Problem **3.17**:

Rodriguez formula gives the Legendre polynomials as

$$P_\ell(x) \;=\; \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell}\,(x^2-1)^\ell. \tag{1}$$

For $\ell = 3$ it evaluates to

$$P_3(x) \;=\; \frac{1}{8\times 6}\,\frac{d^3}{dx^3}\Big((x^2-1)^3 \;=\; x^6-3x^4+3x^2-1\Big) \;=\; \frac{1}{48}\times\big(120x^3-3\times 24x\big) \;=\; \tfrac{5}{2}x^3-\tfrac{3}{2}x. \tag{2}$$

Let's check that this polynomial obeys the Legendre equation for $\ell = 3$:

$$\begin{aligned}(1-x^2)\times P_3''(x) \;-\; 2x\times P_3'(x) \;+\; 3(3+1)\times P_3(x) \;&=\\ &=\; (1-x^2)\times 15x \;-\; 2x\times\frac{15x^2-3}{2} \;+\; 12\times\frac{5x^3-3x}{2}\\ &=\; 15x \;-\; 15x^3 \;-\; 15x^3 \;+\; 3x \;+\; 30x^3 \;-\; 18x\\ &=\; 0.\end{aligned} \tag{3}$$

Or in terms of $g(\theta) = P_3(\cos\theta)$,

$$\begin{aligned} g(\theta) \;&=\; \tfrac{5}{2}\cos^3\theta \;-\; \tfrac{3}{2}\cos\theta,\\ \frac{d}{d\theta}\,g(\theta) \;&=\; -\tfrac{15}{2}\cos^2\theta\sin\theta \;+\; \tfrac{3}{2}\sin\theta,\\ \frac{d^2}{d\theta^2}\,g(\theta) \;&=\;=\; +15\cos\theta\sin^2\theta \;-\; \tfrac{15}{2}\cos^3\theta \;+\; \tfrac{3}{2}\cos\theta,\end{aligned} \tag{4}$$

hence

$$\begin{aligned}\frac{d^2g}{d\theta^2} \;+\; \frac{\cos\theta}{\sin\theta}\times\frac{dg}{d\theta} \;+\; 3(3+1)\times g \;&=\\ =\; 15\cos\theta\sin^2\theta \;-\; \tfrac{15}{2}\cos^3\theta \;+\; \tfrac{3}{2}\cos\theta &\\ -\; \tfrac{15}{2}\cos^3\theta \;+\; \tfrac{3}{2}\cos\theta \;+\; 12\times\tfrac{5}{2}\cos^3\theta \;-\; 12\times\tfrac{3}{2}\cos\theta &\\ =\; 15\cos\theta\sin^2\theta \;+\; 15\cos^3\theta \;-\; 15\cos\theta &\\ =\; 15\cos\theta\times\big(\sin^2\theta \;+\; \cos^2\theta \;-\; 1\big) &\\ =\; 0. &\end{aligned} \tag{5}$$

Finally, let's check the orthogonality of the Legendre polynomials $P_3(x)$ and $P_1(x) = x$:

$$\begin{aligned}
\int_{-1}^{+1} P_3(x) \times P_1(x)\,dx &= \int_{-1}^{+1} \frac{5x^3 - 3x}{2} \times x\,dx = \int_{-1}^{+1} \frac{5x^4 - 3x^2}{2}\,dx \\
&= \int_{-1}^{+1} d\left(\frac{x^5 - x^3}{2}\right) = \left(\frac{x^5 - x^3}{2}\right)\Bigg|_{x=-1}^{x=+1} \\
&= 0 - 0 = 0.
\end{aligned} \tag{6}$$

<u>Problem</u> **<u>3.19</u>**:

(a) See the example on page 32 of my notes on separation of variables: Given the potential on the sphere's of the form

$$V_b(\theta) = k \times \cos(3\theta) \tag{7}$$

for some constant k, we may expand it into Legendre polynomials of $\cos\theta$ as

$$V_b(\theta) = \frac{8k}{5} \times P_3(\cos\theta) - \frac{3k}{5} \times P_1(\cos\theta). \tag{8}$$

(b) For each Legendre polynomial $P_\ell(\cos\theta)$, the potential behaves like r^ℓ inside the sphere and $r^{-\ell-1}$ outside the sphere, *cf.* equations (163–4) on page 31 of my notes. Thus, the potential (8) on the spherical surface extends inside and outside the sphere as

$$\text{inside the sphere} \quad V(r,\theta) = \frac{8k}{5} \times P_3(\cos\theta) \times \frac{r^3}{R^3} - \frac{3k}{5} \times P_1(\cos\theta) \times \frac{r}{R}, \tag{9}$$

$$\text{outside the sphere} \quad V(r,\theta) = \frac{8k}{5} \times P_3(\cos\theta) \times \frac{R^4}{r^4} = \frac{3k}{5} \times P_1(\cos\theta) \times \frac{R^2}{r^2}. \tag{10}$$

(c) The surface charge density on the sphere in question follows from the discontinuity of the radial electric field:

$$\begin{aligned}
\text{inside the sphere}\quad E_r &= -\frac{\partial V}{\partial r} = \frac{8k}{5}\times P_3(\cos\theta)\times\frac{-3r^2}{R^3} - \frac{3k}{5}\times P_1(\cos\theta)\times\frac{-1}{R}\\
&\xrightarrow[r\to R-0]{} -\frac{24k}{5R}\times P_3(\cos\theta) + \frac{3k}{5R}\times P_1(\cos\theta), \qquad (11)\\
\text{outside the sphere}\quad E_r &= -\frac{\partial V}{\partial r} = \frac{8k}{5}\times P_3(\cos\theta)\times\frac{4R^4}{r^5} - \frac{3k}{5}\times P_1(\cos\theta)\times\frac{2R^2}{r^3}\\
&\xrightarrow[r\to R+0]{} +\frac{32k}{5R}\times P_3(\cos\theta) - \frac{6k}{5R}\times P_1(\cos\theta), \qquad (12)
\end{aligned}$$

hence

$$\begin{aligned}
\sigma(\theta) = \epsilon_0\,\mathrm{disc}(E_r@R) &= \frac{56k\epsilon_0}{5R}\times P_3(\cos\theta) - \frac{9k\epsilon_0}{5R}\times P_1(\cos\theta)\\
&= \frac{k\epsilon_0}{R}\Big[28\cos^3\theta - \tfrac{93}{5}\cos\theta\Big].
\end{aligned}\qquad (13)$$

Problem **3.25**:

This problem calls for separation of variables in cylindrical coordinates. It is not one of the textbook examples, but it is discussed in detail in my notes on separation of variables (pages 19–26).

In this problem we are concerned with the outside of a cylindrical pipe, but for a non-trivial asymptotic potential far away from the pipe: Instead of $V(s,\phi)\to 0$ for $s\to\infty$, in this problem we get

$$V \;\to\; -E_0\times x \;=\; -E_0\times s\times\cos\phi. \qquad (14)$$

Here I use the coordinate system where the pipe's axis is the z axis while the electric field $\mathbf{E}_0$ far away from the pipe points in the x direction.

As explained in my notes, a general solution of the 2D Laplace equation in the form

$V(s,\phi) = f(s) \times g(\phi)$ has

$$\begin{aligned} g(\phi) &= A \times \cos(m\phi) \; + \; B \times \sin(m\phi) \\ &\qquad \text{for some integer } m = 0, 1, 2, 3 \ldots, \\ f(s) &= C \times (s/R)^m \; + \; D \times (R/s)^m \quad \text{for } m > 0, \\ f(s) &= C \times \ln(s/R) \; + \; D \quad \text{for } m = 0. \end{aligned} \tag{15}$$

In my notes, for the space outside the cylinder I take $C = 0$ to make sure V stays finite for $s \to \infty$, but for the problem at hand I have a different asymptotics, so I allow for $C \neq 0$. On the other hand, the surface of the conducting pipe must have a uniform potential $V_0(\phi) = \text{const}$, and by the $V(-x,y,z) = -V(x,y,z)$ symmetry of the problem, we take this V_0 to be zero. This gives us a boundary condition $V(s = R, \text{any } \phi) = 0$ and hence $f(s = R) = 0$. To satisfy this boundary condition, we need

$$\begin{aligned} C \; + \; D &= 0 \quad \text{for } m > 0, \\ D &= 0 \quad \text{for } m = 0, \end{aligned} \tag{16}$$

and therefore

$$\begin{aligned} f(s) &= C \times \big((s/R)^m \; - \; (R/s)^m \big) \quad \text{for } m > 0, \\ f(s) &= C \times \ln(s/R) \quad \text{for } m = 0. \end{aligned} \tag{17}$$

Altogether, the solutions of the form $f(s) \times g(\phi)$ are

$$\begin{aligned} V(s,\phi) &= \big((s/R)^m \; - \; (R/s)^m \big) \times \big(A\cos(m\phi) + B\sin(m\phi) \big) \quad \text{for } m > 0, \\ V(s,\phi) &= \ln(s/R) \times A \quad \text{for } m = 0, \end{aligned} \tag{18}$$

while the general solutions of the 2D Laplace equations with $V(s = R) = 0$ are linear combinations of the above, thus

$$V(s,\phi) \; = \; A_0 \ln(s/R) \; + \sum_{m=1}^{\infty} \big(A\cos(m\phi) + B\sin(m\phi) \big) \times \big((s/R)^m \; - \; (R/s)^m \big). \tag{19}$$

Now consider the asymptotic behavior of this potential for $s \to \infty$. For $s \gg R$, we may approximate

$$(s/R)^m \;-\; (R/s)^m \;\approx\; (s/R)^m, \tag{20}$$

hence

$$\text{for } s\to\infty,\quad V(s,\phi) \;\longrightarrow\; A_0\times\ln(s/R) \;+\; \sum_{m=1}^{\infty}\bigl(A\cos(m\phi)+B\sin(m\phi)\bigr)\times(s/R)^m. \tag{21}$$

At the same time, the uniform electric field $\mathbf{E}_0$ far away from the pipe tells us that

$$\text{for } s\to\infty,\quad V(s,\phi) \;\longrightarrow\; -E_0\cos\phi\times s. \tag{22}$$

Matching these two asymptotic expressions tells us that

$$A_1 \;=\; -E_0R,\quad \text{all other } A_m \;=\; 0,\quad \text{all } B_m \;=\; 0. \tag{23}$$

Consequently, everywhere outside the pipe,

$$V(s,\phi) \;=\; -E_0R\times\cos\phi\times\bigl((s/R)\;-\;(R/s)\bigr) \;=\; -E_0\left(s\;-\;\frac{R^2}{s}\right)\times\cos\theta. \tag{24}$$

Finally, the surface charge density on the metal pipe is related to the normal electric field immediately outside the metal. For the round pipe, the normal component of $\mathbf{E}$ is the radial component

$$E_s \;=\; -\frac{\partial V}{\partial s} \;=\; +E_0\left(1\;+\;\frac{R^2}{s^2}\right)\times\cos\theta \;\xrightarrow[s\to R]{}\; +E_0\times 2\times\cos\theta \tag{25}$$

and therefore

$$\sigma(\phi) \;=\; \epsilon_0 E_s(s=R,\phi) \;=\; 2\epsilon_0E_0\times\cos\theta. \tag{26}$$

Problem **3.26**:

This problem also calls for separation of variables in cylindrical coordinates, see pages 19–26 of my notes on separation of variables According to eqs. (121) and (135) of my notes,

$$\begin{aligned} \text{inside the cylinder} \quad V(s,\phi) &= V_0 + \sum_{m=1}^{\infty}\bigl(A_m\cos(m\phi) + B_m\sin(m\phi)\bigr)\times\left(\frac{s}{R}\right)^m, \\ \text{outside the cylinder} \quad V(s,\phi) &= V_0 + \sum_{m=1}^{\infty}\bigl(A_m\cos(m\phi) + B_m\sin(m\phi)\bigr)\times\left(\frac{R}{s}\right)^m, \end{aligned} \tag{27}$$

where the constants V_0, A_m, and B_m follow from expanding the boundary potential $V_b(\phi)$ on the cylinder's surface into a Fourier series:

$$V_b(\phi) = V_0 + \sum_{m=1}^{\infty}\Bigl(A_m\cos(m\phi) + B_m\sin(m\phi)\Bigr), \tag{28}$$

$$A_m = \frac{1}{\pi}\int_0^{2\pi} V_b(\phi)\cos(m\phi)\,d\phi, \quad B_m = \frac{1}{\pi}\int_0^{2\pi} V_b(\phi)\sin(m\phi)\,d\phi, \tag{29}$$

$$V_0 = \frac{1}{2\pi}\int_0^{2\pi} V_b(\phi)\,d\phi. \tag{30}$$

The potential (27) is continuous across the cylinder's surface, but the radial electric field has a discontinuity

Inside the cylinder

$$\begin{aligned} E_s(s,\phi) &= \sum_{m=1}^{+\infty}\bigl(A_m\cos(m\phi) + B_m\sin(m\phi)\bigr)\times\frac{-m\,s^{m-1}}{R^m} \\ &\xrightarrow[s\to R]{} \frac{-1}{R}\sum_{m=1}^{\infty} m\bigl(A_m\cos(m\phi) + B_m\sin(m\phi)\bigr), \end{aligned} \tag{31}$$

Outside the cylinder

$$E_s(s,\phi) \;=\; \sum_{m=1}^{\infty}\bigl(A_m\cos(m\phi)\,+\,B_m\sin(m\phi)\bigr)\times\frac{+m\,R^m}{s^{m+1}}$$
$$\underset{s\to R}{\longrightarrow}\;\frac{+1}{R}\sum_{m=1}^{\infty} m\bigl(A_m\cos(m\phi)\,+\,B_m\sin(m\phi)\bigr), \tag{32}$$

hence the discontinuity at $s=R$ is

$$\mathrm{disc}(E_s) \;=\; +\frac{2}{R}\sum_{m=1}^{\infty} m\bigl(A_m\cos(m\phi)\,+\,B_m\sin(m\phi)\bigr). \tag{33}$$

Physically, this discontinuity stems from the electric charge density on the surface of the cylinder,

$$\mathrm{disc}(E_s) \;=\; \frac{\sigma}{\epsilon_0}\,, \tag{34}$$

hence the ϕ-dependence of this charge density is related to the coefficients A_m and B_m according to

$$\sigma(\phi) \;=\; \frac{2\epsilon_0}{R}\times\sum_{m=1}^{\infty} m\bigl(A_m\cos(m\phi)\,+\,B_m\sin(m\phi)\bigr), \tag{35}$$

$$A_m \;=\; \frac{R}{2\pi m\epsilon_0}\int_0^{2\pi}\sigma(\phi)\,\cos(m\phi)\,d\phi,\quad B_m \;=\; \frac{R}{2\pi m\epsilon_0}\int_0^{2\pi}\sigma(\phi)\,\sin(m\phi)\,d\phi, \tag{36}$$

For the problem at hand, the surface charge density has angular dependence

$$\sigma(\phi) \;=\; a\times\sin(5\phi) \tag{37}$$

for some constant a, and comparing this formula to eq. (35) we immediately identify

$$B_5 \;=\; \frac{aR}{10\epsilon_0}\,,\quad \text{all other } B_m \;=\; 0,\quad \text{all } A_m \;=\; 0. \tag{38}$$

Consequently, plugging these coefficients into the potential (27) gives us

Inside the cylinder

$$V(s,\phi) \;=\; V_0 \;+\; \frac{aR}{10\epsilon_0}\times\frac{s^5}{R^5}\times\sin(5\phi), \tag{39}$$

Outside the cylinder

$$V(s,\phi) \;=\; V_0 \;+\; \frac{aR}{10\epsilon_0}\times\frac{R^5}{s^5}\times\sin(5\phi). \tag{40}$$

The overall constant term V_0 in this potential cannot be determined from the charge density on the cylinder's surface. However, if we further assume the asymptotic condition $V \to 0$ for $s \to \infty$, then we must have $V_0 = 0$.

Problem **3.43**:

(a) Separating variables in spherical coordinates (r, θ, ϕ) and assuming ϕ–independent potentials and charges, we get a solution of the form

$$V(r,\theta) \;=\; P_\ell(\cos\theta)\times\left(A\times r^\ell \;+\; \frac{B}{r^{\ell+1}}\right) \tag{41}$$

for some constant coefficients A and B which depend on the boundary conditions. Thus in general,

Outside the outer shell $(r > b)$

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\times\left(A_\ell\times r^\ell \;+\; \frac{B_\ell}{r^{\ell+1}}\right), \tag{42}$$

Between the inner ball and the outer shell $(a < r < b)$

$$V(r,\theta) \;=\; \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\times\left(C_\ell\times r^\ell \;+\; \frac{D_\ell}{r^{\ell+1}}\right), \tag{43}$$

and all we need to do is to determine the coefficients A_ℓ, B_ℓ, C_ℓ, and D_ℓ from the boundary conditions:

- Far away from the outer shell, the potential must vanish as we go to infinity. To satisfy this boundary condition, we must have **all** $A_\ell = 0$.
- At the surface of the inner ball, the potential is a given constant V_0. In terms of the

expansion (43), this means

$$@r = a, \quad V(\theta) \;=\; V_0 \times P_0(\cos\theta) \;+\; 0 \tag{44}$$

and therefore

$$C_\ell \times a^\ell \;+\; \frac{D_\ell}{a^{\ell+1}} \;=\; \begin{cases} V_0 & \text{for } \ell = 0, \\ 0 & \text{for all other } \ell. \end{cases} \tag{45}$$

- The potential must be continuous across the outer shell at $r = b$, hence

$$A_\ell \times b^\ell \;+\; \frac{B_\ell}{b^{\ell+1}} \;=\; C_\ell \times b^\ell \;+\; \frac{D_\ell}{b^{\ell+1}}\,. \tag{46}$$

- Finally, the discontinuity of the radial electric field at the outer shell is related to the charge density σ at that shell, $\mathrm{disc}(E_r) = \sigma/\epsilon_0$. For the problem at hand $\sigma(\theta) = k\cos\theta$ for some constant k, hence

$$\mathrm{disc}(E_r) \;=\; \frac{k}{\epsilon_0} \times \cos\theta \;=\; \frac{k}{\epsilon_0} \times P_1(\cos\theta). \tag{47}$$

At the same time, for the potentials (42) and (43),

$$\text{for } r > b, \quad E_r(r,\theta) \;=\; \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \times \left(-\ell A_\ell \times r^{\ell-1} \;+\; \frac{(\ell+1)B_\ell}{r^{\ell+2}}\right), \tag{48}$$

$$\text{for } a < r < b, \quad E_r(r,\theta) \;=\; \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \times \left(-\ell C_\ell \times r^{\ell-1} \;+\; \frac{(\ell+1)D_\ell}{r^{\ell+2}}\right), \tag{49}$$

hence the discontinuity at $r = b$ is

$$\mathrm{disc}(E_r) \;=\; \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \times \Big(\ell \times (C_\ell - A_\ell) \times b^{\ell-1} \;+\; (\ell+1) \times (B_\ell - D_\ell) \times b^{-\ell-2}\Big). \tag{50}$$

Comparing this discontinuity to eq. (47), we obtain

$$\ell \times (C_\ell - A_\ell) \times b^{\ell-1} \;+\; (\ell+1) \times (B_\ell - D_\ell) \times b^{-\ell-2} \;=\; \begin{cases} (k/\epsilon_0) & \text{for } \ell = 1, \\ 0 & \text{for all other } \ell. \end{cases} \tag{51}$$

Now, let's solve all these boundary conditions for the coefficients A_ℓ, B_ℓ, C_ℓ and D_ℓ. For $\ell > 1$ all 4 equations relating the coefficients are linear without free terms, so the solution is obviously $A_\ell = B_\ell = C_\ell = D_\ell = 0$. Indeed, combining eqs. (51) and (46) for $\ell \neq 1$ gives us

$$\frac{\ell}{\ell+1} \times (C_\ell - A_\ell) \;=\; \frac{B_\ell - D_\ell}{b^{2\ell+1}} \;=\; -(C_\ell - A_\ell) \quad\Longrightarrow\quad B_\ell \;=\; D_\ell \quad\text{and}\quad C_\ell \;=\; A_\ell. \tag{52}$$

But $A_\ell = 0$ by the asymptotic condition at $r \to \infty$, so we must also have $C_\ell = 0$. Finally, at the inner ball, eq. (45) gives $D_\ell = -a^{2\ell+1} C_\ell = 0$ and hence $B_\ell = 0$ as well.

For $\ell = 0$ we also have $D_0 = B_0$ and $C_0 = A_0 = 0$, but then eq. (45) becomes

$$\frac{D_0}{a} \;=\; V_0, \tag{53}$$

hence $B_0 = D_0 = a \times V_0$.

Finally, for $\ell = 1$ the relevant equations are

$$\begin{aligned} A_1 &= 0, \\ a^3 C_1 \;+\; D_1 &= 0, \\ b^3(C_1 - A_1) \;+\; (D_1 - B_1) &= 0, \\ -b^3(C_1 - A_1) \;+\; 2(D_1 - B_1) &= b^3 \times (k/\epsilon_0), \end{aligned} \tag{54}$$

and solving these linear equations yields

$$A_1 \;=\; 0, \quad B_1 \;=\; +\frac{k}{3\epsilon_0} \times (b^3 - a^3), \quad C_1 \;=\; +\frac{k}{3\epsilon_0}\,, \quad D_1 \;=\; -\frac{k}{3\epsilon_0} \times a^3. \tag{55}$$

It remains to plug all these coefficients into eqs. (42) and (43):

Outside the outer shell ($r > b$)

$$V(r,\theta) \;=\; \frac{aV_0}{r} \;+\; \frac{k}{3\epsilon_0} \times \frac{b^3 - a^3}{r^2} \times \cos\theta, \tag{56}$$

Between the inner ball and the outer shell ($a < r < b$)

$$V(r,\theta) \;=\; \frac{aV_0}{r} \;+\; \frac{k}{3\epsilon_0} \times \left(r \;-\; \frac{a^3}{r^2}\right) \times \cos\theta. \tag{57}$$

This completes part (a) of the problem.

(b) The density of the induced charge on the surface of the conducting inner ball obtains from the radial electric field immediately outside this surface,

$$\sigma(\theta) \;=\; \epsilon_0 \times E_r(a,\theta) \;=\; -\epsilon_0 \times \left.\frac{\partial V}{\partial r}\right|_{r=a}. \tag{58}$$

Taking the derivative of the potential (57) (for $a < r < b$) yields

$$E_r(r,\theta) \;=\; \frac{aV_0}{r^2} \;+\; \frac{k}{3\epsilon_0} \times \left(-1 \;-\; \frac{2a^3}{r^3}\right) \times \cos\theta. \tag{59}$$

Hence, taking the limit $r \to a$ and multiplying by the ϵ_0 yields the charge density on the conducting ball,

$$\sigma(\theta) \;=\; \frac{\epsilon_0 V_0}{a} \;-\; \frac{k}{3} \times 3 \times \cos\theta\,. \tag{60}$$

Physically, the first term here stems from the net electric charge on the ball while the second terms is induced by the charges on the outer shell.

(c) There are two ways to find the net electric charge of the system. One way is to look at the leading $1/r$ term in the potential at large distances from the system,

$$\frac{Q^{\text{net}}}{4\pi\epsilon_0} \;=\; \lim_{r\to\infty}\bigl(r \times v(r)\bigr). \tag{61}$$

A quick glance on the solution (56) for the potential outside the outer shell shows that

$$\lim_{r\to\infty}\bigl(r \times v(r)\bigr) \;=\; aV_0 \tag{62}$$

and therefore

$$Q^{\text{net}} \;=\; 4\pi\epsilon_0 \times a \times V_0. \tag{63}$$

The other way to get the net charge is to simply integrate the given $\sigma = k\cos\theta$ over the outer shell and the solution to part (b) over the surface of the inner ball. Thus

$$Q^{\text{outer shell}} = \int_0^\pi k\cos\theta \times 2\pi b^2 \sin\theta\, d\theta = 0, \tag{64}$$

while

$$Q^{\text{inner ball}} = \int_0^\pi \left(\frac{\epsilon_0 V_0}{a} - k\cos\theta\right) \times 2\pi a^2 \sin\theta\, d\theta = \frac{\epsilon_0 V_0}{a} \times 4\pi a^2 + 0 = 4\pi\epsilon_0 a V_0\,, \tag{65}$$

hence

$$Q^{\text{net}} = Q^{\text{outer shell}} + Q^{\text{inner ball}} = 4\pi\epsilon_0 a V_0\,. \tag{66}$$

We see that both methods yield the same net charge, and this gives us a useful consistency check for our calculations.

Problem **3.55**(a):

Since nothing in the problem depends on the z coordinate (along the square pipe), the problem amounts to solving the 2D Laplace equation $\triangle V(x,y) = 0$ inside the $a \times a$ square subject to boundary conditions

$$\begin{aligned} V(x,y) &= 0 \quad \text{for } (x=0 \text{ or } x=a) \text{ and any } y, \\ V(x,y) &= 0 \quad \text{for } y=0 \text{ and any } x, \\ V(x,y) &= V_0 = \text{const} \quad \text{for } y=a \text{ and any } x. \end{aligned} \tag{67}$$

As explained in the textbook — and also in my notes on the separations of variables — the general solution to the Laplace equation subject to the first 2 boundary conditions here is

$$V(x,y) = \sum_{n=1}^{\infty} A_n \times \sin\frac{n\pi x}{a} \times \sinh\frac{n\pi y}{a} \tag{68}$$

for some constant coefficients A_n. The actual values of these coefficients follow from the last boundary condition at $y = a$:

$$V(x, y = a) \;=\; \sum_{n=1}^{\infty} A_n \times \sin\frac{n\pi x}{a} \times \sinh(n\pi) \;= \text{should be} =\; V_0 \;=\; \text{const.} \tag{69}$$

Hence $A_n \times \sinh(n\pi)$ are the coefficients of Fourier expansion of a constant V_0 into sine waves, thus

$$\begin{aligned}
A_n \times \sinh(n\pi) \;&=\; \frac{2}{a}\int_0^a dx\, V_0 \times \sin\frac{n\pi x}{a} \\
&=\; \frac{2V_0}{n\pi}\int_0^{n\pi} d\xi\, \sin(\xi) \qquad \langle\!\langle\, \text{where } \xi = (n\pi x/a) \,\rangle\!\rangle \\
&=\; \frac{2V_0}{n\pi}\Big(\cos(0) \;-\; \cos(n\pi)\Big) \\
&=\; \frac{2V_0}{n\pi} \times \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}
\end{aligned} \tag{70}$$

Altogether,

$$V(x, y) \;=\; \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{4V_0}{n\pi} \times \frac{\sinh(n\pi y/a)}{\sinh(n\pi)} \times \sin(n\pi x/a). \tag{71}$$

Now consider the surface charges σ on the (inside surface of) the pipe opposite to the V_0 side, *i.e.* at $y = 0$. This surface charge density is related to the normal electric field — the E_y component of the $\mathbf{E}$ immediately outside the metal, thus

$$\sigma(x) \;=\; \epsilon_0 E_y(x, y \to 0) \;=\; -\epsilon_0 \left.\frac{\partial V}{\partial y}\right|_{y=0}. \tag{72}$$

For the potential (71),

$$\begin{aligned}
\frac{\partial V}{\partial y} \;&=\; \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{4V_0}{n\pi} \times \sin(n\pi x/a) \times \frac{(n\pi/a)\cosh(n\pi y/a)}{\sinh(n\pi)} \\
&\xrightarrow[y\to 0]{}\; \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{4V_0}{n\pi} \times \sin(n\pi x/a) \times \frac{(n\pi/a)}{\sinh(n\pi)} \\
&=\; \frac{4V_0}{a} \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{1}{\sinh(n\pi)} \times \sin(n\pi x/a),
\end{aligned} \tag{73}$$

and hence

$$\sigma(x) \;=\; -\frac{4\epsilon_0 V_0}{a} \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{1}{\sinh(n\pi)} \times \sin(n\pi x/a). \tag{74}$$

Integrating this surface charge density over x we find the net charge (or the wall opposite to V_0) per unit of pipe's length as

$$\lambda \;=\; \int_0^a \sigma(x)\,dx \;=\; -\frac{4\epsilon_0 V_0}{a} \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{1}{\sinh(n\pi)} \times \int_0^a \sin(n\pi x/a)\,dx. \tag{75}$$

In this formula

$$\int_0^a \sin(n\pi x/a)\,dx \;=\; \frac{a}{n\pi} \int_0^{n\pi} \sin\xi\,d\xi \;=\; \frac{2a}{n\pi} \quad \text{for odd } n, \tag{76}$$

hence

$$\lambda \;=\; -\frac{8\epsilon_0 V_0}{\pi} \sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{1}{n \times \sinh(n\pi)}\,. \tag{77}$$

I have no idea how to sum the series here analytically, but a numeric sum yields

$$\sum_{n=1,3,5,\ldots}^{\text{odd}} \frac{8}{n \times \sinh(n\pi)} \;\approx\; 0.693147\ldots \;\approx\; \ln(2), \tag{78}$$

and in light of the answer in the textbook I presume the equality here is exact. Thus,

$$\lambda \;=\; -\frac{\ln(2)}{\pi}\,\epsilon_0 V_0\,. \tag{79}$$

Problem **3.55**(b):

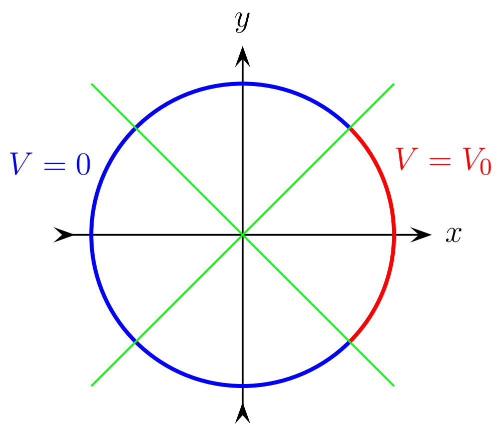
Now consider a round pipe. As explained in my notes on the separation of variables (pages 19–26 for the polar coordinates), inside a cylinder

$$V(s,\phi) \;=\; A_0 \;+\; \sum_{m=1}^{\infty}\bigl(A_m\cos(m\phi)\;+\;B_m\sin(m\phi)\bigr)\times(s/R)^m \tag{80}$$

for some coefficients A_m and B_m, whose values obtain from the boundary potential on the cylinder's surface,

$$\begin{aligned} A_0 \;&=\; \frac{1}{2\pi}\int\limits_0^{2\pi} V_b(\phi)\,d\phi,\\ \text{other } A_m \;&=\; \frac{1}{\pi}\int\limits_0^{2\pi} V_b(\phi)\times\cos(m\phi)\,d\phi,\\ B_m \;&=\; \frac{1}{\pi}\int\limits_0^{2\pi} V_b(\phi)\times\sin(m\phi)\,d\phi. \end{aligned} \tag{81}$$

To make the problem at hand more symmetric, let the x axis (where $\phi=0$) runs through the middle of the V_0 quarter of the pipe. In cross-section



(82)

Consequently,

$$A_0 \;=\; \frac{V_0}{2\pi}\int\limits_{-\pi/4}^{+\pi/4} d\phi \;=\; \frac{V_0}{4}\,, \tag{83}$$

$$A_m = \frac{V_0}{\pi} \int\limits_{-\pi/4}^{+\pi/4} \cos(m\phi)\, d\phi = \frac{2V_0}{m\pi} \times \sin\frac{m\pi}{4}\,, \tag{84}$$

$$B_m = \frac{V_0}{\pi} \int\limits_{-\pi/4}^{+\pi/4} \sin(m\phi)\, d\phi = 0, \tag{85}$$

and therefore inside the pipe

$$V(s,\phi) = \frac{V_0}{4} + \frac{2V_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin(m\pi/4)}{m} \times \cos(m\phi) \times (s/R)^m. \tag{86}$$

The surface charge density on the (inner side of the) pipe is related to the normal electric field just outside the metal, where the normal direction (from the metal out) is opposite to the radial, thus

$$\sigma(\phi) = -\epsilon_0 E_s(s=R,\phi) = +\epsilon_0 \left.\frac{\partial V}{\partial s}\right|_{s=R}, \tag{87}$$

hence for the potential at hand

$$\begin{aligned}\sigma(\phi) &= \frac{2\epsilon_0 V_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin(m\pi/4)}{m} \times \cos(m\phi) \times \frac{ms^{m-1}}{R^m} \\ &\xrightarrow[s\to R]{} \frac{2\epsilon_0 V_0}{\pi R} \sum_{m=1}^{\infty} \sin(m\pi/4) \times \cos(m\phi).\end{aligned} \tag{88}$$

Finally, let's integrate this charge density over the width $R\,d\phi$ of the quadrant opposite to the V_0; in terms of ϕ, this quadrant lies between $\phi = \frac{3\pi}{4}$ and $\phi = \frac{5\pi}{4}$. Thus the net charge of that quadrant per unit of the pipe's length is

$$\lambda = \int\limits_{3\pi/4}^{5\pi/4} \sigma(\phi) \times R\, d\phi = \frac{2\epsilon_0 V_0}{\pi} \sum_{m=1}^{\infty} \sin(m\pi/4) \times \int\limits_{3\pi/4}^{5\pi/4} \cos(m\phi)\, d\phi \tag{89}$$

where

$$\int\limits_{3\pi/4}^{5\pi/4} \cos(m\phi)\,d\phi \;=\; \frac{1}{m}\int\limits_{3m\pi/4}^{5m\pi/4} d\sin(m\phi) \;=\; \frac{1}{m}\big(\sin(5m\pi/4) \,-\, \sin(3m\pi/4)\big) \;=\; \frac{2(-1)^m}{m}\,\sin(m\pi/4). \tag{90}$$

Consequently,

$$\lambda \;=\; \frac{4\epsilon_0 V_0}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\times\sin^2(m\pi/4) \tag{91}$$

where

$$\sin^2(m\pi/4) \;=\; \tfrac{1}{2}\big(1 \,-\, \cos(m\pi/2)\big) \;=\; \frac{2 \,-\, (+i)^m \,-\, (-i)^m}{4}\,, \tag{92}$$

hence

$$\lambda \;=\; \frac{\epsilon_0 V_0}{\pi}\sum_{m=1}^{\infty}\frac{2(-1)^m \,-\, (-i)^m \,-\, (+i)^m}{m}\,, \tag{93}$$

The series here is conditionally convergent, but we can make it absolutely convergent by multiplying each term by η^m for $|\eta| < 1$, and once we some it up, take the limit $\eta \to 1$. Thus,

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{2(-1)^m \,-\, (-i)^m \,-\, (+i)^m}{m}\times\eta^m \;&=\; \sum_{m=1}^{\infty}\frac{1}{m}\times\Big(2(-\eta)^m \,-\, (i\eta)^m \,-\, (-i\eta)^m\Big)\\ &\langle\!\langle \text{ using } \textstyle\sum_{m=1}^{\infty}\frac{x^m}{m} \;=\; -\ln(1-x) \rangle\!\rangle\\ &=\; -2\ln(1+\eta) \;+\; \ln(1-i\eta) \;+\; \ln(1+i\eta)\\ &=\; \ln\frac{1+\eta^2}{(1+\eta)^2}\\ &\xrightarrow[\eta\to 1]{}\; \ln\frac{2}{4} \;=\; -\ln(2),\end{aligned} \tag{94}$$

and therefore

$$\lambda \;=\; -\frac{\ln(2)}{\pi}\,\epsilon_0 V_0\,. \tag{95}$$

The non-textbook problem:

(a) Inside the sphere, the potential is

$$V_{\text{inside}} = kxyz = k \times r\sin\theta\cos\phi \times r\cos\theta\sin\phi \times r\cos\theta \\ = k \times r^3 \times \sin^2\theta\cos\theta \times \tfrac{1}{2}\sin(2\phi). \tag{96}$$

In Cartesian coordinates (x, y, z), this potential obviously obeys the Laplace equations and is regular at the origin, so that in spherical coordinates (r, θ, ϕ) it must expand into a sum or a series of the form

$$V(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times r^{\ell} \times Y_{\ell,m}(\theta, \phi). \tag{97}$$

For the inside potential in question, $V = r^3 \times f(\theta, \phi)$, so only the $\ell = 3$ terms are present in this expansion. Moreover, the ϕ-dependence of the potential is $V \propto \sin(2\phi)$, where $\sin(2\phi) = -\frac{i}{2}e^{+2i\phi} + \frac{i}{2}e^{-2i\phi}$. Since each spherical harmonic $Y_{\ell,m}(\theta, \phi)$ depends on ϕ as $\exp(im\phi)$, we only need the harmonics with $m = \pm 2$. Thus, the only two non-zero terms in the expansion (97) are the $(\ell = 3, m = \pm 2)$ terms, and their coefficient $C_{3,\pm 2}$ obtain from the explicit form of the $Y_{3,\pm 2}(\theta, \phi)$ spherical harmonics: According to the table in Wikipedia,

$$Y_{3,\pm 2}(\theta, \phi) = \sqrt{\frac{105}{32\pi}} \sin^2\theta\cos\theta\exp(\pm 2i\phi), \tag{98}$$

hence

$$\sqrt{\frac{2\pi}{105}} \big(-iY_{3,+2}(\theta, \phi) + iY_{3,-2}(\theta, \phi)\big) = \sin^2\theta\cos\theta \times \tfrac{1}{2}\sin(2\phi), \tag{99}$$

and therefore

$$V_{\text{inside}}(r, \theta, \phi) = \sqrt{\frac{2\pi}{105}}\, k \times r^3 \times \big(-iY_{3,+2}(\theta, \phi) + iY_{3,-2}(\theta, \phi)\big). \tag{100}$$

(b) In general, the inside potential of the form (97) is the solution of the Laplace equation with the boundary condition on the spherical surface

$$V_b(R,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \bigl(C_{\ell,m} R^{\ell}\bigr) \times Y_{\ell,m}(\theta,\phi), \tag{101}$$

and for the same boundary condition, the solution of the Laplace equation for the outside of the sphere is

$$V_{\text{outside}}(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} C_{\ell,m} \times \frac{R^{2\ell+1}}{r^{\ell+1}} \times Y_{\ell,m}(\theta,\phi). \tag{102}$$

For the potential in question,

$$V_{\text{outside}}(r,\theta,\phi) \;=\; \sqrt{\frac{2\pi}{105}}\,k \times \frac{R^7}{r^4} \times \bigl(-iY_{3,+2}(\theta,\phi) \,+\, iY_{3,-2}(\theta,\phi)\bigr), \tag{103}$$

or in Cartesian coordinates

$$V(x,y,z) \;=\; kR^7 \times \frac{xyz}{r^7} \;=\; kR^7 \times \frac{xyz}{[x^2+y^2+z^2]^{7/2}}\,. \tag{104}$$

(c) To simplify our notations, let's summarize the inside-the-sphere and the outside-the-sphere potentials as

$$V(r,\theta,\phi) \;=\; kf(\theta,\phi) \times \begin{cases} r^3 & \text{for } r<R, \\ \frac{R^7}{r^4} & \text{for } r>R. \end{cases} \tag{105}$$

This potential is continuous across the spherical shell, but the radial electric field is not:

$$E_r(r,\theta,\phi) \;=\; -\frac{\partial V}{\partial r} \;=\; kf(\theta,\phi) \times \begin{cases} -3r^2 & \text{for } r<R, \\ +4\frac{R^7}{r^5} & \text{for } r>R. \end{cases} \tag{106}$$

The discontinuity here is related to the surface charge density on the sphere,

$$\begin{aligned} \sigma(\theta,\phi) \;&=\; \epsilon_0 \,\text{disc}\, E_r(@r=R) \;=\; \epsilon_0 \lim_{\epsilon\to+0}\bigl(E_r(R+\epsilon,\theta,\phi) \,-\, E_r(R-\epsilon,\theta,\phi)\bigr) \\ &=\; \epsilon_0 \times kf(\theta,\phi) \times \Bigl((-3R^2) \,-\, (+4R^2) \;==\; 7R^2\Bigr) \\ &=\; -7kR^2\epsilon_0 \times f(\theta,\phi). \end{aligned} \tag{107}$$

Specifically,

$$\sigma(\theta,\phi) \;=\; -7kR^2\epsilon_0 \times \sin^2\theta\cos\theta \times \tfrac{1}{2}\sin(2\phi) \;=\; -7kR^2\epsilon_0 \times \frac{xyz}{R^3}\,. \qquad (108)$$