside}\\\text{the ball}}} \mathbf{E}^2(\mathbf{r})\, d^3\text{Vol} &= \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \times \int\limits_R^\infty \frac{1}{r^4} \times 4\pi r^2\, dr \\ &= \frac{Q^2}{4\pi\epsilon_0^2} \times \int\limits_R^\infty \frac{dr}{r^2} \\ &= \frac{Q^2}{4\pi\epsilon_0^2} \times \frac{1}{R}\,. \end{aligned} \tag{7}$$

Altogether,

$$
\begin{aligned}
\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}^2(\mathbf{r})\, d^3\mathrm{Vol} \;&=\; \iiint\limits_{\substack{\text{inside}\\ \text{the ball}}} \mathbf{E}^2(\mathbf{r})\, d^3\mathrm{Vol} \;+\; \iiint\limits_{\substack{\text{outside}\\ \text{the ball}}} \mathbf{E}^2(\mathbf{r})\, d^3\mathrm{Vol} \\
&=\; \frac{1}{20\pi} \times \frac{Q^2}{\epsilon_0^2\, R} \;+\; \frac{1}{4\pi} \times \frac{Q^2}{\epsilon_0^2\, R} \\
&=\; \frac{6}{20\pi} \times \frac{Q^2}{\epsilon_0^2\, R}\,,
\end{aligned}
\tag{8}
$$

and consequently the potential energy

$$
\begin{aligned}
U \;&=\; \frac{\epsilon_0}{2} \times \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}^2(\mathbf{r})\, d^3\mathrm{Vol} \\
&=\; \frac{\epsilon_0}{2} \times \frac{6}{20\pi} \times \frac{Q^2}{\epsilon_0^2\, R} \\
&=\; \frac{3}{20\pi} \times \frac{Q^2}{\epsilon_0\, R}\,.
\end{aligned}
\tag{9}
$$

Note the agreement of this energy with the energy calculated in part (a).

(c) The textbook equation (2.44) gives us yet another way of calculating the electrostatic potential energy. This time, we integrate the $\mathbf{E}^2$ only over the volume occupied by the charges but we also add a surface term,

$$
U \;=\; \frac{\epsilon_0}{2} \left(\iiint\limits_{\mathcal{V}} \mathbf{E}^2\, d^3\mathrm{Vol} \;+\; \iint\limits_{\mathcal{S}} V\mathbf{E} \cdot \mathbf{d^2A} \right). \tag{2.44}
$$

For the problem at hand, $\mathcal{V}$ is the solid ball while $\mathcal{S}$ is its spherical surface. At that surface,

$$
V \;=\; V(R) \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{R}\,, \quad \mathbf{E} \;=\; E(R)\,\hat{\mathbf{r}} \quad \text{where} \quad E(R) = \frac{Q}{4\pi\epsilon_0} \times \frac{1}{R^2}\,. \tag{10}
$$

thus $V\mathbf{E}$ has a constant magnitude over the sphere and its direction is always $\perp$ to the sphere. Consequently, the surface integral becomes simply

$$\iint\limits_{\text{sphere}} V\mathbf{E}\cdot\mathbf{d^2A} \;=\; V(R)\times E(R)\times\text{Area(sphere)} \;=\; \frac{Q}{4\pi\epsilon_0\,R}\times\frac{Q}{4\pi\epsilon_0\,R^2}\times 4\pi R^2 \;=\; \frac{Q^2}{4\pi\epsilon_0^2\,R}\,. \tag{11}$$

As to the volume integral over the ball, we have already computed it in eq. (6). Thus, altogether

$$\iiint\limits_{\text{ball}} \mathbf{E}^2\,d^3\text{Vol} \;+\; \iint\limits_{\text{sphere}} V\mathbf{E}\cdot\mathbf{d^2A} \;=\; \frac{1}{20\pi}\times\frac{Q^2}{\epsilon_0^2\,R} \;+\; \frac{1}{4\pi}\times\frac{Q^2}{\epsilon_0^2\,R} \;=\; \frac{6}{20\pi}\times\frac{Q^2}{\epsilon_0^2\,R}\,, \tag{12}$$

and therefore eq. (2.44) yields

$$U \;=\; \frac{\epsilon_0}{2}\times\frac{6}{20\pi}\times\frac{Q^2}{\epsilon_0^2\,R} \;=\; \frac{3}{20\pi}\times\frac{Q^2}{\epsilon_0\,R}\,. \tag{13}$$

Again, this energy agrees with the results of parts (a) and (b).

Problem **2.35**:

Let's build up the charged solid ball one infinitesimally thin spherical shell at a time by bringing the net charge of that shell from infinitely far away. Throughout the process, the volume charge density of the ball we are building is held to constant

$$\rho \;=\; \frac{Q_{\text{ultimate}}}{\frac{4\pi}{3}R^3_{\text{ultimate}}} \tag{14}$$

Consider the shell of radius r and thickness dr. The net charge of this shell is

$$dQ \;=\; \rho\times 4\pi r^2\,dr. \tag{15}$$

At the time we assemble this shell, the ball has net charge

$$Q(r) \;=\; \frac{4\pi r^3}{3}\times\rho\,, \tag{16}$$

hence the electrostatic potential at the ball's surface relative to the infinity is

$$V(\text{surface}) \;-\; V(\infty) \;=\; \frac{Q(r)}{4\pi\epsilon_0\,r} \;=\; \frac{\rho r^2}{3\epsilon_0}\,. \tag{17}$$

Consequently, bringing the extra charge dQ from the infinity to the surface of the ball takes work

$$dW = dQ \times \Big(V(\text{surface}) - V(\infty)\Big) = 4\pi\rho r^2\,dr \times \frac{\rho r^2}{3\epsilon_0} = \frac{4\pi\rho^2}{3\epsilon_0} \times r^4\,dr. \tag{18}$$

The net work for building the whole ball of radius $R_{\text{ultimate}} = R$ obtains by integrating this formula

$$\begin{aligned} W_{\text{net}} &= \int dW \\ &= \int\limits_0^R \frac{4\pi\rho^2}{3\epsilon_0} \times r^4\,dr \\ &= \frac{4\pi\rho^2}{3\epsilon_0} \times \frac{R^5}{5}\,. \end{aligned} \tag{19}$$

Rewriting this formula in terms of the net charge $Q_{\text{ultimate}} = Q$ of the completed ball, we obtain

$$W_{\text{net}} = \frac{4\pi R^5}{15\epsilon_0} \times \left(\rho = \frac{3Q}{4\pi R^3}\right)^2 = \frac{3}{20\pi} \times \frac{Q^2}{\epsilon_0\,R}\,. \tag{20}$$

Finally, the electrostatic potential energy of the ball is precisely the net work of assembling the ball, that is, starting with infinitesimal pieces spread out at the infinity and moving them inward against the electrostatic repulsion forces. Thus,

$$U = W_{\text{net}} = \frac{3}{20\pi} \times \frac{Q^2}{\epsilon_0\,R}\,. \tag{21}$$

By inspection, this energy agrees with what we had calculated in problem **2.34** by three different methods.

<u>Problem **2.37**</u>:

First, let me remind you a bit of theory I explained in class on Thursday 2/8.

Consider two charges, Q_1 and Q_2, not necessarily point-like. By itself, Q_1 would create the electric field $\mathbf{E}_1(\mathbf{r})$, and likewise, Q_2 by itself would create the field $\mathbf{E}_2(\mathbf{r})$. By the superposition principle, the net field of the two charges is

$$\mathbf{E}(\mathbf{r}) \;=\; \mathbf{E}_1(\mathbf{r}) \;+\; \mathbf{E}_2(\mathbf{r}), \tag{22}$$

so the net electrostatic energy of the system is

$$U_{\text{net}} \;=\; \frac{\epsilon_0}{2}\iiint \mathbf{E}^2(\mathbf{r})\,d^3\text{Vol} \;=\; \frac{\epsilon_0}{2}\iiint \big(\mathbf{E}_1^2 \,+\, \mathbf{E}_2^2 \,+\, 2\mathbf{E}_1\cdot\mathbf{E}_2\big)\,d^3\text{Vol} \tag{23}$$

where the volume integrals are over the whole space. In other words,

$$U_{\text{net}} \;=\; U_1^{\text{self}} \;+\; U_2^{\text{self}} \;+\; U_{12}^{\text{int}} \tag{24}$$

where

$$U_1^{\text{self}} \;=\; \frac{\epsilon_0}{2}\iiint \mathbf{E}_1^2\,d^3\text{Vol}, \quad U_2^{\text{self}} \;=\; \frac{\epsilon_0}{2}\iiint \mathbf{E}_2^2\,d^3\text{Vol}, \tag{25}$$

$$U_{12}^{\text{int}} \;=\; \epsilon_0\iiint \mathbf{E}_1\cdot\mathbf{E}_2\,d^3\text{Vol}. \tag{26}$$

Physically, the U_1^{self} is the self-interaction of the first charge — that is, the work of assembling that charge from infinitesimal bits. Likewise, the U_2^{self} is the self-interaction energy of the second charge. Finally, the U_{12}^{int} is the energy due to electrostatic forces between the two charges, regardless of the self-energy of the two charges themselves. That is, suppose we have already assembled the charges Q_1 and Q_2, but we keep them infinitely far away from each other. Then the U_{12}^{int} energy stores the work of bringing these two charges from ∞ to their ultimate locations near each other

For the point charges Q_1 and Q_2, the electrostatic self-energies U_1^{self} and U_2^{self} are infinite, but the interaction energy should be finite; specifically, we should have

$$U_{12}^{\text{int}} \;=\; \frac{Q_1Q_2}{4\pi\epsilon_0\,R_{12}} \tag{27}$$

where R_{12} is the distance between the two charges. The purpose of this exercise is to show that the interaction energy defined according to eq. (27) indeed agrees with this formula.

And now, let's calculate. Let's choose our coordinate system such that the first charge Q_1 is at the origin, $\mathbf{r}_1 = \mathbf{0}$, while the second charge sits on the z axis, $\mathbf{r}_2 = (0, 0, z_2)$ where $z_2 = +R_{12}$, the distance between the charges. Then at some generic point $\mathbf{r}$, the electric fields are

$$\mathbf{E}_1(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3}, \quad \mathbf{E}_2(\mathbf{r}) = \frac{Q_2}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_2}{|\mathbf{r} - \mathbf{r}_2|^3}, \tag{28}$$

hence

$$\epsilon_0 \, \mathbf{E}_1 \cdot \mathbf{E}_2 = \frac{Q_1 Q_2}{16\pi^2 \epsilon_0} \frac{\mathbf{r}^2 - \mathbf{r} \cdot \mathbf{r}_2}{|\mathbf{r}|^2 \, |\mathbf{r} - \mathbf{r}_2|^3}. \tag{29}$$

In spherical coordinates (r, θ, ϕ) for the $\mathbf{r}$, we have

$$\begin{aligned} \mathbf{r}^2 &= r^2, \\ \mathbf{r}^2 - \mathbf{r} \cdot \mathbf{r}_2 &= r^2 - r\cos\theta \times z_2 = r(r - z_2 \cos\theta), \\ |\mathbf{r} - \mathbf{r}_2|^2 &= \mathbf{r}^2 + \mathbf{r}_2^2 - 2\mathbf{r} \cdot \mathbf{r}_2 = r^2 + z_2^2 - 2r z_2 \cos\theta, \end{aligned} \tag{30}$$

hence

$$\frac{\mathbf{r}^2 - \mathbf{r} \cdot \mathbf{r}_2}{|\mathbf{r}|^2 \, |\mathbf{r} - \mathbf{r}_2|^3} = \frac{r(r - z_2 \cos\theta)}{r^3 (r^2 + z_2^2 - 2r z_2 \cos\theta)^{3/2}} = \frac{-1}{r^2} \times \frac{\partial}{\partial r} \frac{1}{\sqrt{r^2 + z_2^2 - 2r z_2 \cos\theta}}. \tag{31}$$

Plugging this formula into eq. (29) and integrating over the whole space, we obtain

$$\begin{aligned} U_{12}^{\text{int}} &= \iiint \epsilon_0 \, \mathbf{E}_1 \cdot \mathbf{E}_2 \, d^3\text{Vol} \\ &= \int_0^\infty dr \, r^2 \iint d^2\Omega(\theta, \phi) \, \frac{Q_1 Q_2}{16\pi^2 \epsilon_0} \times \frac{-1}{r^2} \times \frac{\partial}{\partial r} \frac{1}{\sqrt{r^2 + z_2^2 - 2r z_2 \cos\theta}} \\ &= -\frac{Q_1 Q_2}{16\pi^2 \epsilon_0} \iint d^2\Omega(\theta, \phi) \int_0^\infty dr \, \frac{\partial}{\partial r} \frac{1}{\sqrt{r^2 + z_2^2 - 2r z_2 \cos\theta}}. \end{aligned} \tag{32}$$

Let's do the radial integral first. For any fixed (θ, ϕ) we obtain

$$\begin{aligned} \int_0^\infty dr \, \frac{\partial}{\partial r} \frac{1}{\sqrt{r^2 + z_2^2 - 2r z_2 \cos\theta}} &= \left. \frac{1}{\sqrt{r^2 + z_2^2 - 2r z_2 \cos\theta}} \right|_{r=0}^{r=\infty} \\ &= \frac{1}{\sqrt{\infty}} - \frac{1}{\sqrt{z_2^2}} = -\frac{1}{z_2} = -\frac{1}{R_{12}}, \end{aligned} \tag{33}$$

regardless of the angular coordinates (θ, ϕ). Consequently,

$$\begin{aligned} U_{12}^{\text{int}} &= -\frac{Q_1 Q_2}{16\pi^2\epsilon_0} \iint d^2\Omega \left(\frac{-1}{R_{12}} = \text{const}\right) = +\frac{Q_1 Q_2}{16\pi^2\epsilon_0\, R_{12}} \times \iint d^2\Omega \\ &= +\frac{Q_1 Q_2}{16\pi^2\epsilon_0\, R_{12}} \times 4\pi = +\frac{Q_1 Q_2}{4\pi\epsilon_0\, R_{12}}\,, \end{aligned} \tag{34}$$

in perfect agreement with eq. (27). *Quod erat demonstrandum.*

Problem **2.60**:

The conducting shell is neutral on the whole, but its inner and outer surfaces carry charges induced by the point charge inside the inner cavity. Specifically, for the point charge q being at the center of the shell, there is induced charge $-q$ uniformly distributed over the inner surface of the shell, and also induced charge $+q$ uniformly distributed over the outer surface; in terms of surface charge densities,

$$\sigma^{\text{inner}} = -\frac{q}{4\pi a^2} = \text{const}, \quad \sigma^{\text{outer}} = +\frac{q}{4\pi b^2} = \text{const}. \tag{35}$$

But when we move the point charge to ∞ through a tiny hole in the shell, the induced charge $-q$ on the inner surface flows towards the hole, then flows along the hole's surface to the outer surface, and eventually cancels the $+q$ charge that used to be there. By the time the point charge approaches the ∞, the induced charges on both surfaces of the shell vanish altogether.

By the energy conservation, the work of moving the point charge to ∞ is due to removal of the potential energy of interactions between the point charge with the induced charges on the shell, and also between the induced charges themselves. In terms of the net potential energies,

$$\begin{aligned} W &= U[\text{point charge by itself}] - U[\text{point charge plus induced charges}] \\ &= -U[\text{everything except point charge's self-interaction}] \end{aligned} \tag{36}$$

where the second line stems from the point charge's self-interaction not caring whether that charge is inside the shell or by itself at ∞.

Let me give two ways for calculating the energy difference (36). **Method #1:**

$$W = -U + U^{\rm self}_{\rm point\,charge} = -\frac{1}{2}\sum_{i,j}{}' Q_i \times V_j@(i) \tag{37}$$

where i runs over the 3 charged objects in the system — the point charge, the inner surface of the shell, and the outer surface, — and the prime on the sum indicates skipping the self-interaction of the point charge but including the self-interactions of the shell's surfaces. Specifically,

$$\begin{aligned} W = -\Big(& Q_{\rm inner} \times V_{\rm point}(a) + Q_{\rm outer} \times V_{\rm point}(b) + Q_{\rm outer} \times V_{\rm inner}(b) \\ & + \frac{1}{2} Q_{\rm inner} \times V_{\rm inner}(a) + \frac{1}{2} Q_{\rm outer} \times V_{\rm outer}(b) \Big). \end{aligned} \tag{38}$$

In this sum,

$$\begin{aligned} Q_{\rm inner} \times V_{\rm point}(a) &= -\frac{q^2}{4\pi\epsilon_0} \times \frac{1}{a}\,, \\ Q_{\rm outer} \times V_{\rm point}(b) &= +\frac{q^2}{4\pi\epsilon_0} \times \frac{1}{b}\,, \\ Q_{\rm outer} \times V_{\rm inner}(b) &= -\frac{q^2}{4\pi\epsilon_0} \times \frac{1}{b}\,, \\ Q_{\rm inner} \times V_{\rm inner}(a) &= +\frac{q^2}{4\pi\epsilon_0} \times \frac{1}{a}\,, \\ Q_{\rm outer} \times V_{\rm outer}(a) &= +\frac{q^2}{4\pi\epsilon_0} \times \frac{1}{b}\,, \end{aligned} \tag{39}$$

so assembling all the terms together, we get

$$W = -\frac{q^2}{4\pi\epsilon_0} \times \left(\frac{-1}{a} + \frac{+1}{b} + \frac{-1}{b} + \frac{1}{2} \times \frac{+1}{a} + \frac{1}{2} \times \frac{1}{b} \right) = +\frac{q^2}{4\pi\epsilon_0} \times \left(\frac{1}{2a} - \frac{1}{2b} \right). \tag{40}$$

Note the positive sign of this net work: putting the charge inside the conducting shell reduces the overall potential energy, so removing the charge back to ∞ increases the energy back to its original value.

Method #2: Use

$$U = \frac{\epsilon_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}^2 \, d^3\text{Vol}, \tag{41}$$

hence the difference between the energy of an isolated point charge and a point charge inside the conducting shell is

$$W = \frac{\epsilon_0}{2} \iiint\limits_{\substack{\text{whole}\\ \text{space}}} \Big(\mathbf{E}^2_{\text{standalone}}(\mathbf{r}) - \mathbf{E}^2_{\text{in shell}}(\mathbf{r}) \Big) d^3\text{Vol}. \tag{42}$$

For simplicity, instead of moving the charge from inside the shell to ∞, let's keep the charge fixed at the origin while we move the shell in the opposite direction: from centered at the origin to centered at the ∞. For the shell centered on the charge,

$$\mathbf{E}^2(\mathbf{r}) = \begin{cases} \left(\dfrac{q}{4\pi\epsilon_0} \dfrac{1}{r^2}\right)^2 & \text{outside the shell, } r > b, \\ 0 & \text{within the shell, } a < r < b, \\ \left(\dfrac{q}{4\pi\epsilon_0} \dfrac{1}{r^2}\right)^2 & \text{inside the shell, } r < a, \end{cases} \tag{43}$$

while after we remove the shell to infinity

$$\mathbf{E}^2(\mathbf{r}) = \left(\frac{q}{4\pi\epsilon_0} \frac{1}{r^2}\right)^2 \quad \text{everywhere.} \tag{44}$$

Consequently, subtracting the $\mathbf{E}^2$ for the two situations at similar positions $\mathbf{r}$ relative to the point charge, we have

$$\Big(\mathbf{E}^2_{\text{standalone}}(\mathbf{r}) - \mathbf{E}^2_{\text{in shell}}(\mathbf{r}) \Big) = \begin{cases} \left(\dfrac{q}{4\pi\epsilon_0} \dfrac{1}{r^2}\right)^2 & \text{within the shell, } a < b < b, \\ 0 & \text{everywhere else.} \end{cases} \tag{45}$$

Hence, plugging this difference into the integral (42), we have

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \iiint\limits_{\substack{\text{within the}\\ \text{shell only}}} \left(\frac{q}{4\pi\epsilon_0}\,\frac{1}{r^2}\right)^2 \times d^3\text{Vol} \\ &= \frac{q^2}{32\pi^2\epsilon_0} \times \int\limits_a^b \frac{1}{r^4} \times 4\pi r^2\,dr \\ &= \frac{q^2}{8\pi\epsilon_0} \times \left(\frac{1}{a} - \frac{1}{b}\right). \end{aligned} \tag{46}$$

<u>Problem **2.43**</u>:

Consider a long piece (length $L \gg a, b$) of the coaxial tubes in question. Suppose the inner tube has charge $+Q$ while the outer tube has charge $-Q$ (both, within the length L). Then, by the Gauss Law, the electric field vanishes inside the inner tube or outside the outer tube, while between the tubes the field is

$$\mathbf{E} = \frac{Q/L}{2\pi\epsilon_0}\,\frac{\hat{\mathbf{s}}}{s} \tag{47}$$

where s is the cylindrical radial coordinate. Integrating this field between the two tubes, we obtain the potential difference

$$V = V_{\text{outer}} - V_{\text{inner}} = \int\limits_a^b ds\,\hat{\mathbf{s}}\cdot\mathbf{E} = \frac{Q/L}{2\pi\epsilon_0} \times \int\limits_a^b \frac{ds}{s} = \frac{Q/L}{2\pi\epsilon_0} \times \ln\frac{b}{a}\,. \tag{48}$$

Note that this potential difference — *i.e.*, the voltage on the capacitor — is proportional to the stored charge Q, so we can recast this formula in terms of the capacitance C:

$$\frac{1}{C} = \frac{V}{Q} = \frac{1/L}{2\pi\epsilon_0} \times \ln\frac{b}{a}\,, \tag{49}$$

hence

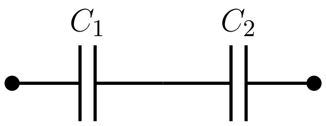
$$C = L \times \frac{2\pi\epsilon_0}{\ln(b/a)}\,. \tag{50}$$

Numerically, $2\pi\epsilon_0 = 56 \cdot 10^{-12}$ F/m $= 56$ pF/m, so the capacitance per unit length of the two coaxial tubes is

$$\frac{C}{L} = \frac{56\ \mathrm{pF/m}}{\ln(b/a)}. \tag{51}$$

<u>The non-textbook problem</u>:

(a) In the serial circuit



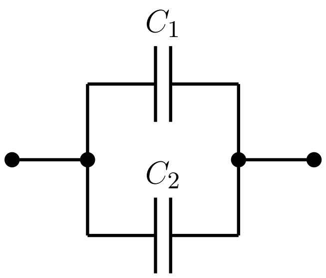
the same time-dependent current $I(t)$ flows through both capacitors, so they acquire the same charge $Q = \int I\,dt$. Likewise, when the capacitors discharge, the same charge flows from both capacitors, so the charge flowing through the outside wire is Q rather than $2Q$. As to the voltages of the charged capacitors, the first capacitor has $V_1 = Q/C_1$, the second capacitor has $V_2 = Q/C_2$, and in the serial circuit these two voltages add up to the net voltage between the outside wires

$$V_{\text{net}} = V_1 + V_2 = Q \times \left(\frac{1}{C_1} + \frac{1}{C_2}\right). \tag{52}$$

Hence, the equivalent capacitance of the serial circuit is

$$C_{\text{serial}} = \frac{Q}{V_{\text{net}}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \frac{C_1 C_2}{C_1 + C_2}. \tag{53}$$

On the other hand, in the parallel circuit



The currents through the two capacitors are separate so they acquire separate charges Q_1 and Q_2. Moreover, when the capacitors discharge, their charges flow separately through the outside wires, thus

$$Q_{\text{net}} = Q_1 + Q_2 \,. \tag{54}$$

Also, in the parallel circuit the voltages on the two capacitors are equal to each other and to the outside voltage,

$$V_1 = V_2 = V. \tag{55}$$

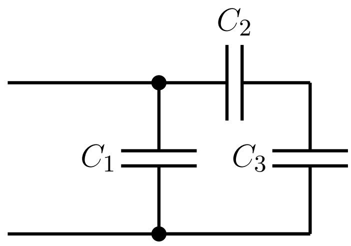
Consequently, the charges of the two capacitors are $Q_1 = C_1 \times V$, $Q_2 = C_2 \times V$, and the net charge is

$$Q_{\text{net}} = Q_1 + Q_2 = (C_1 + C_2) \times V. \tag{56}$$

Thus, the equivalent capacitance of the parallel circuit is

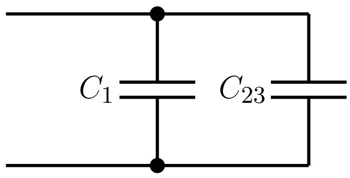
$$C_{\text{parallel}} = \frac{Q_{\text{net}}}{V} = C_1 + C_2 \,. \tag{57}$$

For more complicated — but finite — capacitor circuits, the equivalent capacitance follows by recursive application of eqs. (53) and (57) via the *subcircuit rule: Any subcircuit may be replaced with a single capacitor of the same equivalent capacitance.* For example, in the three-capacitor circuit



(58)

we may replace the serial subcircuit of C_2 and C_3 with a single capacitor

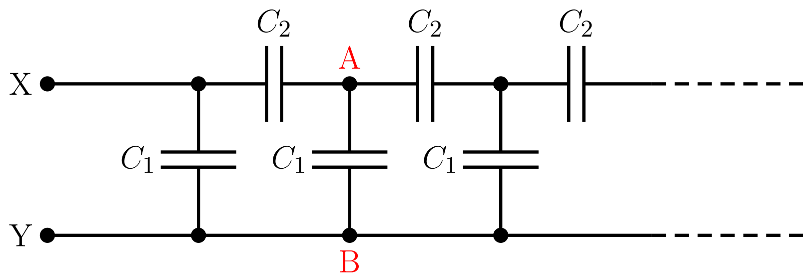


$$C_{23} = \frac{C_2 C_3}{C_2 + C_3}$$

and then the whole circuit simplifies to a parallel circuit of net capacitance

$$C_{\text{net}} = C_1 + C_{23} = C_1 + \frac{C_2 C_3}{C_2 + C_3} \tag{59}$$

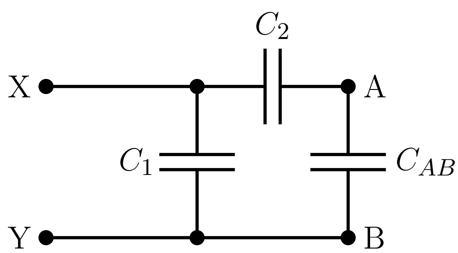
(b) Now consider the infinite ladder circuit



The key to solving this circuit is its very infinity, which means that the sub-circuit comprising everything to the right of the first capacitor pair — *i.e.*, everything to the right of points A and B — is completely equivalent to the whole circuit. Consequently, the equivalent capacity of the complete circuit between points X and Y is equal to the equivalent capacity of the subcircuit to the right of A and B,

$$C_{XY} = C_{AB}\,. \tag{60}$$

To make use of this relation, we need an independent relation between the C_{CY} and C_{AB} capacitances, and we can get it from the subcircuit rule. Indeed, let's replace the entire infinite subcircuit to the right of the points A and B with a single capacitor of capacitance C_{AB}, whatever that capacitance happens to be, thus



(61)

The resulting 3-capacitor circuit looks just like the example (58) I made at the end of part (a), so its equivalent capacitance obtains similarly to eq. (59), namely

$$C_{XY} = C_1 + \frac{C_2 C_{AB}}{C_2 + C_{AB}}\,. \tag{62}$$

Together, eqs. (60) and (62) give us two algebraic relations between two unknown capacitances C_{XY} and C_{AB}. To solve them for the C_{XY} we plug eq. (60) into eq. (62) to obtain

$$C_{XY} \;=\; C_1 \;+\; \frac{C_2 C_{XY}}{C_2 + C_{XY}}\,, \tag{63}$$

hence

$$(C_{XY} - C_1) \times (C_2 + C_{XY}) \;=\; C_2 C_{XY}\,, \tag{64}$$

and therefore

$$C_{XY}^2 \;-\; C_1 \times C_{XY} \;-\; C_1 C_2 \;=\; 0. \tag{65}$$

This quadratic equation has a unique positive root, thus

$$C_{XY} \;=\; \frac{C_1 \;+\; \sqrt{C_1^2 + 4C_1C_2}}{2}\,. \tag{66}$$

In particular, for $C_1 = 1\,\mu\mathrm{F}$ and $C_2 = 2\,\mu\mathrm{F}$ the answer is $C_{XY} = 2\,\mu\mathrm{F}$.

<u>Problem **3.3**</u>:

Preamble: The way various differential operators work in the spherical or cylindrical coordinates is explained in the textbook section §1.4. Specifically, the Laplacian operator in such coordinates acts according to the textbook equations (1.73) and (1.82), namely

$$\begin{aligned}\triangle F(r,\theta,\phi) \;&=\; \frac{1}{r^2} \times \frac{\partial}{\partial r}\left(r^2 \times \frac{\partial F}{\partial r}\right) \;+\; \frac{1}{r^2\,\sin\theta} \times \frac{\partial}{\partial\theta}\left(\sin\theta \times \frac{\partial F}{\partial\theta}\right)\\ &\quad+\; \frac{1}{r^2\,\sin^2\theta} \times \frac{\partial^2 F}{\partial\phi^2}\,,\end{aligned} \tag{1.73}$$

$$\triangle F(s,\phi,z) \;=\; \frac{1}{s} \times \frac{\partial}{\partial s}\left(s \times \frac{\partial F}{\partial s}\right) \;+\; \frac{1}{s^2} \times \frac{\partial^2 F}{\partial\phi^2} \;+\; \frac{\partial^2 F}{\partial z^2}\,. \tag{1.82}$$

In particular, for the spherically symmetric functions which in spherical coordinates depend only on the radius r,

$$\triangle F(r\text{ only}) \;=\; \frac{1}{r^2} \times \frac{d}{dr}\left(r^2 \times \frac{dF}{dr}\right) \;=\; \frac{d^2 F}{dr^2} \;+\; \frac{2}{r} \times \frac{dF}{dr}\,. \tag{67}$$

Likewise, for the axially symmetric functions which in cylindrical coordinates depend only on s,

$$\triangle F(s\text{ only}) \;=\; \frac{1}{s}\times\frac{d}{ds}\left(s\times\frac{dF}{ds}\right) \;=\; \frac{d^2F}{ds^2}\;+\;\frac{1}{s}\times\frac{dF}{ds}\,. \tag{68}$$

(a) Suppose a spherically symmetric potential $V(r\,\text{only})$ obeys the Laplace equation $\triangle V(r)\equiv 0$, hence in light of eq. (67)

$$\triangle V(r) \;=\; V''(r)\;+\;\frac{2}{r}\times V'(r) \;=\; 0. \tag{69}$$

To find the general solution of this equations, we start with the derivative $V'(r)=dV/dr$ which obeys

$$\frac{dV'}{dr}\;+\;\frac{2V'}{r}\;=\;0. \tag{70}$$

To solve this first-order equation, we recast it in terms of the differentials dV' and dr, thus

$$\begin{aligned} \frac{dV'}{V'} \;&=\; -2\frac{dr}{r}\,, \\ d\big(\ln V'\big) \;&=\; d\big(-2\ln r\big), \\ \ln V'(r) \;&=\; \text{const}\;-\;2\ln(r), \\ V'(r) \;&=\; \frac{\text{const}}{r^2} \end{aligned} \tag{71}$$

Let's call the constant in the last formula here $-A$.

Now, given the $V'(r)$, solving for the $V(r)$ itself is just the matter of integration:

$$\begin{aligned} \frac{dV}{dr} \;&=\; \frac{-A}{r^2}\,, \\ V(r) \;&=\; \int\frac{-A}{r^2}\,dr \;=\; \frac{A}{r}\;+\;B, \end{aligned} \tag{72}$$

where B some other constant.

Altogether, the most general spherically symmetric potential — in any spherical shell without the electric charges in it — has form

$$V(r) \;=\; \frac{A}{r}\;+\;B \tag{73}$$

for some constants A and B.

(b) Now consider an axially symmetric electric potential $V(s)$. In any interval of S where there are no electric charges, the potential obeys the Laplace equation $\triangle V(s) \equiv 0$, or in light or eq. (68),

$$\triangle V(s) \;=\; V''(s) \;+\; \frac{1}{s} \times V'(s) \;=\; 0. \tag{74}$$

To find the most general solution to this equation, we first rewrite it as a first-order equation for the derivative $V'(s) = dV/ds$,

$$\frac{dV'}{ds} \;+\; \frac{V}{s} \;=\; 0. \tag{75}$$

In terms of the differentials dV' and ds, this equation becomes

$$\begin{aligned} \frac{dV'}{V'} \;&=\; -\frac{ds}{s}\,, \\ d\bigl(\ln V'\bigr) \;&=\; d\bigl(-\ln s\bigr), \\ \ln V'(s) \;&=\; -\ln s \;+\; \text{const}, \\ V'(s) \;&=\; \frac{\text{const}}{\text{s}}\,. \end{aligned} \tag{76}$$

It remains to give this constant a name — say A — and integrate to get the $V(s)$ itself:

$$V(s) \;=\int V'(s)\,ds \;=\int \frac{A}{s}\,ds \;=\; A \times \ln(s) \;+\; \text{const}. \tag{77}$$

Altogether, the most general axi-symmetric potential — in the interval of s where there are no electric charges — have form

$$V(s) \;=\; A \times \ln(s) \;+\; B \tag{78}$$

for some constants A and B.

Problem **3.1**:

All the hard calculations you need for this problem are explained in detail in my notes on electrostatic theorems, pages 6–9. In particular, eqs. (30) and (31) give the mean potential over a sphere due to a point charge: For the charge outside the sphere

$$V_{\text{mean}} = \frac{Q}{4\pi\epsilon_0} \times \frac{1}{r_q} = V(\text{center}) \tag{79}$$

while for the charge inside the sphere

$$V_{\text{mean}} = \frac{Q}{4\pi\epsilon_0} \times \frac{1}{R} \neq V(\text{center}). \tag{80}$$

Note however that for the charge inside the sphere, its contribution to the average potential does not depend on r_q as long as $r_q < R$: it does not matter where exactly we put the charge inside the sphere as long as it's inside. Consequently, for any number of charges inside the sphere — or for any continuous charge inside the sphere — their net contribution to the average potential is simply

$$V_{\text{mean}} = \frac{Q_{\text{net}}}{4\pi\epsilon_0 R}. \tag{81}$$

At the same time, the contribution of any outside charges to the average potential equals to the contribution of the same charges to the potential at the center. Thus altogether,

$$V_{\text{mean}}[\text{sphere}] = V^{\text{due to outside charges}}(\text{center}) + \frac{Q^{\text{net}}_{\text{inside}}}{4\pi\epsilon_0 R}. \tag{82}$$

Problem **3.4**:

First, let me relate the average electric field *vector* over a sphere to the averaged potential over the same sphere. To set up the notations, let $\mathbf{c}$ be the radius-vector of the sphere's center. Then a generic point on the sphere's surface is located at

$$\mathbf{r} \;=\; \mathbf{c} \;+\; R\mathbf{n} \tag{83}$$

where R is the sphere's radius while $\mathbf{n}$ is a generic unit vector. Averaging the potential over the sphere is equivalent to averaging over the direction of that unit vector, thus

$$V_{\rm avg}(\mathbf{c},R) \;=\; \frac{1}{4\pi R^2}\iint\limits_{\rm sphere} V(\mathbf{r})\,d^2A \;=\; \frac{1}{4\pi}\iint V(\mathbf{c}+R\mathbf{n})\,d^2\Omega(\mathbf{n}) \tag{84}$$

where $d^2\Omega(\mathbf{n})$ is the infinitesimal solid angle for the direction of $\mathbf{n}$. Similarly, for the electric field, the average over the sphere amounts to

$$\mathbf{E}_{\rm avg}(\mathbf{c},R) \;=\; \frac{1}{4\pi R^2}\iint\limits_{\rm sphere} \mathbf{E}(\mathbf{r})\,d^2A \;=\; \frac{1}{4\pi}\iint \mathbf{E}(\mathbf{c}+R\mathbf{n})\,d^2\Omega(\mathbf{n}). \tag{85}$$

Note that here we average the electric field as a vector — that is, average each component separately — instead of averaging the magnitude $E=|\mathbf{E}|$. For the average magnitude, this problem simply would not work!

The key to this problem is the relation between the average field vector and the average potential: The average field is simply (minus) the gradient of the average potential WRT the center location $\mathbf{c}$:

$$\mathbf{E}_{\rm avg}(\mathbf{c},R) \;=\; -\nabla_c V_{\rm avg}(\mathbf{c},R) \;=\; -\frac{\partial V_{\rm avg}}{\partial c_x}\,\hat{\mathbf{x}} \;-\; \frac{\partial V_{\rm avg}}{\partial c_y}\,\hat{\mathbf{y}} \;-\; \frac{\partial V_{\rm avg}}{\partial c_z}\,\hat{\mathbf{z}} \tag{86}$$

where all the partial derivatives are taken at fixed sphere's radius R. Indeed, in the integral

$$V_{\rm avg}(\mathbf{c},R) \;=\; \frac{1}{4\pi}\iint V(\mathbf{c}+R\mathbf{n})\,d^2\Omega(\mathbf{n}) \tag{87}$$

the variables $\mathbf{c}=(c_x,c_y,c_z)$ are completely independent from R or from the integrations variables parametrizing the direction $\mathbf{n}$ (for example, the two angular coordinates θ and ϕ). In particular, the ranges of the integration variables are completely independent from

(c_x, c_y, c_z). Consequently, the derivatives of the integral WRT to the (c_x, x_y, c_z) are equal to the integrals of the derivatives,

$$\begin{aligned}
\frac{\partial}{\partial c_x} V_{\rm avg}(\mathbf{c}, R) &= \frac{1}{4\pi} \iint \frac{\partial V(\mathbf{c} + R\mathbf{n})}{\partial c_x}\, d^2\Omega(\mathbf{n}), \\
\frac{\partial}{\partial c_y} V_{\rm avg}(\mathbf{c}, R) &= \frac{1}{4\pi} \iint \frac{\partial V(\mathbf{c} + R\mathbf{n})}{\partial c_y}\, d^2\Omega(\mathbf{n}), \\
\frac{\partial}{\partial c_z} V_{\rm avg}(\mathbf{c}, R) &= \frac{1}{4\pi} \iint \frac{\partial V(\mathbf{c} + R\mathbf{n})}{\partial c_z}\, d^2\Omega(\mathbf{n}),
\end{aligned} \tag{88}$$

or in vector notations,

$$\nabla_{\mathbf{c}} V_{\rm avg}(\mathbf{c}, R) = \frac{1}{4\pi} \iint \nabla_{\mathbf{c}} V(\mathbf{c} + R\mathbf{n})\, d^2\Omega(\mathbf{n}),$$

where all the partial derivatives on the RHS are taken for fixed $\mathbf{n}$ before we integrate. Thus, at the differentiation time, the shift vector $\mathbf{r} - \mathbf{c} = R\mathbf{n}$ is held constant, so the derivatives are simply

$$\left(\frac{\partial V(\mathbf{r} = \mathbf{c} + R\mathbf{n})}{\partial c_x} \right)_{R\mathbf{n}}^{\rm fixed} = \left. \frac{\partial V}{\partial x} \right|_{@\mathbf{r}=\mathbf{c}+r\mathbf{n}}, \tag{89}$$

and likewise for the y and z derivatives. In vector notations, this means

$$\Big(\nabla_{\mathbf{c}} V(\mathbf{c} + R\mathbf{n}) \Big)_{R\mathbf{n}}^{\rm fixed} = \nabla V|_{\mathbf{r}=\mathbf{c}+R\mathbf{n}} = -\mathbf{E}(\mathbf{r} = \mathbf{c} + R\mathbf{n}) \tag{90}$$

since the ordinary gradient of the potential is simply (minus) the electric field vector. Consequently, plugging this formula back into the integral (88), we arrive at

$$\nabla_{\mathbf{c}} V_{\rm avg}(\mathbf{c}, R) = -\frac{1}{4\pi} \iint \nabla_{\mathbf{c}} \mathbf{E}(\mathbf{r} = \mathbf{c} + R\mathbf{n})\, d^2\Omega(\mathbf{n}) = -\mathbf{E}_{\rm avg}(\mathbf{c}, R). \tag{91}$$

This completes the proof of eq. (86).

(a) Now that we have eq. (86), part (a) of the problem follows trivially from the *mean value theorem for the potential: If there are no charges inside a sphere of radius R, then*

$$V_{\rm avg}(\mathbf{c},R) \;=\; V@(\text{the center }\mathbf{c}). \tag{92}$$

Indeed, given eqs. (86) and (92), the mean electric field vector over the sphere is

$$\mathbf{E}_{\rm avg}(\mathbf{c},R) \;=\; -\nabla_{\mathbf{c}}V_{\rm avg}(\mathbf{c},R) \;=\; -\nabla_{\mathbf{c}}V(\mathbf{c}) \;=\; +\mathbf{E}(\mathbf{c}). \tag{93}$$

(b) Now suppose there are some charges inside the sphere. As we saw in problem **3.1**, the contribution of such charge to the mean potential on the sphere does not depend on where they are placed inside the sphere, as long as they are inside it,

$$V(\mathbf{c},R)[\text{due to inside charges}] \;=\; \frac{Q^{\rm net}_{\rm inside}}{4\pi\epsilon_0 R}\,. \tag{94}$$

So if we move the center of the sphere just a little bit and do not cross any charges, this mean potential is not going to change at all. Consequently, by eq. (86),

$$\mathbf{E}_{\rm avg}(\mathbf{c},R)[\text{due to inside charges}] \;=\; -\nabla_{\mathbf{c}}V(\mathbf{c},R)[\text{due to inside charges}] \;=\; 0, \tag{95}$$

the inside charges do not contribute to the mean electric field vector.

More generally, suppose there are charges both inside and outside the sphere, *but there are no charges right at the spherical surface itself.* In this case,

$$V_{\rm avg}(\mathbf{c},R) \;=\; V^{\text{due to outside charges}}(\mathbf{c}) \;+\; \frac{Q^{\rm net}_{\rm inside}}{4\pi\epsilon_0 R}\,, \tag{96}$$

and if we displace the center $\mathbf{c}$ by an infinitesimal $d\mathbf{c}$, the net charge inside the sphere would not change, $dQ^{\rm net}_{\rm inside}=0$. Consequently,

$$\nabla_{\mathbf{c}}V_{\rm avg}(\mathbf{c},R) \;=\; \nabla V^{\text{due to outside charges}}(\mathbf{c}) \;+\; 0, \tag{97}$$

and hence *the mean electric field vector on the sphere equals to the electric field at the center due to the outside charges only.*

However, if there are any charges right at the spherical surface — or continuous charges on lines, surfaces, or volumes crossed by the sphere — then even an infinitesimal motion of the sphere's center may change the net charge inside the sphere. Consequently, the mean potential due to the inside charges would change with the sphere's displacement, and by eq. (86) this would contribute to the mean electrical field vector. Thus, *the charges strictly inside the sphere do not contribute to the mean electric field vector, but the charges right at the spherical surface do contribute.*