Problem **3.50**:

(a) The electric fields $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r})$ due to each charge distribution obey the respective equations

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}) &= -\nabla V_1(\mathbf{r}), \quad \nabla\cdot\mathbf{E}_1(\mathbf{r}) = \frac{\rho_1(\mathbf{r})}{\epsilon_0}, \\ \mathbf{E}_2(\mathbf{r}) &= -\nabla V_2(\mathbf{r}), \quad \nabla\cdot\mathbf{E}_2(\mathbf{r}) = \frac{\rho_2(\mathbf{r})}{\epsilon_0}. \end{aligned} \tag{1}$$

Consequently,

$$\mathbf{E}_1\cdot\mathbf{E}_2 = -\mathbf{E}_1\cdot\nabla V_2 = -\nabla\cdot(V_2\mathbf{E}_1) + V_2\,\nabla\cdot\mathbf{E}_1 = -\nabla\cdot(V_2\mathbf{E}_1) - \frac{V_2\rho_1}{\epsilon_0}, \tag{2}$$

and in likewise manner

$$\mathbf{E}_1\cdot\mathbf{E}_2 = -\nabla\cdot(V_1\mathbf{E}_2) - \frac{V_1\rho_2}{\epsilon_0}. \tag{3}$$

Comparing the last two equations gives us

$$V_1\rho_2 - V_2\rho_1 = \epsilon_0\nabla\cdot(V_2\mathbf{E}_1 - V_1\mathbf{E}_2), \tag{4}$$

hence after integration over some volume $\mathcal{V}$ and applying the Gauss theorem

$$\iiint_{\mathcal{V}} V_1\rho_2\,d^3\mathrm{Vol} - \iiint_{\mathcal{V}} V_2\rho_1\,d^3\mathrm{Vol} = \epsilon_0\iiint_{\mathcal{V}}\nabla\cdot(V_2\mathbf{E}_1 - V_1\mathbf{E}_2)d^3\mathrm{Vol} = \epsilon_0\iint_{\mathcal{S}}(V_2\mathbf{E}_1 - V_1\mathbf{E}_2)\cdot\mathbf{d^2A} \tag{5}$$

where the surface $\mathcal{S}$ is the boundary of the volume $\mathcal{V}$. Suppose $\mathcal{V}$ is a ball of very large radius R so $\mathcal{S}$ is the spherical surface of that radius. In the limit of $R\to\infty$, the potentials V_1 and V_2 scale as $1/R$, the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ scale as $1/R^2$, the sphere's area scales as R^2, so the surface integral on the RHS of eq. (5) scales as $1/R$ and eventually vanishes for $R=\infty$. At the same time, volume integrals on the LHS of eq. (5) expand to integrals over the whole space, thus

$$\iiint_{\substack{\text{whole}\\\text{space}}} V_1\rho_2\,d^3\mathrm{Vol} = \iiint_{\substack{\text{whole}\\\text{space}}} V_2\rho_1\,d^3\mathrm{Vol}, \tag{6}$$

quod erat demonstrandum.

(b) In the Green's Reciprocity Theorem (6), the two charge distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ do not have to coexist. Instead, we may first turn on just the $\rho_1(\mathbf{r})$ distribution, measure the potential $V_1(\mathbf{r})$ it creates and record it for later analysis, then turn off the $\rho_1(\mathbf{r})$ distribution and turn on the $\rho_2(\mathbf{r})$ distributions, measure and record the potential $V_2(\mathbf{r})$ it creates, then turn off the second distribution, and finally calculate the integrals (6) from the recorded measurements. But however we do it, the integrals would have to come out equal.

For the problem at hand, we are going to use the distributions which are definitely not simultaneous. The first distribution comprises surface charges on both conductors, such that the net charge of the conductor a is Q while the conductor b is neutral on the whole (but has $\sigma \neq 0$ due to induced charges). We do not know the details of the surface charge densities for either conductor, but we know that the net potential $V_1(\mathbf{r})$ they create is constant over each conductor, namely V_{aa} over the conductor a and V_{ab} over the conductor b.

Likewise, the second distribution also goes over the two conductors' surfaces, but this time the conductor b has net charge Q while the conductor a is neutral on the whole. Again, we do not know the details of the surface charge densities, but we do know that the net potential $V_2(\mathbf{r})$ they create is constant over each conductor, namely V_{ba} over the conductor a and V_{bb} over the conductor b.

Now let's calculate the integrals in eq. (6). For each distribution, the charges are confined to the two conductor's surfaces, and the potential is locally constant over each surface, hence

$$\begin{aligned}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} V_1\rho_2\, d^3\text{Vol} &= \iint\limits_{\text{cond.}\,a} V_1\sigma_2\, d^2A \;+\; \iint\limits_{\text{cond.}\,b} V_1\sigma_2\, d^2A \\ &= V_{aa}\times \iint\limits_{\text{cond.}\,a} \sigma_2\, d^2A \;+\; V_{ab}\times \iint\limits_{\text{cond.}\,b} \sigma_2\, d^2A \\ &= V_{aa}\times Q_a^{\text{net}}[\text{distribution 2}] \;+\; V_{ab}\times Q_b^{\text{net}}[\text{distribution 2}] \\ &= V_{aa}\times 0 \;+\; V_{ab}\times Q.\end{aligned} \tag{7}$$

Similarly,

$$\iiint\limits_{\substack{\text{whole}\\ \text{space}}} V_2\rho_1\, d^3\text{Vol} \;=\; V_{ba}\times Q \;+\; V_{bb}\times 0. \tag{8}$$

Plugging these two formulae into the Green's Reciprocity Theorem (6), we immediately obtain

$$V_{aa} \times 0 \ + \ V_{ab} \times Q \ = \ V_{ba} \times Q \ + \ V_{bb} \times 0 \tag{9}$$

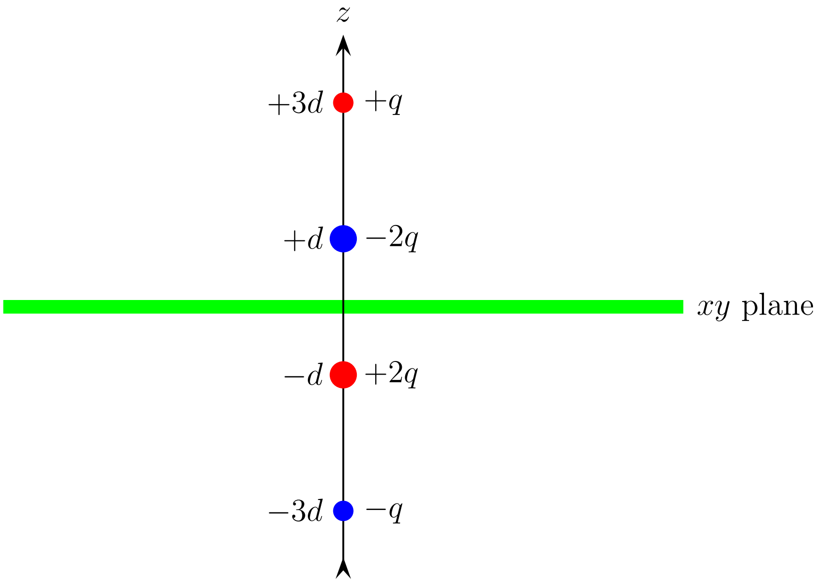
and therefore

$$V_{ab} \ = \ V_{ba} \, , \tag{10}$$

quod erat demonstrandum.

Problem **3.7**:

Both charges above the plane give rise to the mirror-image charges below the plane. Altogether, the electric field above the plane looks like the field due to 4 point charges as shown on the picture below



(11)

Consequently, the force on the top charge is the net Coulomb force form all the other charges — including the second real charge and both image charges. Thus,

$$\begin{aligned} F_z^{\text{net}} &= \frac{-2q^2}{4\pi\epsilon_0} \times \frac{1}{(2d)^2} \ + \ \frac{+2q^2}{4\pi\epsilon_0} \times \frac{1}{(4d)^2} \ + \ \frac{-q^2}{4\pi\epsilon_0} \times \frac{1}{(6d)^2} \\ &= \frac{q^2}{4\pi\epsilon_0 \, d^2} \left(-\frac{2}{4} + \frac{2}{16} - \frac{1}{36} \right) \\ &= -\frac{29}{72} \times \frac{q^2}{4\pi\epsilon_0 \, d^2} \, , \end{aligned} \tag{12}$$

where the − sign indicates the downward direction of the net force.

Problem **3.10**:

(a) For any collection of charges above the ground — which we take to be a flat conducting plane — we may get the net electric field above the ground (and only above the ground) by the image charge method. We simply combine the Coulomb fields of the actual charges above the ground with Coulomb fields of their mirror images below the ground.

This works not only for the point charges but also for the line charges, surface charges, *etc.* In particular, for a uniformly charged wire at height h above the ground, the mirror image is simply a 'wire' of opposite charge density $-\lambda$ located at depth h below the ground (*i.e.*, at $z = -h$), and the net field above the ground is simply the combined field of the two wires.

For a single uniformly charged wire, the electric potential is

$$V(s) \;=\; -\frac{\lambda}{2\pi\epsilon_0}\ln(s) \;+\; \text{const} \;=\; -\frac{\lambda}{4\pi\epsilon_0}\times\ln(s^2) \;+\; \text{const} \tag{13}$$

where s is the distance from the wire. For the wire in question

$$V_{\text{wire}}^{\text{real}}(y,z) \;=\; -\frac{\lambda}{4\pi\epsilon_0}\times\ln\bigl(y^2+(z-h)^2\bigr) \;+\; \text{const},$$

while its image below the ground creates the potential

$$V_{\text{wire}}^{\text{image}}(y,z) \;=\; +\frac{\lambda}{4\pi\epsilon_0}\times\ln\bigl(y^2+(z+h)^2\bigr) \;+\; \text{const}.$$

Adding up the two potentials, we arrive at

$$\begin{aligned} V_{\text{net}}(y,z) \;&=\; V_{\text{wire}}^{\text{real}}(y,z) \;+\; V_{\text{wire}}^{\text{image}}(y,z) \\ &=\; \frac{\lambda}{4\pi\epsilon_0}\times\Bigl(\ln(y^2+(z+h)^2) \;-\; \ln(y^2+(z-h)^2)\Bigr) \;+\; \text{const} \\ &=\; \frac{\lambda}{4\pi\epsilon_0}\times\ln\left(\frac{y^2+(z+h)^2}{y^2+(z-h)^2}\right) \;+\; \text{const}. \end{aligned} \tag{14}$$

In principle, the constant here can be anything — it depends on the reference point for the potential, — but if we want the ground-level potential to be zero, then we should set the constant term to zero. Thus,

$$\text{for any } z \geq 0, \quad V(y,z) \;=\; \frac{\lambda}{4\pi\epsilon_0} \times \ln\left(\frac{y^2+(z+h)^2}{y^2+(z-h)^2}\right)\;; \tag{15}$$

indeed, by inspection this potential vanishes for $z = 0$.

(b) Above the ground, the potential is given by the formula (15), and the electric field follows from its gradient. Below the ground, there is no electric field, and the potential is zero. The discontinuity between the electric fields above and below the ground is due to the surface charge density $\sigma(x,y)$ — or in our case, simply $\sigma(y)$ — which obtains as

$$\sigma \;=\; \epsilon_0 \times \operatorname{disc} E_\perp \;=\; \epsilon_0 \times E_\perp(z \to +0) \;=\; -\epsilon_0 \left(\frac{\partial V}{\partial z}\right)_{@z\to+0}. \tag{16}$$

In other words, starting with the potential (15) above the ground, take its derivative WRT z, evaluate it for $z = 0$, and multiply by $-\epsilon_0$. Thus,

$$\frac{\partial V}{\partial z} \;=\; \frac{\lambda}{4\pi\epsilon_0} \times \left(\frac{2(z+h)}{y^2+(z+h)^2} - \frac{2(z-h)}{y^2+(z-h)^2}\right), \tag{17}$$

which for $z = 0$ becomes

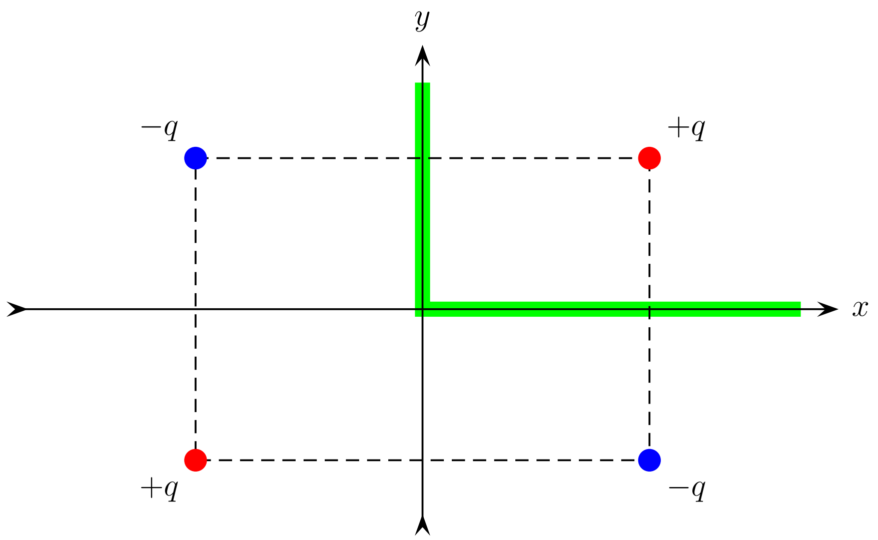
$$\left(\frac{\partial V}{\partial z}\right)_{z=0} \;=\; \frac{\lambda}{4\pi\epsilon_0} \times \frac{4h}{y^2+h^2}\,. \tag{18}$$

Consequently, the surface charge density at the ground level is

$$\sigma(y) \;=\; -\frac{\lambda}{\pi} \times \frac{h}{y^2+h^2}\,. \tag{19}$$

Problem **3.11**:

(a) The two conducting planes acts as mirrors for the charges. The two mirrors joined at 90° angle create 3 images of any object: one image in the mirror A, one image in the mirror B, and one image-of-the-image. Note that the image in B of the image in A is exactly at the same place as the image in A of the image in B, so there is only one image-of-the-image, and there is no need to consider images-of-images-of-images, *etc.*, *etc.* Altogether, we have 4 charges



(20)

Note the signs of these charges: the images in a single plane have opposite signs to the original charge, while the image-of-the-image has charge $+q$ due to double sign flip.

The net charge configuration — 1 real charge plus 3 image charges — obviously obeys the Poisson equation for the upper right quadrant as well as the $V \to 0$ asymptotic condition for $r \to \infty$. To show that it obeys the boundary conditions $V = 0$ on the two conducting half-planes, we simply use the symmetries of the 4 charge system extended to the whole space, namely (1) $x \to -x$, $y \to +y$, $Q \to -Q$, and hence $V \to -V$, and (2) $x \to +x$, $y \to -y$, $Q \to -Q$, and hence $V \to -V$. By these symmetries $V(x, 0) = 0$ for any x and $V(0, y) = 0$ for any y, in perfect agreement with the boundary conditions.

Of course, there are no actual image charges beyond the conducting planes, there are only the surface charges on the planes themselves. But *in the upper right quadrant between the planes*, the electric field is the same as if there were point image charges instead of the surface charges on the planes.

(b) The net force on the real point charge is $Q\times\mathbf{E}$ due to all the other charges in the systems, namely the surface charges on the 'mirror' planes. But since the net electric field due to all those surface charges looks exactly like the Coulomb field of the 3 image charges, the net force on the real point charge is simply the net Coulomb force from the 3 image charges:

$$\mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\,, \tag{21}$$

$$\mathbf{F}_1 = -\frac{q^2}{4\pi\epsilon_0}\,\frac{\hat{\mathbf{y}}}{(2b)^2}\,, \tag{22}$$

$$\mathbf{F}_2 = -\frac{q^2}{4\pi\epsilon_0}\,\frac{\hat{\mathbf{x}}}{(2a)^2}\,, \tag{23}$$

$$\mathbf{F}_3 = +\frac{q^2}{4\pi\epsilon_0}\,\frac{(2a)\hat{\mathbf{x}} + (2b)\hat{\mathbf{y}}}{\left((2a)^2 + (2b)^2\right)^{3/2}}\,, \tag{24}$$

In components,

$$F_x^{\text{net}} = -\frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{a^2} - \frac{a}{(a^2+b^2)^{3/2}}\right), \tag{25}$$

$$F_y^{\text{net}} = -\frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{b^2} - \frac{b}{(a^2+b^2)^{3/2}}\right). \tag{26}$$

(c) The force (26) performs work when the real charge moves around. At the same time, the image charges mirror the real charge's motion, but that does not take any additional forces, hence no extra work. Thus, the net infinitesimal work is simply

$$\begin{aligned} dW &= \mathbf{F}\cdot d\mathbf{r}_{\text{real}} = F_x\,da + F_y\,db \\ &= -\frac{q^2}{16\pi\epsilon_0}\left(\frac{da}{a^2} + \frac{db}{b^2} - \frac{ada + bdb}{(a^2+b^2)^{3/2}}\right) \\ &= +\frac{q^2}{16\pi\epsilon_0}\times d\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{\sqrt{a^2+b^2}}\right). \end{aligned} \tag{27}$$

Integrating this work as the charge moves from infinity to its final position at $(x=a, y=b)$, we find the potential energy

$$U = -W_{\text{net}} = -\frac{q^2}{16\pi\epsilon_0}\times\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{\sqrt{a^2+b^2}}\right). \tag{28}$$

Note: this potential energy is 4 times smaller than the naive potential energy of 4 point charges,

$$\begin{aligned} U_{\text{naive}} &= -\frac{q^2}{4\pi\epsilon_0}\times\frac{1}{2a}\times 2_{\text{pairs}} - \frac{q^2}{4\pi\epsilon_0}\times\frac{1}{2b}\times 2_{\text{pairs}} + \frac{q^2}{4\pi\epsilon_0}\times\frac{1}{2\sqrt{a^2+b^2}}\times 2_{\text{pairs}} \\ &= -\frac{q^2}{4\pi\epsilon_0}\times\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{\sqrt{a^2+b^2}}\right). \end{aligned} \tag{29}$$

But this naive energy equals to the integral over the electric field square

$$U_{\text{naive}} = \frac{\epsilon_0}{2}\iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{E}^2\, d^3\text{Vol} \tag{30}$$

over the whole space, while in reality the electric field of the 4 charges exists only in the quadrant between the planes; outside that quadrant $\mathbf{E} = 0$. Thus physically, we should integrate (30) only over the top right quadrant, which leads to the real potential energy being only a quarter of the naive energy,

$$U = \frac{1}{4}U_{\text{naive}}\,. \tag{31}$$

(d) Finally, consider what happens when the two conducting planes intersect at an angle $\alpha \neq 90^\circ$. In this case, the image in plane B of the image in plane A is at a different place than the image in plane A of the image in plane B. And beyond these two images-of-images, there will be images of images of images, *etc.*, *etc.*.

For a rational angle α — or rather, for α being a rational fraction of 360° — the images of images of images ... would eventually coincide, and we end up with a finite set of image charges. For example, for $\alpha = 360^\circ/(2N)$ (N being a whole number), we end up with $2N$ charges altogether, including 1 real charge and $2N-1$ image charges. Among the image charges, N would have charge $-q$ and $N-1$ charge $+q$.

In a situation like this, the force on the real charge and the potential energy obtain along the lines of parts (b) and (c) of this problem (for $\alpha = 90^\circ = 360^\circ/4$). But I am not doing it here in any detail, and I do not expect you to do it in your homework.

On the other hand, for an irrational α — or rather irrational $\alpha/360^\circ$ — the images of images of images ... of the original charge never converge to a finite set. Instead, we get an infinite set of image charges forming an irrational pattern that's useless for any calculations. For this sorry situation, the image charge method does not work.

<u>Problem **3.8**</u>:

Preamble: Consider a point charge Q outside a grounded conducting sphere. The potential $V(\mathbf{r})$ for this system obeys the Poisson equation

$$\Delta V(\mathbf{r}) \;=\; -\frac{Q}{\epsilon_0}\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_q) \quad \text{for } |\mathbf{r}| > R, \tag{32}$$

the boundary condition

$$V(r,\theta,\phi) \;=\; 0 \quad \text{for } r = R, \tag{33}$$

and the asymptotic condition

$$V(r,\theta,\phi) \;\to\; 0 \quad \text{for } r \to \infty. \tag{34}$$

The image charge method — with an image charge Q_i at some location $\mathbf{r}_i$ inside the sphere — produces potential

$$V(\mathbf{r}) \;=\; \frac{Q}{4\pi\epsilon_0} \times \frac{1}{|\mathbf{r}-\mathbf{r}_q|} \;+\; \frac{Q_i}{4\pi\epsilon_0} \times \frac{1}{|\mathbf{r}-\mathbf{r}_i|} \tag{35}$$

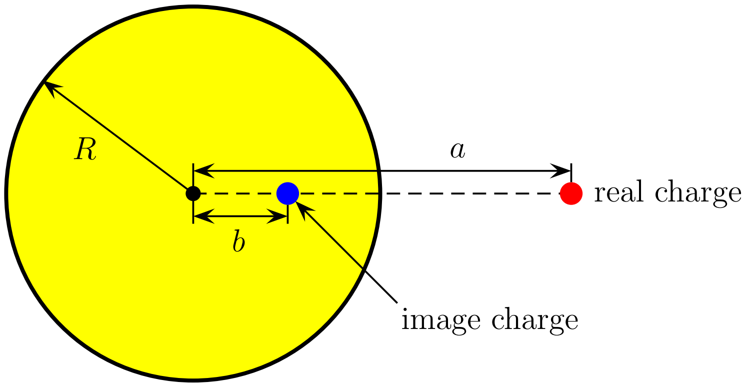
which automatically satisfies the Poisson equation (32) (for the outside of the sphere only) and the asymptotic condition (34). All we need to check is the boundary condition (33) for the potential (35), and the textbook claims that it is satisfied for the image charge

$$Q_i \;=\; -\frac{R}{a} \times Q \tag{36}$$

located at distance

$$b \;=\; \frac{R^2}{a} \tag{37}$$

from the sphere's center and on the same ray (from the center) as the real charge:



(38)

(a) Our first task is to verify that the potential (35) indeed vanishes all over the sphere's surface for the image charge (36) located according to eqs. (37) and (38). Let's pick a generic point P on the sphere's surface. Since the real charge and the image charge sit on the same ray (from the sphere's center), the radius vector $\mathbf{r}_p$ of the point P makes the same angle θ with the radius vectors $\mathbf{r}_q$ and $\mathbf{r}_i$ of the two charges. Consequently,

$$|\mathbf{r}_p - \mathbf{r}_q|^2 \;=\; R^2 \,+\, a^2 \,-\, 2Ra\,\cos\theta \quad \text{and} \quad |\mathbf{r}_p - \mathbf{r}_i|^2 \;=\; R^2 \,+\, b^2 \,-\, 2Rb\,\cos\theta \tag{39}$$

for the same angle θ. Moreover, since $b = R^2/a$ we have

$$|\mathbf{r}_p - \mathbf{r}_i|^2 \;=\; R^2 + \frac{R^4}{a^2} - \frac{2R^3}{a}\cos\theta \;=\; \frac{R^2}{a^2}\times\Bigl(a^2 + R^2 - 2Ra\,\cos\theta\Bigr) \;=\; \frac{R^2}{a^2}\times|\mathbf{r}_p - \mathbf{r}_q|^2 \tag{40}$$

and hence

$$\frac{Q_i}{|\mathbf{r}_p - \mathbf{r}_i|} \;=\; \frac{a}{R}\times\frac{Q_i}{|\mathbf{r}_p - \mathbf{r}_q|}\,. \tag{41}$$

Finally, in light of eq. (36) for the value of the image charge Q_i.

$$\frac{a}{R}\times Q_i \;=\; -Q \quad\Longrightarrow\quad \frac{Q_i}{|\mathbf{r}_p - \mathbf{r}_i|} \;=\; -\frac{Q}{|\mathbf{r}_p - \mathbf{r}_q|}\,, \tag{42}$$

which means that for any point P on the sphere (and only on the sphere) the potential vanishes,

$$V(P) \;=\; \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{|\mathbf{r}_p - \mathbf{r}_q|} + \frac{Q_i}{|\mathbf{r}_p - \mathbf{r}_i|}\right) \;=\; 0. \tag{43}$$

Quod erat demonstrandum.

(b) Let's use spherical coordinates (r, θ, ϕ) where the $\theta = 0$ axis points towards the real and the image charges. Then, outside the sphere, the potential is given by

$$V(r, \theta, \phi) \;=\; \frac{Q}{4\pi\epsilon_0} \times \left(\frac{1}{\sqrt{a^2 + r^2 - 2ra\cos\theta}} \;-\; \frac{(R/a)}{\sqrt{(b^2 + r^2 - 2rb\cos\theta}} \right). \tag{44}$$

The surface charge density on the conducting sphere follows from the normal (*i.e.*, radial) electric field right outside the sphere,

$$\sigma(\theta, \phi) \;=\; \epsilon_0 E_r(r, \theta, \phi) \;=\; -\epsilon_0 \frac{\partial V(r, \theta, \phi)}{\partial r} \quad @\, r = R + 0. \tag{45}$$

In light of eq. (44),

$$-\epsilon_0 \frac{\partial V}{\partial r} \;=\; +\frac{Q}{4\pi} \left(\frac{r - a\cos\theta}{[a^2 + r^2 - 2ra\cos\theta]^{3/2}} \;-\; \frac{R}{a} \times \frac{r - b\cos\theta}{[b^2 + r^2 - 2rb\cos\theta]^{3/2}} \right), \tag{46}$$

so let's evaluate this formula for $r = R$. Thanks to $b = R^2/a$,

$$b^2 \,+\, R^2 \,-\, 2Rb\cos\theta \;=\; \frac{R^2}{a^2} \times \left(a^2 + R^2 - 2Ra\cos\theta\right), \tag{47}$$

$$R \,-\, b\cos\theta \;=\; \frac{R}{a} \times \left(a - R\cos\theta\right), \tag{48}$$

$$\frac{R}{a} \times \frac{r - b\cos\theta}{[b^2 + R^2 - 2Rb\cos\theta]^{3/2}} \;=\; \frac{a}{R} \times \frac{a - R\cos\theta}{[a^2 + R^2 - 2Ra\cos\theta]^{3/2}}\,, \tag{49}$$

and therefore

$$\begin{aligned} -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} \;&=\; \frac{Q}{4\pi} \times \frac{1}{[a^2 + R^2 - 2Ra\cos\theta]^{3/2}} \times \\ &\quad \times \left((R - a\cos\theta) \;-\; \frac{a}{R} \times (a - R\cos\theta) \;=\; \frac{R^2 - a^2}{R} \right). \end{aligned} \tag{50}$$

Thus altogether, the charge density on the sphere's surface is

$$\sigma(\theta, \phi) \;=\; -\frac{Q}{4\pi} \times \frac{a^2 - R^2}{R} \times \frac{1}{[a^2 + R^2 - 2Ra\cos\theta]^{3/2}}\,. \tag{51}$$

Now let's integrate this charge density to get the net charge *induced* on the sphere. In spherical coordinates,

$$Q_{\text{ind}} \;=\; \iint\limits_{\text{sphere}} \sigma\, d^2A \;=\; \int\limits_0^{2\pi} d\phi \int\limits_0^{\theta} d\theta \times R^2 \sin\theta \times \sigma(\theta,\phi) \;=\; 2\pi R^2 \int\limits_{-1}^{+1} d(\cos\theta)\,\sigma(\cos\theta), \tag{52}$$

where the last equality follows from the ϕ-independence of the charge density and $d(\cos\theta) = -\sin\theta\, d\theta$. For the charge density as in eq. (51),

$$\begin{aligned} Q_{\text{ind}} \;&=\; 2\pi R^2 \int\limits_{-1}^{+1} dc\, \frac{-Q(a^2-R^2)}{4\pi R} \times \frac{1}{[R^2+a^2-2Rac]^{3/2}} \\ &\qquad \langle\!\langle \text{ where } c \text{ stands for } \cos\theta \,\rangle\!\rangle \\ &=\; -\frac{QR(a^2-R^2)}{2} \int\limits_{-1}^{+1} \frac{dc}{[R^2+a^2-2Rac]^{3/2}} \\ &=\; -\frac{QR(a^2-R^2)}{2} \int\limits_{c=-1}^{c=+1} \frac{1}{Ra}\, d\left(\frac{1}{\sqrt{R^2+a^2-2Rac}}\right) \\ &=\; -\frac{Q(a^2-R^2)}{2a} \times \left[\frac{1}{\sqrt{R^2+a^2-2Ra}} - \frac{1}{\sqrt{R^2+a^2+2Ra}}\right], \end{aligned} \tag{53}$$

where the expression in $[\cdots]$ evaluates to

$$[\cdots] \;=\; \left[\frac{1}{|a-R|} - \frac{1}{a+R}\right] \;=\; \frac{2R}{a^2-R^2} \quad (\text{for } R < a). \tag{54}$$

Consequently,

$$Q_{\text{ind}} \;=\; -\frac{Q(a^2-R^2)}{2a} \times \frac{2R}{a^2-R^2} \;=\; -Q \times \frac{R}{a}\,, \tag{55}$$

— which is exactly the same as the image charge Q_i. Thus, similar to the conducting plane, the net induced charge on the grounded spherical surface agrees with the apparent image charge!

(c) The net force on the real charge outside the grounded sphere is $\mathbf{F} = Q\mathbf{E}$ where $\mathbf{E}$ is due to the induced surface charges. But outside the sphere $\mathbf{E}$ due to surface charges is the same as the Coulomb field of the image charge, thus the magnitude of the force is

$$|F| = \frac{|QQ_i|}{4\pi\epsilon_0} \times \frac{1}{(a-b)^2} = \frac{Q^2}{4\pi\epsilon_0} \times \frac{R}{a} \times \frac{a^2}{(a^2-R^2)^2} = \frac{Q^2}{4\pi\epsilon_0} \times \frac{aR}{(a^2-R^2)^2}\,, \tag{56}$$

and its direction is towards the sphere's center. As the real charge is brought from the ∞ to a finite distance a from the sphere, this force performs positive work

$$\begin{aligned} W &= \int_a^\infty F(a')\,da' = \frac{Q^2}{4\pi\epsilon_0}\int_a^\infty \frac{Ra'\,da'}{(a'^2-R^2)^2} \\ &= \frac{Q^2}{4\pi\epsilon_0} \times \frac{R}{2} \int_{a'=a}^{a'=\infty} \frac{d(a'^2-R^2)}{(a'^2-R^2)^2} \\ &= \frac{Q^2 R}{8\pi\epsilon_0} \times \frac{1}{a^2-R^2}\,. \end{aligned} \tag{57}$$

which means the potential energy of the real point charge outside of the grounded sphere is

$$U = -W = -\frac{Q^2}{8\pi\epsilon_0} \times \frac{R}{a^2-R^2}\,. \tag{58}$$

PS: Curiously,

$$U = \frac{1}{2} \times \frac{QQ_i}{4\pi\epsilon_0(a-b)} = \frac{1}{2}U_{\text{naive}}\,. \tag{59}$$

That is, the true potential energy is $\frac{1}{2}$ of the naive potential energy one might expect for the two point charges, one real and one image. The reason the true potential energy is less (in absolute value) than the naive energy is clear: while it take work to move the real charge, the image charge moves by itself without any extra work. What's not so clear is the ratio U/U_{naive} is simply $\frac{1}{2}$, just like for the conducting plane. For the plane, that factor $\frac{1}{2}$ follows from the upside-downside symmetry, but the sphere does not have any obvious inside-outside symmetry, so why the similar $\frac{1}{2}$ factor?

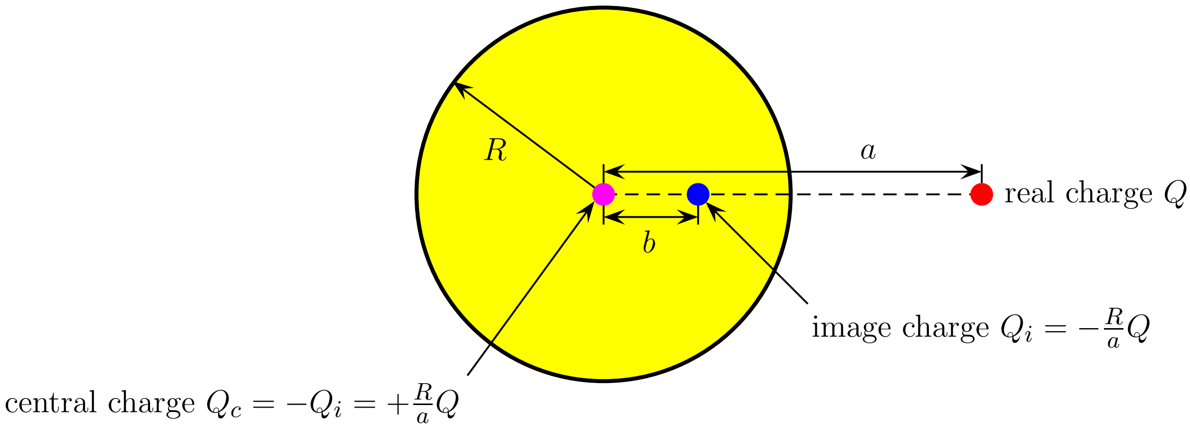
It turns out, the conducting sphere does have a non-obvious inside-outside symmetry $r \to R^2/r$, it's a special case of the *conformal symmetry* of the electromagnetic field. But the working of this symmetry is way outside the undergraduate class material, so let me stop here. In any case, I do not expect you to offer any explanations of $U = \frac{1}{2}U_{\text{naive}}$ in your homework.

<u>Problem **3.9**</u>:
If the conducting sphere is not grounded, then the boundary condition for the potential on the sphere's outer surface is $V = \text{const}$ but not necessarily $V = 0$. In terms of the surface charge density, this calls for

$$\sigma_{\text{ungrounded}}(\theta, \phi) \;=\; \sigma_{\text{grounded}}(\theta, \phi) \;+\; \text{const}, \tag{60}$$

where the constant is adjusted to keep the overall charge of the sphere whatever it was before we brought in the outside point charge, for example $Q_{\text{net}} = 0$.

In terms of the electric field outside the sphere, the constant term in the surface charge density is equivalent to the same net charge at the center. In terms of the image charge method, this means that the surface charges induced on the ungrounded conducting sphere by the outside point charge are equivalent to *two* point charges inside the sphere: the image charge $Q_i = -(R/a)Q$ located similarly to the image in the grounded sphere, and the central charge $Q_c = -Q_i = +(R/a)Q$ located at the center. Altogether, we have



(61)

This picture is for the neutral un-grounded sphere. For the charged sphere, the central charge would be $Q_c = Q_{\text{net}} - Q_i$ instead of simply $Q_c = -Q_i$.

Now consider the net force on the real point charge Q. As usual, $\mathbf{F} = Q\mathbf{E}$ [due to all other charges], which amounts to $\mathbf{F} = Q\mathbf{E}$[due to Q_i and Q_c]. In other words, its the net Coulomb force between Q and the image+central charges,

$$F = \frac{QQ_i}{4\pi\epsilon_0(a-b)^2} + \frac{QQ_c}{4\pi\epsilon_0\, a^2} = -\frac{Q^2R}{4\pi\epsilon_0 a}\left(\frac{1}{(a-b)^2} - \frac{1}{a^2}\right), \tag{62}$$

where the minus sign indicates the direction towards the sphere's center. After a bit of algebra, we obtain

$$\frac{1}{(a-b)^2} - \frac{1}{a^2} = \frac{R^2(2a^2-R^2)}{a^2(a^2-R^2)^2}, \tag{63}$$

hence

$$F = -\frac{Q^2}{4\pi\epsilon_0} \times \frac{R^3(2a^2-R^2)}{a^3(a^2-R^2)^2}. \tag{64}$$

For the real charge very near the sphere, $a = R+d$ for $d \ll R$, this force may be approximated as

$$F \approx -\frac{Q^2}{4\pi\epsilon_0} \times \frac{1}{4d^2}, \tag{65}$$

similar to the point charge near a conducting plane, while for the the charge very far from the sphere, $a \gg R$, we have a very different behavior, namely

$$F \approx -\frac{Q^2}{4\pi\epsilon_0} \times \frac{2R^3}{a^5}. \tag{66}$$

<u>The non-textbook problem</u>:

(a) First, let's take another look at the image of a charge outside a conducting sphere, *cf.* the two previous problems. Basically, the induced charges on the spherical surface create the electric field which inside the sphere cancels the field of the real outside charge Q, while outside the sphere it looks like the field of the image charge Q_i,

$$\begin{aligned} \text{inside the sphere} \quad \mathbf{E}^{\text{induced}}(x,y,z) &= -\mathbf{E}^{\text{real charge}}(x,y,z), \\ \text{outside the sphere} \quad \mathbf{E}^{\text{induced}}(x,y,z) &= \mathbf{E}^{\text{image charge}}(x,y,z). \end{aligned} \tag{67}$$

Now let's take away the original charge Q and put a new charge $Q' = -Q_i$ inside the sphere, precisely where the image charge used to be. Suppose the surface charge density $\sigma(\theta,\phi)$

of the induced charges on the sphere remains exactly the same as before, then outside the sphere

$$\mathbf{E}^{\text{induced}}(x,y,z) \;=\; \mathbf{E}^{\text{old image}\,Q_i}(x,y,z) \;=\; -\mathbf{E}^{\text{new charge}\,Q'}(x,y,z), \tag{68}$$

so the net electric field outside the sphere is zero. Of course, this is precisely what the surface charges need to do, so we conclude that the new $\sigma(\theta,\phi)$ induced by the inside charge $Q' = -Q_i$ is indeed the same as the old $\sigma(\theta,\phi)$ induced by the outside charge Q.

Consequently, inside the sphere, the electric field of the induced charges looks like the field of the new image charge $Q'_i = -Q$ located precisely where the original charge Q used to be! And this confirms the image charge method for the inside charges.

Specifically, the real charge and its image lie on the same ray from the center of the sphere and have reciprocal distances from the center:

$$b^{\text{inside}}_{\text{charge}} \times a^{\text{outside}}_{\text{charge}} \;=\; R^2_{\text{sphere}} \tag{69}$$

regardless of which charge is real and which is the image.

As to the value of the image charge, for the outside real charge we have

$$Q_i \;=\; -\frac{R}{a}\times Q \;=\; -\frac{b}{R}\times Q \quad\Longrightarrow\quad Q \;=\; -\frac{R}{b}\times Q_i. \tag{70}$$

For the inside real charge, we exchange the real charge and the image, and also the signs of both charges, thus

$$Q'_i \;=\; -Q \;=\; +\,\frac{R}{b}\times Q_i \;=\; -\frac{R}{b}\times Q', \tag{71}$$

or equivalently

$$Q'_i \;=\; -\frac{a}{R}\times Q'. \tag{72}$$

(b) Inside the sphere, the electric field of the induced charges on the surface looks exactly like the field of the image charge. Consequently, the force on the charge Q' is simply the Coulomb force of the image charge Q'_i. Since the two charges have opposite signs, this force is attractive; thus its direction is towards the image charge and *away from the sphere's center!*

As to the magnitude,

$$F = \frac{|Q'Q'_i|}{4\pi\epsilon_0} \times \frac{1}{(a-b)^2} = \frac{Q'^2}{4\pi\epsilon_0} \times \frac{R}{b} \times \frac{1}{\left[(R^2/b) - b\right]^2} = \frac{Q'^2}{4\pi\epsilon_0} \times \frac{Rb}{(R^2-b^2)^2}. \tag{73}$$

This force vanishes when the charge Q' is at the sphere's center, but it becomes large when ithe charge approaches the spherical surface from the inside.

(c) The inside surface of a conducting shell does not care if the shell is grounded or not. Regardless of what happens outside the inner surface, the induced charges on that surface must cancel the electric field of the inside charge Q' everywhere outside the inner surface, so their field inside the cavity should look the field of the image charge. So the image charge does not care if the sphere is grounded or not, and the force on the inside charge is exactly the same in both cases.

The only difference between the grounded and un-grounded spheres concerns the electric field *outside the outer surface of the shell.* If the shell is grounded, the outside surface is neutral and the outside field is zero. On the other hand, for a neutral un-grounded shell, the outside surface has charge $+Q'$ (to cancel the $-Q'$ charge of the inner surface), and there is outside field which looks like the Coulomb field of a point charge $+Q'$ at the center of the sphere.

Problem **3.13**:

The potential in the infinite slot is explained in detail in the textbook example **3.3** and also in my notes on the separation of variables method. According to the textbook equations (3.30) and (3.34), the potential inside the slot is

$$V(x,y) = \sum_{n=1}^{\infty} A_n \times \sin\left(\frac{n\pi y}{a}\right) \times \exp\left(-\frac{n\pi x}{a}\right), \tag{3.30}$$

where a is the slot's width, and the coefficients A_n follow from expanding the boundary potential $V_b(y)$ at $x = 0$ into the Fourier series,

$$A_n = \frac{2}{a}\int_0^a V_b(y)\times\sin\left(\frac{n\pi y}{a}\right)dy \quad\Longrightarrow\quad V_b(y) = \sum_{n=1}^{\infty} A_n\times\sin\left(\frac{n\pi y}{a}\right). \tag{3.34}$$

For the problem at hand, $V_b(y)$ is a piece-wise constant function with a jump at $x = \frac{a}{2}$,

$$v_0(y) = \begin{cases} +V_0 & \text{for } 0 < y < \frac{a}{2}, \\ -V_0 & \text{for } \frac{a}{2} > y < a. \end{cases} \tag{74}$$

Plugging this boundary potential into eq. (3.34), we obtain

$$\begin{aligned} A_n &= +\frac{2V_0}{a}\int_0^{a/2}\sin\left(\frac{n\pi y}{a}\right)dy - \frac{2V_0}{a}\int_{a/2}^{a}\sin\left(\frac{n\pi y}{a}\right)dy \\ &= +\frac{2V_0}{a}\times\frac{a}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) - \frac{2V_0}{a}\times\frac{a}{n\pi}\left(\cos\frac{n\pi}{2}-\cos(\pi n)\right) \\ &= \frac{2V_0}{n\pi}\times\left(1+\cos(n\pi)-2\cos\frac{n\pi}{2}\right) \\ &= \frac{2V_0}{n\pi}\times\begin{cases} 4 & \text{if } n\equiv 2 \pmod 4, \\ 0 & \text{otherwise.}\end{cases} \end{aligned} \tag{75}$$

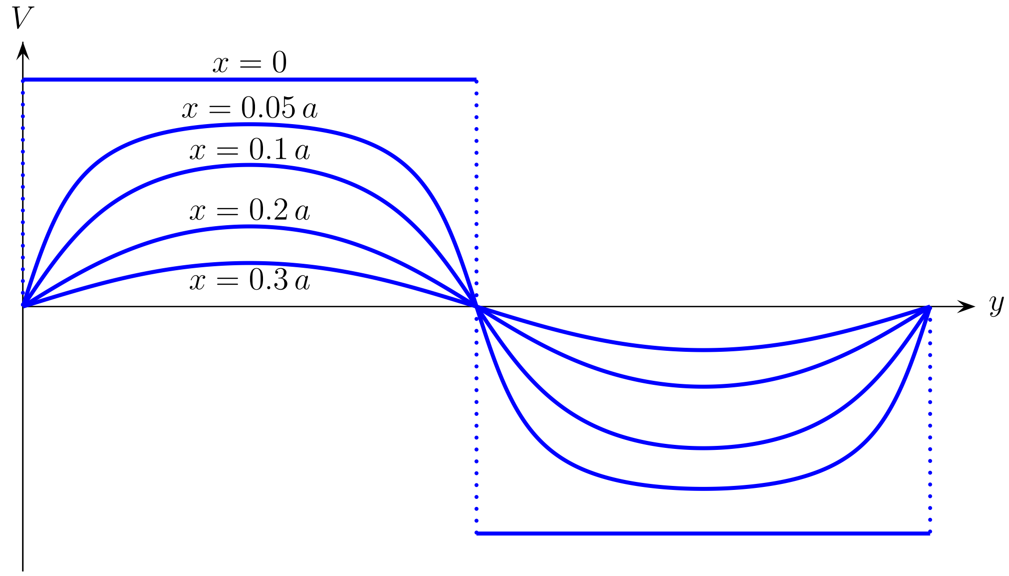
Thus, in the expansion (3.30) we may restrict n to $n = 4m+2$ for $m = 0, 1, 2, 3, \ldots$; for all other n the coefficients C_n are zero. Consequently, the series (3.30) becomes

$$V(x,y) = \frac{2V_0}{\pi}\sum_{m=0}^{\infty}\frac{4}{4m+2}\times\sin\left(\frac{(4m+2)\pi y}{a}\right)\times\exp\left(-\frac{(4m+2)\pi y}{a}\right). \tag{76}$$

Note that this series looks just like the series in the textbook equation (3.36), except for rescaling the coordinates, $x \to 2x$, $y \to 2y$. Consequently, we may sum up the series (76) in exactly the same way as in eq. (3.37), namely

$$V(x,y) = \frac{2V_0}{\pi}\times\arctan\left(\frac{\sin(2\pi y/a)}{\sinh(2\pi x/a)}\right). \tag{77}$$

Graphically, it looks like the square wave (74) at small $x \ll a$, but for larger x it starts looking more and more like the sine wave $\sin(2\pi y/a)$ with exponentially decreasing amplitudes. Here is the slice-plot of the potential $V(y)$ for few fixed x ranging from 0 to $0.3\,a$:



Problem **3.14**:

The electric charge density σ on a surface of a conductor is related to the electric field immediately outside the conductor,

$$\sigma \;=\; \epsilon_0 \times E_\perp[\text{just outside the conductor}]. \tag{78}$$

For the problem at hand, the textbook eqs. (3.36–37) give us the potential inside the slot,

$$V(x,y) \;=\; \frac{4V_0}{\pi}\sum_{\text{odd}\,n\geq 1}\frac{1}{n}\times\sin\left(\frac{n\pi y}{a}\right)\times\exp\left(-\frac{n\pi x}{a}\right) \tag{3.36}$$

$$= \frac{2V_0}{\pi}\,\arctan\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right), \tag{3.37}$$

from which we may obtain the electric field anywhere inside the slot, and in particular just outside the conducting strip at the slot's left end. Consequently, eq. (78) will gives us the charge density $\sigma(y)$ on the *right side* of the vertical conducting strip at $x=0$ — the side facing the inside of the slot.

Presumably, these is also some other charge density on the left side of the vertical strip facing the outside world, but to find that density we would need to know the electric field outside the slot. That task is beyond the present homework, so I am not going to write the solution here.

Going back to the potential (3.37) and taking the $\partial/\partial x$ derivative, we have

$$\frac{d}{dx}\arctan\frac{A}{B(x)} = \frac{d\arctan(A/B)}{d(A/B)} \times \frac{d(A/B(x))}{dx} = \frac{1}{1+(A/B)^2} \times \frac{-AB'}{B^2} = -\frac{AB'}{A^2+B^2} \tag{79}$$

hence for $A = \sin(\pi y/a)$ and $B(x) = \sinh(\pi x/a)$,

$$\frac{\partial}{\partial x}\arctan\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right) = -\frac{\sin(\pi y/a) \times (\pi/a)\cosh(\pi x/a)}{\sin^2(\pi y/a) + \sinh^2(\pi x/a)} \tag{80}$$

and therefore

$$E_x(x,y) = -\frac{\partial V}{\partial x} = +\frac{2V_0}{a} \times \frac{\sin(\pi y/a) \times \cosh(\pi x/a)}{\sin^2(\pi y/a) + \sinh^2(\pi x/a)}. \tag{81}$$

For $x \to 0$ this formula simplifies to

$$E_x(0,y) = \frac{2V_0}{a} \times \frac{1}{\sin(\pi y/a)}, \tag{82}$$

so according to eq. (78), the charge density on the right side of the vertical strip is

$$\sigma(y) = \frac{2\epsilon_0 V_0}{a} \times \frac{1}{\sin(\pi y/a)}. \tag{83}$$

<u>Alternative</u> derivation of eq. (82):
Suppose we did not have the analytical formula (3.37) for the potential inside the slot but only the un-summed series (3.36). In this case, we may take the x–derivative term-by-term to get a series for the electric field, then simplify that series for $x = 0$, and finally try to sum

it up. Thus, starting with the potential (3,36), we have

$$\begin{aligned}
E_x(x,y) &= -\frac{\partial}{\partial x}\left[\frac{4V_0}{\pi}\sum_{\text{odd}\,n\ge1}\frac{1}{n}\times\sin\left(\frac{n\pi y}{a}\right)\times\exp\left(-\frac{n\pi x}{a}\right)\right] \\
&= -\frac{4V_0}{\pi}\sum_{\text{odd}\,n\ge1}\frac{1}{n}\times\sin\left(\frac{n\pi y}{a}\right)\times\left[\frac{d}{dx}\exp\left(-\frac{n\pi x}{a}\right) = -\frac{n\pi}{a}\times\exp\left(-\frac{n\pi x}{a}\right)\right] \\
&= +\frac{4V_0}{a}\sum_{\text{odd}\,n\ge1}\sin\left(\frac{n\pi y}{a}\right)\times\exp\left(-\frac{n\pi x}{a}\right),
\end{aligned} \tag{84}$$

which for $x=0$ simplifies to

$$E_x(0,y) = \frac{4V_0}{a}\sum_{\text{odd}\,n\ge1}\sin\left(\frac{n\pi y}{a}\right). \tag{85}$$

To sum up this Fourier series we use

$$\sin(n\xi) = \operatorname{Im}\bigl(e^{in\xi}\bigr) = \operatorname{Im}\Bigl(\bigl(e^{i\xi}\bigr)^n\Bigr) \quad \text{for } \xi=\pi(y/a). \tag{86}$$

Consequently,

$$\begin{aligned}
\sum_{\text{odd}\,n\ge1}\sin(n\xi) &= \operatorname{Im}\left(\sum_{\text{odd}\,n\ge1}\bigl(e^{i\xi}\bigr)^n\right) \\
&= \operatorname{Im}\left(\sum_{m=0}^{\infty}\bigl(e^{i\xi}\bigr)^{2m+1} = \frac{e^{i\xi}}{1-\bigl(e^{i\xi}\bigr)^2} = \frac{1}{e^{-i\xi}-e^{+i\xi}} = \frac{1}{-2i\sin(\xi)}\right) \\
&= \frac{+1}{2\sin(\xi)},
\end{aligned} \tag{87}$$

and therefore

$$E_x(0,y) = \frac{2V_0}{a}\times\frac{1}{\sin(\pi y/a)}, \tag{88}$$

in perfect agreement with eq. (82).

Problem **3.15**:

(a) Using the separation of variables method in Cartesian coordinates (x, y) [for the problem at hand, the potential does not depend on z], we look for solutions of the Laplace equation (and some of the boundary conditions) in the form

$$V(x,y) = \sum_n C_n \times f_n(x) \times g_n(y). \tag{89}$$

Similar to the infinite slot example discussed in class, we have

$$g_n(y) = \sin(n\pi y/a) \quad \text{for integer } n = 1, 2, 3, \ldots. \tag{90}$$

hence $f_n(x)$ obeys $f_n''(x) = +(n\pi/a)^2 f_n(x)$ and therefore

$$C_n f_n(x) = A_n \exp(+n\pi x/a) + B_n \exp(-n\pi x/a) \tag{91}$$

for some constant coefficients A_n and B_n. For the infinite slot, the asymptotic condition at $x \to \infty$ requires $A_n = 0$, but for the rectangular pipe we have a different condition. Specifically, since the boundary condition at the left wall $x = 0$ is $V = 0$, we need $f_n(0) = 0$, which calls for $A_n + B_n = 0$. In other words, up to an overall coefficient,

$$f_n(x) = \sinh(n\pi x/a), \tag{92}$$

hence

$$V(x,y) = \sum_{n=1}^{\infty} C_n \times \sinh(n\pi x/a) \times \sin(n\pi y/a). \tag{93}$$

The coefficients C_n in this series are determined from the boundary condition at the right wall $x = b$:

$$\sum_{n=1}^{\infty} C_n \sinh(n\pi b/a) \times \sin(n\pi y/a) = \text{given } V_b(y). \tag{94}$$

Thus, Fourier transforming this boundary potential $V_b(y)$, we immediately obtain

$$C_n \times \sinh(n\pi b/a) = \frac{2}{a} \int_0^a da \, \sin(n\pi y/a) \times V_b(y). \tag{95}$$

Once we evaluate all these integrals, find all the coefficients C_n, plug them into eq. (93), and sum the series, we would obtain the potential $V(x, y)$ over the entire interior of the pipe.

(b) Now consider the special case $V_b(y) = V_0 = \text{const}$. Fourier transforming this constant into a series of sine waves, we obtain

$$\begin{aligned} \frac{2}{a}\int_0^a dy\, \sin(n\pi y/a) \times V_0 &= \frac{2V_0}{a} \times \left[\frac{-\cos(n\pi y/a)}{(n\pi/a)}\right]_{y=0}^{y=a} \\ &= \frac{2V_0}{n\pi} \times (1 - \cos(n\pi) \;=\; \frac{2V_0}{n_\pi} \times \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n, \end{cases} \end{aligned} \tag{96}$$

and hence

$$C_{\text{odd}\,n} \;=\; \frac{(4/n\pi)V_0}{\sinh(n\pi b/a)}, \quad C_{\text{even}\,n} \;=\; 0. \tag{97}$$

Plugging these coefficient into the series (93), we arrive at

$$V(x,y) \;=\; \sum_{\text{odd}\, n=1,3,5,\ldots} \frac{4V_0}{n\pi} \times \frac{\sinh(n\pi x/a)}{\sinh(n\pi b/a)} \times \sin(n\pi y). \tag{98}$$

I don't know how to sum this series analytically, and I don't expect you to do it in your homework (or even try to do it). So let me simply sum the series numerically for $b = \frac{1}{2}a$ and present a 3D plot of the solution: