<u>Problem **3.29**</u>:

The system shown on textbook figure 3.31 has zero net charge but non-zero dipole moment pointing up. Indeed,

$$\begin{aligned} \mathbf{p}_{\text{net}} &= \sum Q_i \mathbf{r}_i \;=\; +q(0,0,-a) \;+\; 3q(0,0,+a) \;-\; 2q(0,-a,0) \;-\; 2q(0,+a,0) \\ &= qa\big(0,(0+0+2-2),(-1+3+0+0)\big) \;=\; (0,0,+2qa). \end{aligned} \tag{1}$$

There are also non-zero quadrupole moment, octupole moment, *etc.*, but at long distances $r \gg a$ the lowest non-zero multipole — in this case, the dipole — dominates the potential. Thus,

$$V(x,y,z)_{\text{net}} \;\approx\; V_{\text{dipole}}(x,y,z) \;=\; \frac{p=2qa}{4\pi\epsilon_0} \times \frac{\cos\theta}{r^2} \;=\; \frac{p=2qa}{4\pi\epsilon_0} \times \frac{z}{[x^2+y^2+z^2]^{3/2}}\,. \tag{2}$$

<u>Problem **3.30**</u>:

(a) A spherical shell with charge density $\sigma(\theta,\phi) = k\cos\theta$ has zero net charge but non-zero dipole moment. By axial symmetry, this dipole moment points in the z direction and has value

$$\begin{aligned} p_z &= \iint z \times \sigma \times d^2A \;=\; \iint R\cos\theta \times k\cos\theta \times R^2 \sin\theta\, d\theta\, d\phi \\ &= 2\pi R^3 k \times \int_{-1}^{+1} \cos^2\theta\, d(\cos\theta) \;=\; 2\pi R^3 k \times \frac{2}{3}\,. \end{aligned} \tag{3}$$

(b) At large distances from the sphere, the potential is dominated by the lowest multipole moment, which for the sphere in question is the dipole. Thus, for $r \gg R$,

$$V(r,\theta,\phi) \;\approx\; V_{\text{dipole}}(r,\theta,\phi) \;=\; \frac{p_z = \frac{4}{3}\pi R^3 k}{4\pi\epsilon_0} \times \frac{\cos\theta}{r^2} \;=\; \frac{R^3 k}{3\epsilon_0} \times \frac{\cos\theta}{r^2}. \tag{4}$$

For comparison, using the separation of variable method described in textbook §3.3.2

and in my notes (pages 26–36, especially 33–36), we have

$$V(r,\theta,\phi) \;=\; \sum_{\ell=0}^{\infty} C_\ell \times \left(\frac{R}{r}\right)^{\ell+1} \times P_\ell(\cos\theta) \quad \text{for all } r>R, \text{exactly}, \tag{5}$$

where the coefficients C_ℓ obtain as

$$C_\ell \;=\; \frac{R}{2\epsilon_0} \int\limits_0^\pi \sigma(\theta)\, P_\ell(\cos\theta)\, \sin\theta\, d\theta. \tag{6}$$

In particular, for $\sigma = k\cos\theta \;=\; kP_1(\cos\theta)$, we have

$$C_1 \;=\; \frac{R}{2\epsilon_0} \times \frac{2k}{3} \;=\; \frac{kR}{3\epsilon_0}\,, \quad \text{while all other } C_\ell \;=\; 0,$$

hence

$$V(r,\theta,\phi) \;=\; \frac{kR}{3\epsilon_0} \times \frac{R^2}{r^2} \times \cos\theta \;=\; \frac{kR^3}{3\epsilon_0} \times \frac{\cos\theta}{r^2}\,, \quad \text{for all } r>R, \text{exactly}. \tag{7}$$

In other words, the dipole approximation (4) to the outside-the-sphere potential happens to be exact. From the multipole expansion point of view, this means that *all the electric multipoles other than the dipole happen to vanish.*

Problem **3.53**:

In the textbook exercise 3.8 — and also in my notes — we considered a conducting sphere placed in a uniform external electric field $\mathbf{E}_0$, and we saw that the net field outside the spheres was the sum of the external field plus a pure dipole moment induced in the sphere. In terms of the potential,

$$V(r,\theta) \;=\; -E_0 \times r\cos\theta \;+\; E_0 R^3 \times \frac{\cos\theta}{r^2} \tag{8}$$

where the first term is the external field while the second term is due to the the induced dipole moment

$$p \;=\; 4\pi\epsilon_0 R^3 E_0. \tag{9}$$

Note that this formula is exact for all $r>R$, so there are no higher multipoles induced in the sphere, just the dipole.

To obtain the same result using the image charge method, we treat the uniform external field as if it is created by a pair of very distant point charges, $+Q$ at $\mathbf{r}_+ = (0,0,-a)$ and $-Q$ at $(0,0,+a)$ for $a \gg R$. At the origin, the field created by these two charges is

$$\mathbf{E}_{\text{ext}}(\mathbf{0}) \;=\; \frac{2Q}{4\pi\epsilon_0\,a^2}\,\hat{\mathbf{z}}, \tag{10}$$

and while this field is not exactly uniform, it varies over the distances comparable to a, so when we focus on much shorter distances the field (10) looks approximately uniform. To make it more uniform, let's increase a while at the same time increasing the charge Q according to

$$Q \;=\; 2\pi\epsilon_0 \times E_0 \times a^2. \tag{11}$$

As we do this, the electric field (10) at the origin remains fixed at $\mathbf{E}_{\text{ext}}(\mathbf{0}) = E_0\hat{\mathbf{z}}$, but it becomes more and more uniform since it now varies over the increasingly large distance scales $O(a)$. Thus, in the $a \to \infty$ limit (while the charge $\pm Q$ grow according to eq. (11)), the electric field dues to two distant charges becomes *uniform* $\mathbf{E}(\mathbf{r}) \equiv E_0\hat{\mathbf{z}}$.

Now consider the induced charges on the sphere and the electric field they create. For a single point charge outside the sphere, the field of the induced charges looks like the field of the image charge inside the sphere. The value of this image charge is

$$Q^{\text{image}} \;=\; -\frac{R}{a} \times Q^{\text{real}} \tag{12}$$

and it is located on the same ray from the center of the sphere as the real charge but at distance

$$b \;=\; \frac{R^2}{a} \tag{13}$$

from the center. For the un-grounded sphere, the image charge is compensated by the extra charge at the center of the sphere.

For the problem at hand, we have two real charges $\pm Q$ at the same distance a, so there are two image charges,

$$-\frac{QR}{a}\ \text{at}\ (0,0,-b)\quad\text{and}\quad +\frac{QR}{a}\ \text{at}\ (0,0,+b). \tag{14}$$

Note that the net image charge is zero, so there is no need for the compensating charge at the sphere's center.

In the limit $a\to\infty$ we have $b\to 0$, so the two image charges get very close to each other and form a dipole. The dipole moment of the two image charges points in the $+z$ direction, while its magnitude is

$$p\;=\;2b\times\frac{QR}{a}\;=\;\frac{2R^2}{a}\times\frac{QR}{a}\;=\;\frac{2QR^3}{a^2}\,, \tag{15}$$

and since we keep Q/a^2 ratio fixed as we take $a\to\infty$, this dipole moment stays fixed. Specifically, according to eq. (11),

$$p\;=\;\frac{2R^3}{a^2}\times 2\pi\epsilon_0 E_0\,a^2\;=\;4\pi\epsilon_0 R^3\,E_0\,. \tag{16}$$

Moreover, in the $a\to\infty$ limit and hence $b\to 0$, the two image charges become an ideal dipole: finite dipole moment but infinitesimally short distance between the opposite charges. Such dipoles have negligible quadrupole, octupole, and higher multipole moment, and only the dipole moment is present; that's why they are called *ideal dipoles* or *pure dipoles.*

For the sphere in question, this means that the electric field due to induced charges on the sphere's surface is the field of a pure dipole (16). Thus, combining the external field with the induced dipole field, we have

$$V_{\text{net}}(\mathbf{r})\;=\;V_{\text{ext}}(\mathbf{r})+V_{\text{dipole}}(\mathbf{r})\;=\;-E_0\,z+\frac{p}{4\pi\epsilon_0}\,\frac{\cos\theta}{r^2}\;=\;-E_0\,r\cos\theta+E_0R^3\times\frac{\cos\theta}{r^2}\,. \tag{17}$$

exactly as in eq. (8).

The non-textbook problem:

By symmetry, the solid ball in question has zero net charge but it has a non-zero dipole moment pointing from the south pole to the north pole (assuming $\rho_0 > 0$). Indeed, the charge density of the ball can be written as

$$\rho(r,\theta,\phi) \;=\; \rho_0 \times \operatorname{sign}(\cos\theta) \times \begin{cases} 1 & \text{for } r<R, \\ 0 & \text{for } r>R, \end{cases} \tag{18}$$

hence the dipole moment

$$\begin{aligned} p_z \;&=\; \iiint\limits_{\text{ball}} d^3\mathrm{Vol}\, \rho(\mathbf{r}) \times z \\ &=\; \int_0^R dr\, r^2 \int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\phi\, \rho_0 \operatorname{sign}(\cos\theta) \times r\cos\theta \\ &=\; \rho_0 \times \int_0^R dr\, r^3 \times \int_0^\pi d\theta\, \sin\theta\cos\theta \operatorname{sign}(\cos\theta) \times 2\pi \\ &=\; 2\pi\rho_0 \times \frac{R^4}{4} \times \int_{-1}^{+1} dc\, c\operatorname{sign}(c) \end{aligned} \tag{19}$$

where the last integral over $c = \cos\theta$ evaluates to

$$\int_{-1}^{+1} dc\, c\operatorname{sign}(c) \;=\; \int_0^{+1} dc\, c \;-\; \int_{-1}^{0} dc\, c \;=\; \frac{+1}{2} \;-\; \frac{-1}{2} \;=\; 1. \tag{20}$$

Thus, the ball in question has dipole moment

$$\mathbf{p} \;=\; \tfrac{\pi}{2}\rho_0 R^4\, \hat{\mathbf{z}}. \tag{21}$$

There are also higher multipole moments for the odd $\ell = 3, 5, 7, \ldots$, but far away from the ball the potential is dominated by the dipole moment,

$$V(r,\theta) \;\approx\; \frac{p}{4\pi\epsilon_0} \times \frac{\cos\theta}{r^2} \;=\; \frac{\rho_0 R^4}{8\epsilon_0} \times \frac{\cos\theta}{r^2}\,. \tag{22}$$

Problem **3.57**:

By energy conservation, the charged particle would move at constant speed along some orbit only if its potential energy — and hence the electrostatic potential of the dipole — is constant along the orbit. By the axial symmetry of the dipole's potential

$$V = \frac{p}{4\pi\epsilon_0} \times \frac{\cos\theta}{r^2}, \tag{23}$$

the circular orbits which are parallel to the xy plane and centered somewhere on the z axis automatically obey this condition. In the cylindrical coordinates (z, s, ϕ) such orbits are parametrized by

$$s = S = \text{const}, \quad z = Z = \text{const}, \quad \phi \text{ runs from } 0 \text{ to } 2\pi. \tag{24}$$

In fact, there are no other circular orbits over which the dipole potential remains constant, but proving this fact is a much harder exercise. For the purposes of this homework, let's simply assume that the orbit has geometry (24) for some S and Z.

Next, circular motion at constant speed needs a centripetal force $F = mv^2/S$, which must come from the dipole's electric field. Thus, along the orbit we need

$$q\mathbf{E} = -\frac{mv^2}{S}\,\hat{\mathbf{s}}; \tag{25}$$

note that the direction of this force must be $\perp$ to the z axis and towards that axis. On the other hand, as explained in my notes on electric dipoles, the electric field of the dipole is

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0\, r^3}. \tag{26}$$

In cylindrical coordinates, the numerator here becomes

$$3p\,\frac{z}{r}\left(\frac{z}{r}\,\hat{\mathbf{z}} + \frac{s}{r}\,\hat{\mathbf{s}}\right) - p\hat{\mathbf{z}} = \frac{p}{r^2 = s^2 + z^2}\left((3z^2 - r^2 = 2z^2 - s^2)\hat{\mathbf{z}} + 3sz\hat{\mathbf{s}}\right), \tag{27}$$

hence

$$\mathbf{E}(z, s, \phi) = \frac{p}{4\pi\epsilon_0}\,\frac{(2z^2 - s^2)\hat{\mathbf{z}} + 3zs\hat{\mathbf{s}}}{[z^2 + s^2]^{5/2}}. \tag{28}$$

To keep this field within the plane of the particle's orbit we need $E_z = 0$ and hence

$$2z^2 \; - \; s^2 \; = \; 0. \tag{29}$$

Furthermore, this field must point towards the z axis rather then away from it, which calls foe $E_s < 0$ and hence $z < 0$. Thus, given the radius S of the circular orbit, its vertical coordinate should be

$$Z \; = \; -\frac{S}{\sqrt{2}}\,. \tag{30}$$

Given this orbit geometry, the centripetal force due to the dipole's electric field is

$$F \; = \; q|E_s| \; = \; \frac{qp}{4\pi\epsilon_0} \times \frac{3S^2/\sqrt{2}}{[(3/2)S^2]^{5/2}} \; = \; \frac{4}{3\sqrt{3}}\,\frac{qp}{4\pi\epsilon_0\,S^3}\,, \tag{31}$$

which sets the particle's orbital speed so that

$$F \; = \; \frac{mv^2}{S}\,, \tag{32}$$

hence

$$v \; = \; \frac{2}{3^{3/4}}\sqrt{\frac{qp}{4\pi\epsilon_0 m}\,\frac{1}{S}}\,. \tag{33}$$

Finally, the angular momentum and the net energy of the particle follow easily from the above data. The angular momentum is simply

$$L \; = \; S \times mv \; = \; \frac{2}{3^{3/4}}\sqrt{\frac{qpm}{4\pi\epsilon_0}} \tag{34}$$

regardless of the orbital radius S, the kinetic energy is

$$K \; = \; \frac{mv^2}{2} \; = \; \frac{2}{3\sqrt{3}}\,\frac{qp}{4\pi\epsilon_0}\,\frac{1}{S^2}\,, \tag{35}$$

and the potential energy is

$$U \; = \; qV \; = \; \frac{qp}{4\pi\epsilon_0}\left(\frac{\cos\theta}{r^2} \; = \; \frac{z}{[s^2+z^2]^{3/2}}\right) \; = \; \frac{qp}{4\pi\epsilon_0}\,\frac{-S/\sqrt{2}}{((3/2)S^2)^{3/2}} \; = \; -\frac{2}{3\sqrt{s}}\,\frac{qp}{4\pi\epsilon_0}\,\frac{1}{S^2}\,. \tag{36}$$

Note that the particles kinetic energy and the potential energy have equal magnitudes and opposite signs, so the total mechanical energy of the charged particle is zero,

$$K + U = 0. \tag{37}$$

Problem **4.4**:

The atom is neutral on the whole, but in the electric field $\mathbf{E}$ of the point charge q the atom acquires a dipole moment

$$\mathbf{p} = \alpha\mathbf{E}. \tag{38}$$

Moreover, the electric field of the point charge is not uniform, hence the dipole moment (38) feels a net force

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} = \alpha(\mathbf{E}\cdot\nabla)\mathbf{E}. \tag{39}$$

Specifically, for

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}, \tag{40}$$

we have

$$\mathbf{E}\cdot\nabla = \frac{q}{4\pi\epsilon_0\, r^2}\,\frac{\partial}{\partial r}\bigg|^{\text{fixed }\theta,\phi}, \tag{41}$$

hence

$$\mathbf{F} = \alpha\frac{q}{4\pi\epsilon_0\, r^2}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{\alpha q^2}{(4\pi\epsilon_0)^2\, r^2}\frac{-2\hat{\mathbf{r}}}{r^3} = -\frac{2\alpha q^2}{(4\pi\epsilon_0)^2}\frac{\hat{\mathbf{r}}}{r^5}. \tag{42}$$

Note: the direction of this force is $-\hat{\mathbf{r}}$, which means directly towards the charge; in other words, the charge *attracts* a neutral but polarizable atom with the force

$$F = \frac{2\alpha q^2}{(4\pi\epsilon_0)^2}\times\frac{1}{r^5} \tag{43}$$

which decreases with distance like $1/r^5$.

Problems **4.5** and **4.29**:

For the sake of definiteness, let the first dipole sit at the coordinate origin $\mathbf{r}_1 = (0,0,0)$ and point in the $+\hat{\mathbf{y}}$ direction, while the second dipole sits at point $\mathbf{r}_2(+r,0,0)$ and points in the $+\hat{\mathbf{x}}$ direction, as shown on the diagram below:

(44)

4.5(a) The first dipole creates electric field

$$\mathbf{E}_1(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \,-\, \mathbf{p}_1}{|\mathbf{r}|^3}\,. \tag{45}$$

In particular, at the location of the second dipole

$$\mathbf{p}_1\cdot\hat{\mathbf{r}}_2 \;=\; 0 \quad\Longrightarrow\quad \frac{3(\mathbf{p}_1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \,-\, \mathbf{p}_1}{|\mathbf{r}|^3} \;=\; \frac{-\mathbf{p}_1}{r^3} \tag{46}$$

and therefore

$$\mathbf{E}_1(\mathbf{r}_2) \;=\; -\frac{p_1}{4\pi\epsilon_0\,r^3}\,\hat{\mathbf{y}}. \tag{47}$$

This field creates the torque on the second dipole (relative to its own center)

$$\vec{\tau}_2 \;=\; \mathbf{p}_2\times\mathbf{E}_1(\mathbf{r}_2) \;=\; (p_2\hat{\mathbf{x}})\times\frac{-p_1\hat{\mathbf{y}}}{4\pi\epsilon_0\,r^3} \;=\; \frac{p_1p_2}{4\pi\epsilon_0\,r^3}\,(-\hat{\mathbf{x}}\times\hat{\mathbf{y}}) \;=\; -\frac{p_1p_2}{4\pi\epsilon_0\,r^3}\,\hat{\mathbf{z}}. \tag{48}$$

Note: on the diagram (44), the direction of this torque is *clockwise.*

4.5(b) To find the torque on the first dipole, we need the electric field of the second dipole at the first dipole's location. In general,

$$\mathbf{E}_2(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_2\cdot\hat{\mathbf{r}}')\hat{\mathbf{r}}' \,-\, \mathbf{p}_2}{|\mathbf{r}'|^3} \tag{49}$$

where $\mathbf{r}' = \mathbf{r} - \mathbf{r}_2$. At the location of the first dipole $\mathbf{r}' = (-r, 0, 0)$, hence $|\mathbf{r}'| = r$, $\hat{\mathbf{r}}' = -\hat{\mathbf{x}}$, and since $\mathbf{p}_2 = (+p_2, 0, 0)$, we have

$$\mathbf{p}_2\cdot\mathbf{r}' \;=\; -p_2 \quad\Longrightarrow\quad 3(\mathbf{p}_2\cdot\hat{\mathbf{r}}')\hat{\mathbf{r}}' \,-\, \mathbf{p}_2 \;=\; +3p_2\hat{\mathbf{x}} \,-\, p_2\hat{\mathbf{x}} \;=\; +2p_2\hat{\mathbf{x}} \tag{50}$$

and consequently

$$\mathbf{E}_2(\mathbf{r}_1) \;=\; +\frac{2p_2}{4\pi\epsilon_0\, r^3}\,\hat{\mathbf{x}}. \tag{51}$$

Therefore. the torque on the first dipole (relative to its own center) is

$$\tau_1 \;=\; \mathbf{p}_1\times\mathbf{E}_2(\mathbf{r}_1) \;=\; (p_1\hat{\mathbf{y}})\times\frac{2p_2}{4\pi\epsilon_0\, r^3}\,\hat{\mathbf{x}} \;=\; \frac{2p_1p_2}{4\pi\epsilon_0\, r^3}\,(\hat{\mathbf{y}}\times\hat{\mathbf{x}}) \;=\; -\frac{2p_1p_2}{4\pi\epsilon_0\, r^3}\,\hat{\mathbf{z}} \tag{52}$$

Note: this torque is twice as strong as the torque on the second dipole. Also, both torques have the same direction — clockwise on the diagram (44) — so they do not add up to zero!

4.29(a) Let me first derive a general formula for the force between two electric dipoles, verify the Third Law of Newton, and then apply the formula to the dipoles at hand.

The electric field of the first dipole at the location of the second dipole is

$$\mathbf{E}_1(\mathbf{r}_2) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_1\cdot\hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12} \,-\, \mathbf{p}_1}{r_{12}^3} \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_1\cdot\mathbf{r}_{12})\mathbf{r}_{12} \,-\, r_{12}^2\mathbf{p}_1}{r_{12}^5} \tag{53}$$

where $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$. Consequently, the force on the second dipole is

$$\mathbf{F}_{1\,\mathrm{on}\,2} \;=\; (\mathbf{p}_2\cdot\nabla)\mathbf{E}_1(\mathbf{r}_2) \tag{54}$$

where the gradient ∇ is with respect to the second dipole's location $\mathbf{r}_2$, while the first dipole's location $\mathbf{r}_1$ is fixed. However, since the field $\mathbf{E}_1(\mathbf{r}_2)$ depends only on the difference

$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$, we may just as well take the gradient with respect to the $\mathbf{r}_{12}$. Thus,

$$\nabla(\mathbf{p}_1 \cdot \mathbf{r}_{12}) \;=\; \mathbf{p}_1 \quad\Longrightarrow\quad (\mathbf{p}_2 \cdot \nabla)(\mathbf{p}_1 \cdot \mathbf{r}_{12}) \;=\; \mathbf{p}_2 \cdot \mathbf{p}_1\,, \tag{55}$$

$$(\mathbf{p}_2 \cdot \nabla)\mathbf{r}_{12} \;=\; \mathbf{p}_2\,, \tag{56}$$

$$(\mathbf{p}_2 \cdot \nabla)\big(3(\mathbf{p}_1 \cdot \mathbf{r}_{12})\mathbf{r}_{12} - r_{12}^2\mathbf{p}_1\big) \;=\; 3(\mathbf{p}_2 \cdot \mathbf{p}_1)\mathbf{r}_{12} + 3(\mathbf{p}_1 \cdot \mathbf{r}_{12})\mathbf{p}_2 - 2(\mathbf{p}_2 \cdot \mathbf{r}_{12})\mathbf{p}_1\,, \tag{57}$$

$$(\mathbf{p}_2 \cdot \nabla)\,\frac{1}{r_{12}^5} \;=\; -\frac{5(\mathbf{p}_2 \cdot \mathbf{r}_{12})}{r_{12}^7}\,, \tag{58}$$

hence

$$\begin{aligned}
\mathbf{F}_{1\,\mathrm{on}\,2} &= (\mathbf{p}_2 \cdot \nabla)\left[\frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_1 \cdot \mathbf{r}_{12})\mathbf{r}_{12} - r_{12}^2\mathbf{p}_1}{r_{12}^5}\right] \\
&= \frac{1}{4\pi\epsilon_0}\left(\begin{aligned} &\frac{3(\mathbf{p}_2 \cdot \mathbf{p}_1)\mathbf{r}_{12} + 3(\mathbf{p}_1 \cdot \mathbf{r}_{12})\mathbf{p}_2 - 2(\mathbf{p}_2 \cdot \mathbf{r}_{12})\mathbf{p}_1}{r_{12}^5} \\ &-\frac{5(\mathbf{p}_2 \cdot \mathbf{r}_{12})}{r_{12}^7}\big[3(\mathbf{p}_1 \cdot \mathbf{r}_{12})\mathbf{r}_{12} - r_{12}^2\mathbf{p}_1\big]\end{aligned}\right) \\
&= \frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_2 \cdot \mathbf{p}_1)\hat{\mathbf{r}}_{12} + 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{12})\mathbf{p}_2 + 3(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{12})\mathbf{p}_1 - 15(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{12})(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12}}{r_{12}^4}\,.
\end{aligned} \tag{59}$$

This force decreases as $1/r_{12}^4$ with the distance between the dipoles, and it has rather complicated dependence on the directions of the two dipole moments relative to the line connecting the dipoles. However, this direction dependence has two important features: The bottom line of eq. (59) is symmetric WRT $\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$ and odd with respect to the direction vector $\hat{\mathbf{r}}_{12}$. Consequently, when we exchange the roles of the two dipoles, we obtain

$$\begin{aligned}
\mathbf{p}_1 \leftrightarrow \mathbf{p}_2\,,\quad \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \quad&\Longrightarrow\quad \mathbf{r}_{12} \to \mathbf{r}_{21} = -\mathbf{r}_{12} \quad\Longrightarrow \\
&\Longrightarrow\quad r_{12} \to r_{21} = r_{12} \quad\text{but}\quad \hat{\mathbf{r}}_{12} \to \hat{\mathbf{r}}_{21} = -\mathbf{r}_{12}\,,
\end{aligned} \tag{60}$$

hence

$$\begin{aligned}
\mathbf{F}_{2\,\mathrm{on}\,1} &= \frac{1}{4\pi\epsilon_0}\,\frac{3(\mathbf{p}_1 \cdot \mathbf{p}_2)\hat{\mathbf{r}}_{21} + 3(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{21})\mathbf{p}_1 + 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{21})\mathbf{p}_2 - 15(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{21})(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{21})\hat{\mathbf{r}}_{21}}{r_{21}^4} \\
&= \frac{1}{4\pi\epsilon_0}\,\frac{-3(\mathbf{p}_2 \cdot \mathbf{p}_1)\hat{\mathbf{r}}_{12} - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{12})\mathbf{p}_2 - 3(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{12})\mathbf{p}_1 + 15(\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{12})(\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12}}{r_{12}^4} \\
&= -\mathbf{F}_{1\,\mathrm{on}\,2}\,.
\end{aligned} \tag{61}$$

Thus, we see that the force between two electric dipoles duly obeys the Third Law of Newton:

The force of the second dipole on the first dipole has the same magnitude but opposite direction from the force of the first dipole on the second dipole.

Finally, let's apply the general formula (59) to the two dipoles on the diagram (44). Given $\mathbf{p}_1 = (0, p_1, 0)$, $\mathbf{p}_2 = (p_2, 0, 0)$ and $\hat{\mathbf{r}}_{12} = (1, 0, 0)$, we have:

$$(\mathbf{p}_2 \cdot \mathbf{p}_1) \;=\; 0, \quad (\mathbf{p}_1 \cdot \hat{\mathbf{r}}_{12}) \;=\; 0, \quad (\mathbf{p}_2 \cdot \hat{\mathbf{r}}_{12}) \;=\; p_2\,, \tag{62}$$

hence

$$3(\mathbf{p}_2\cdot\mathbf{p}_1)\hat{\mathbf{r}}_{12} + 3(\mathbf{p}_1\cdot\hat{\mathbf{r}}_{12})\mathbf{p}_2 + 3(\mathbf{p}_2\cdot\hat{\mathbf{r}}_{12})\mathbf{p}_1 - 15(\mathbf{p}_1\cdot\hat{\mathbf{r}}_{12})(\mathbf{p}_2\cdot\hat{\mathbf{r}}_{12})\hat{\mathbf{r}}_{12} \;=\; 3p_2\mathbf{p}_1 \;=\; 3p_2p_1\,\hat{\mathbf{y}} \tag{63}$$

and therefore, the force of the first dipole on the second dipole is

$$\mathbf{F}_{1\,\mathrm{on}\,2} \;=\; +\frac{3p_1p_2}{4\pi\epsilon_0\,r^4}\,\hat{\mathbf{y}} \tag{64}$$

while the force of the second dipole on the first dipole is

$$\mathbf{F}_{2\,\mathrm{on}\,1} \;=\; -\frac{3p_1p_2}{4\pi\epsilon_0\,r^4}\,\hat{\mathbf{y}}. \tag{65}$$

4.29(b) Finally, let's consider the torques on the dipoles relative to a common point, namely the origin of the coordinate system. In general, the net torque on a body relative to some point depends on the choice of that reference point according to

$$\vec{\tau}^{\mathrm{net}}[\text{relative to } A] \;-\; \vec{\tau}^{\mathrm{net}}[\text{relative to } B] \;=\; (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}^{\mathrm{net}} \tag{66}$$

where $\mathbf{F}^{\mathrm{net}}$ is the net force acting on the body in question. In particular, for the two dipole in question, we have

$$\begin{aligned} \vec{\tau}_1[\text{relative to origin}] \;&=\; \vec{\tau}_1[\text{relative to dipole itself}] \;+\; \mathbf{r}_1 \times \mathbf{F}^{\mathrm{net}}_{\mathrm{on}\,1}\,, \\ \vec{\tau}_2[\text{relative to origin}] \;&=\; \vec{\tau}_2[\text{relative to dipole itself}] \;+\; \mathbf{r}_2 \times \mathbf{F}^{\mathrm{net}}_{\mathrm{on}\,2}\,. \end{aligned} \tag{67}$$

Since the first dipole happens to sit at the origin of our coordinate system, $\mathbf{r}_1 = \mathbf{0}$, its torque relative to the origin is exactly as we have calculated in problem **4.5**(b):

$$\vec{\tau}_1[\text{relative to origin}] \;=\; \vec{\tau}_1[\text{relative to dipole itself}] \;=\; -\frac{2p_1p_2}{4\pi\epsilon_0\,r^3}\,\hat{\mathbf{z}}. \tag{68}$$

On the other hand, the second dipole sits at $\mathbf{r}_2 = (r, 0, 0) = r\hat{\mathbf{x}}$, so its torque relative to the origin has an extra term in addition to the torque relative to its own center we have calculate in problem **4.5**(a):

$$\begin{aligned}
\vec{\tau}_2[\text{relative to origin}] &= \vec{\tau}_2[\text{relative to dipole itself}] \;+\; \mathbf{r}_2\times\mathbf{F}_{1\,\text{on}\,2}\\
&= -\frac{p_1p_2}{4\pi\epsilon_0\,r^3}\,\hat{\mathbf{z}} \;+\; (r\hat{\mathbf{x}})\times\left(\frac{3p_1p_2}{4\pi\epsilon_0\,r^4}\,\hat{\mathbf{y}}\right)\\
&= \frac{p_1p_2}{4\pi\epsilon_0\,r^3}\Big(-\hat{\mathbf{z}} \,+\, 3\hat{\mathbf{x}}\times\hat{\mathbf{y}} \;=\; -\hat{\mathbf{z}}\,+\,3\hat{\mathbf{z}} \;=\; +2\hat{\mathbf{z}}\Big)\\
&= +\frac{2p_1p_2}{4\pi\epsilon_0\,r^3}\,\hat{\mathbf{z}}.
\end{aligned}$$

Note: unlike the clockwise torque relative to the dipole's own center, the torque relative to the origin is counterclockwise; it also has a smaller magnitude.

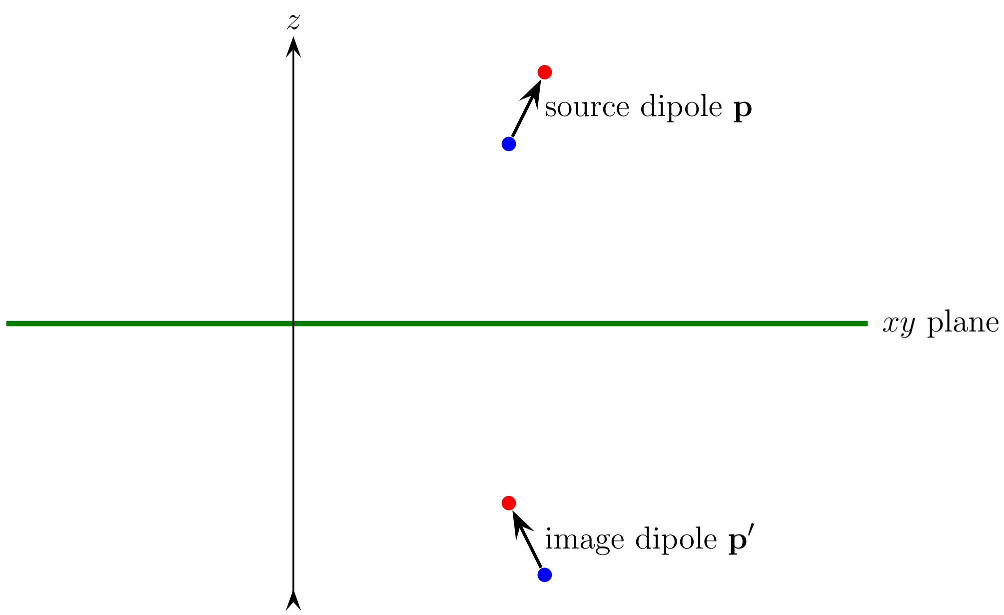
But most importantly, the torques on the two dipoles relative to the same reference point — the origin — are equal in magnitude and opposite in direction,

$$\vec{\tau}_{1\,\text{on}\,2}[\text{relative to origin}] \;+\; \vec{\tau}_{2\,\text{on}\,1}[\text{relative to origin}] \;=\; \vec{0}. \tag{69}$$

This is the rotational analogue of the Newton's Third Law, which assures that the net angular momentum of the two dipoles is conserved.

<u>Problem **4.6**</u>:

The electric field of the dipole leads to surface charges on the conducting plane, and the field of those conducting charges can be accounted by the mirror image of the dipole in the plane:



(70)

Note that the image charges have opposite signs while the mirror reflection reverses the z coordinate but not the x or y coordinates, thus

$$q \;\to\; -q, \quad x \;\to\; +x, \quad y \;\to\; +y, \quad z \;\to\; -z. \tag{71}$$

Consequently — as you see on the diagram (70) — the image dipole moment $\mathbf{p}'$ has the same z component as the original dipole moment $\mathbf{p}$ but opposite x and y components,

$$p'_z \;=\; +p_z \quad \text{but} \quad p'_x \;=\; -p_x \quad \text{and} \quad p'_y \;=\; -p_y\,, \tag{72}$$

so it is tilted from the vertical in the opposite direction.

Now suppose the source dipole — and hence also the image dipole — have very small sizes compared to the distance $r = 2z$ between them, so we may approximate both dipoles as ideal dipoles. Then the torque on the source dipole (relative to its own center) is

$$\vec{\tau} \;=\; \mathbf{p}\times\mathbf{E}' \tag{73}$$

where $\mathbf{E}'$ is the electric field of the image dipole at the location of the source dipole. In vector notations

$$\mathbf{E}' \;=\; \frac{1}{4\pi\epsilon_0\,r^3}\Big(3(\mathbf{p}'\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \;-\; \mathbf{p}'\Big) \tag{74}$$

where $\mathbf{r} = 2z\,\hat{\mathbf{z}} \implies \hat{\mathbf{r}} = \hat{\mathbf{z}}$. In components,

$$\begin{aligned}
(\mathbf{p}'\cdot\hat{\mathbf{r}}) \;&=\; p'_z \;=\; p_z\,, && (75)\\
3(\mathbf{p}'\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} \;-\; \mathbf{p}' \;&=\; 3p'_z\,\hat{\mathbf{z}} \;-\; p'_x\,\hat{\mathbf{x}} \;-\; p'_y\,\hat{\mathbf{y}} \;-\; p'_z\,\hat{\mathbf{z}} &&\\
&=\; 2p'_z\,\hat{\mathbf{z}} \;-\; p'_x\,\hat{\mathbf{x}} \;-\; p'_y\,\hat{\mathbf{y}} &&\\
&=\; 2p_z\,\hat{\mathbf{z}} \;+\; p_x\,\hat{\mathbf{x}} \;+\; p_y\,\hat{\mathbf{y}}, && (76)\\
\mathbf{E}' \;&=\; \frac{1}{4\pi\epsilon_0\,(2z)^3}\,\big(p_x\,\hat{\mathbf{x}} + p_y\,\hat{\mathbf{y}} \;+\; 2p_z\,\hat{\mathbf{z}}\big). && (77)
\end{aligned}$$

Note the factor of 2 (marked in red) in front of the $p_z\,\hat{\mathbf{z}}$ term — it makes the electric field $\mathbf{E}'$ non-parallel to the source dipole $\mathbf{p}$, and that what causes the torque on the dipole:

$$\begin{aligned}
\vec{\tau} &= \mathbf{p}\times\mathbf{E}' \\
&= \frac{1}{4\pi\epsilon_0\,(2z)^3}\bigl(p_x\,\hat{\mathbf{x}}+p_y\,\hat{\mathbf{y}}+p_z\,\hat{\mathbf{z}}\bigr)\times\bigl(p_x\,\hat{\mathbf{x}}+p_y\,\hat{\mathbf{y}}+2p_z\,\hat{\mathbf{z}}\bigr) \\
&= \frac{1}{4\pi\epsilon_0\,(2z)^3}\bigl(p_x\,\hat{\mathbf{x}}+p_y\,\hat{\mathbf{y}}+p_z\,\hat{\mathbf{z}}\bigr)\times\bigl(p_z\,\hat{\mathbf{z}}\bigr) \\
&= \frac{1}{4\pi\epsilon_0\,(2z)^3}\bigl(+p_yp_z\,\hat{\mathbf{x}}-p_xp_z\,\hat{\mathbf{y}}\bigr).
\end{aligned}\tag{78}$$

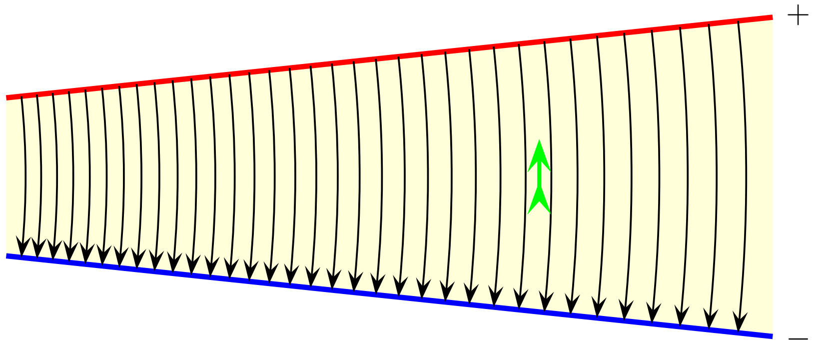
For the dipole tilted from the $+z$ axis through angle ϕ towards the $+x$ axis, we have

$$p_x = p\sin\phi,\quad p_y = 0,\quad p_z = p\cos\phi \quad\Longrightarrow\quad \vec{\tau} = \frac{p^2}{4\pi\epsilon_0\,(2z)^3}\bigl(-\sin\phi\cos\phi\,\hat{\mathbf{y}}\bigr).\tag{79}$$

For $\phi < 90^\circ$, the direction of this torque is $-\hat{\mathbf{y}}$ — *i.e.*, counterclockwise in the xz plane, — which means that the dipole is torqued back to the upward direction. But for tilt angle $\phi > 90^\circ$, the torque flips directions and starts further tilting the dipole to the downward direction.

The torque vanishes when the dipole is either vertical or horizontal. The vertical directions — both up and down — are stable: when the dipole is tilted away from vertical, the torque will twist it back to the vertical (up or down, whichever is closer). On the other hand, the horizontal directions are unstable: for any tilt away from the horizontal, the torque will twist it further away from the horizontal towards the vertical.

<u>Problem **4.30**</u>:
Let's start with a picture of the electric field lines between the not-quite-parallel plates:



(80)

The direction of $\mathbf{E}$ is approximately downward, while its magnitude increases as we go left (where the plates are closer to each other) and decreases as we go right. At the same time, the dipole moment $\mathbf{p}$ (green arrow on figure (80)) is pointing up (*cf.* the textbook figure **4.33**). This dipole moment is anti-parallel to the electric field, so its potential energy is

$$U(x) \;=\; -\mathbf{p}\cdot\mathbf{E}(x) \;=\; +pE(x). \tag{81}$$

This energy increases when the dipole moves left and decreases when the dipole moves right, which means a non-zero net force on the dipole pushing it to the right.

Thus, the net force on the dipole shown on figure **4.33** pushes it to the right.

ALTERNATIVE SOLUTION, without using the potential energy. The textbook equation (4.5) gives the net force on an ideal dipole in a non-uniform electric field,

$$\mathbf{F} \;=\; (\mathbf{p}\cdot\nabla)\mathbf{E}. \tag{82}$$

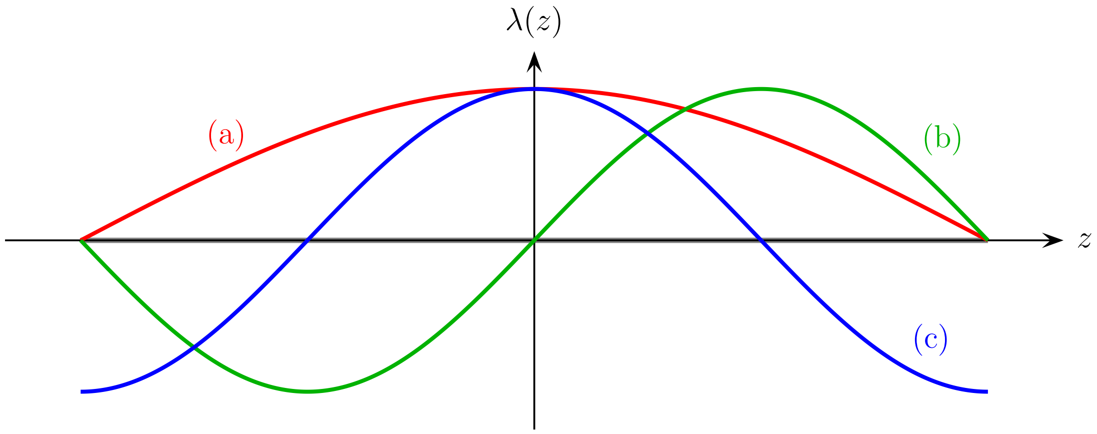
For the dipole pointing in the $+y$ direction, this means

$$\mathbf{F} \;=\; p\,\frac{\partial}{\partial y}\,\mathbf{E}(x,y). \tag{83}$$

At first blush, the electric field depicted on figure (80) looks vertical and y independent, but a closer look shows that the field lines are slightly bent and follow circular arcs rather than straight lines. Mathematically, it means that $\mathbf{E}(x,y)$ has a small x component which depends on y: For positive y the E_x is positive while for negative y the E_x is negative. Consequently, the derivative $\partial E_x/\partial y$ is positive, which means that the force (83) has a positive x component, $F_x > 0$. As to the y component of the force, the derivative $\partial E_y/\partial y$ vanishes by symmetry (for the dipole sitting on the x axis), hence $F_y = 0$. Thus, the force (83) pushes the dipole to the right.

Problem **3.46**:

First, let me plot the three charge densities (a), (b), (c) along the rod:



(a) By inspection of the red line on this plot, the density $\lambda_a(z)$ is positive all along the rod, so there is a non-zero net charge (AKA the monopole moment)

$$Q_q \;=\; \int \lambda_a(z)\,dz \;=\; \int\limits_{-a}^{+a} k\cos\frac{\pi z}{2a}\,dz \;=\; \frac{2ka}{\pi}\left(\sin\frac{+\pi}{2}\;-\;\sin\frac{-\pi}{2}\right) \;=\; \frac{4}{\pi}\,ka. \tag{84}$$

Hence, the leading term in the potential at large distances from the rod is

$$V_a(r,\theta) \;\approx\; \frac{Q_{\rm net}}{4\pi\epsilon_0}\times\frac{1}{r} \;=\; \frac{ka}{\pi^2\epsilon_0}\times\frac{1}{r}\,. \tag{85}$$

(b) By inspection of the green curve on the plot, the density $\lambda_b(z)$ is antisymmetric WRT $z\to -z$, hence there is no net charge but there is a net dipole moment

$$\begin{aligned} p \;=\; \int z\times\lambda_b(z)\,dz \;&=\; \int\limits_{-a}^{+a} z\times k\sin(\pi z/a)\,dz \;=\; \frac{ka^2}{\pi^2}\int\limits_{-\pi}^{+\pi} t\,\sin(t)\,dt \qquad \langle\!\langle\,\text{where } t=\pi z/a\,\rangle\!\rangle\\ &=\; \frac{ka^2}{\pi^2}\times\Big[\sin(t)\;-\;t\cos(t)\Big]_{-\pi}^{+\pi} \;=\; \frac{ka^2}{\pi^2}\times\big((+\pi)\;-\;(-\pi)\big) \;=\; \frac{2ka^2}{\pi}\,. \end{aligned} \tag{86}$$

Consequently, the leading term in the potential at large distances from the rod is

$$V_b(r,\theta) \;=\; \frac{p}{4\pi\epsilon_0}\times\frac{\cos\theta}{r^2} \;=\; \frac{ka^2}{2\pi^2\epsilon_0}\times\frac{\cos\theta}{r^2}\,. \tag{87}$$

(c) Finally, according to the blue line on the plot, the $\lambda_c(z)$ distribution is symmetric rather than antisymmetric WRT $z \to -z$, so there's no dipole moment nor any higher multipole moment with an odd ℓ; only the even–ℓ multipoles are allowed. However, for $\ell = 0$ the monopole moment — *i.e.*, the net charge of this distribution — is zero,

$$Q_{\text{net}} = \int \lambda_c(z)\,dz = \int_{-a}^{+a} k\cos(\pi z/a)\,dz = \frac{ka}{\pi}\Big[-\sin(\pi z/a)\Big]_{z=-a}^{z=+a} = 0. \tag{88}$$

Consequently, the leading multipole of the distribution (c) should be the quadrupole moment.

Now let's calculate the quadrupole moment tensor

$$\mathcal{Q}_{ij} = \int\left(\tfrac{3}{2}r_i r_j - \tfrac{1}{2}r^2\delta_{i,j}\right) \times \lambda_c(z)\,dz. \tag{89}$$

Along the rod $x = y = 0$, hence

$$\left(\tfrac{3}{2}r_i r_j - \tfrac{1}{2}r^2\delta_{i,j}\right) = \begin{cases} +z^2 & \text{for } i = j = z, \\ -\frac{1}{2}z^2 & \text{for } i = j = x \text{ or } i = j = y, \\ 0 & \text{for } i \neq j, \end{cases} \tag{90}$$

which means that

$$\mathcal{Q}_{zz} = \int z^2 \times \lambda_c(z)\,dz, \tag{91}$$

$$\mathcal{Q}_{xx} = \mathcal{Q}_{yy} = -\tfrac{1}{2}\mathcal{Q}_{zz}\,, \tag{92}$$

$$\text{all other } \mathcal{Q}_{ij} = 0. \tag{93}$$

Consequently, the numerator of the quadrupole potential

$$V_{\text{quadrupole}}(\mathbf{r}) = \frac{\sum_{i,j}\mathcal{Q}_{i,j}\hat{r}_i\hat{r}_j}{4\pi\epsilon_0\,r^3} \tag{94}$$

has form

$$\begin{aligned} \sum_{i,j}\mathcal{Q}_{i,j}\hat{r}_i\hat{r}_j &= \mathcal{Q}_{zz} \times \frac{z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2}{r^2} \\ &= \mathcal{Q}_{zz} \times \left(\cos^2\theta - \tfrac{1}{2}\sin^2\theta = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) \\ &= \mathcal{Q}_{zz} \times P_2(\cos\theta), \end{aligned} \tag{95}$$

which leads to

$$V_{\text{quadrupole}}(r,\theta) \;=\; \frac{\mathcal{Q}_{zz}}{4\pi\epsilon_0}\times\frac{P_2(\cos\theta)}{r^3}\,. \tag{96}$$

To complete this calculation, we need to evaluate the integral in eq. (91):

$$\begin{aligned}
\mathcal{Q}_{zz} \;&=\; \int z^2\times\lambda_c(z)\,dz \;=\; \int\limits_{-a}^{+a} z^2\times k\cos(\pi z/a)\,dz \\
&=\; \frac{ka^3}{\pi^3}\int\limits_{-\pi}^{+\pi} t^2\cos(t)\,dt \qquad \langle\!\langle\,\text{where } t=\pi z/a\,\rangle\!\rangle \\
&=\; \frac{ka^3}{\pi^3}\times\Big[t^2\sin(t)\;+\;2t\cos(t)\;-\;2\sin(t)\Big]_{t=-\pi}^{t=+\pi} \;=\; \frac{ka^3}{\pi^3}\times\big[(-2\pi)\;-\;(+2\pi)\;=\;-4\pi\big] \\
&=\; -\frac{4ka^3}{\pi^2}\,.
\end{aligned} \tag{97}$$

Therefore, the leading term in the potential at large distances from the rod is

$$V(r,\theta) \;\approx\; V_{\text{quadrupole}}(r,\theta) \;=\; -\frac{ka^3}{\pi^3\epsilon_0}\times\frac{P_2(\cos\theta)}{r^3}\,. \tag{98}$$

<u>Problem **3.27**</u>:

Far away from the ball in question, the potential is dominated by the lowest–ℓ multipole with a non-zero moment. By axial symmetry of the ball,

$$V(r,\theta) \;=\; \frac{\mathcal{M}_\ell}{4\pi\epsilon_0}\times\frac{P_\ell(\cos\theta)}{r^{\ell+1}} \;\longrightarrow\; \frac{\mathcal{M}_\ell}{4\pi\epsilon_0}\times\frac{(\pm1)^\ell}{r^{\ell+1}} \quad \text{for } \theta=0 \text{ or } \theta=\pi. \tag{99}$$

So let's start evaluating the monopole, dipole, quadrupole, *etc.*, moments of the charged ball until we find a non-zero moment.

The charge density

$$\rho(r,\theta) \;=\; k\times\frac{R(R-2r)}{r^2}\times\sin\theta\times\begin{cases}1 & \text{for } r<R,\\ 0 & \text{for } r>R,\end{cases} \tag{100}$$

is rather singular at the origin, but the net charge of the ball in question is finite. In fact, the net charge — AKA, the monopole moment — is zero:

$$\begin{aligned}
Q^{\text{net}} &= \iiint\limits_{\text{ball}} \rho\, d^3\text{Vol} \\
&= \int\limits_0^R k\,\frac{R(R-2r)}{r^2} \times r^2\,dr \times \int\limits_0^\pi \sin\theta \times \sin\theta\,d\theta \times \int\limits_0^{2\pi} d\phi \\
&= kR \times \int\limits_0^R (R-2r)\,dr \times \int\limits_0^\pi \sin^2\theta\,d\theta \times 2\pi \\
&= kR \times 0 \times \frac{\pi}{2} \times 2\pi \;=\; 0
\end{aligned} \tag{101}$$

since the radial integral happens to vanish:

$$\int\limits_0^R (R-2r)\,dr \;=\; R\times R \;-\; 2\times\frac{R^2}{2} \;=\; 0. \tag{102}$$

The dipole moment $\mathbf{p}$ of the ball also vanishes due to symmetries of the charge density (100): The axial symmetry kills the p_x and the p_y components of the dipole moment, while the reflection symmetry $z \leftrightarrow -z$ (or in spherical coordinates, $\theta \leftrightarrow \pi-\theta$) kills the p_z component.

By the same symmetry, all the higher 2^ℓ–poles with odd ℓ also vanish, so only the even–ℓ multipoles may contribute to the potential.

Thus, out next order of business is to calculate the quadrupole moment of the charge density (100). By axial symmetry, the only independent component of the quadrupole tensor is

$$\begin{aligned}
\mathcal{Q}_{zz} \;=\; \mathcal{M}_2 &= \iiint\limits_{\text{ball}} \Big(\tfrac{3}{2}z^2 - \tfrac{1}{2}r^2 \;=\; r^2\times P_2(\cos\theta)\Big) \times \rho\,d^3\text{Vol} \\
&= \int\limits_0^R dr\,r^2 \int\limits_0^\pi d\theta\,\sin\theta \int\limits_0^{2\pi} d\phi\,\left[\frac{r^2(3\cos^2\theta-1)}{2} \times k\,\frac{R(R-2r)}{r^2}\,\sin\theta\right] \\
&= \int\limits_0^R r^2\,\frac{kR(R-2r)}{r^2}\,r^2\,dr \times \int\limits_0^\pi \frac{3\cos^2\theta-1}{2}\,\sin^2\theta\,d\theta \times 2\pi,
\end{aligned} \tag{103}$$

where the θ integral evaluates to

$$\begin{aligned}\int_0^\pi \frac{3\cos^2\theta-1}{2}\times\sin^2\theta\,d\theta \;&=\; \int_0^\pi \frac{3\cos(2\theta)+3-2}{4}\times\frac{1-\cos(2\theta)}{2}\,d\theta\\ &\quad\langle\!\langle\,\text{changing variable to }\theta_2=2\theta\,\rangle\!\rangle\\ &=\;\frac{1}{16}\int_0^{2\pi} d\theta_2\,(3\cos\theta_2+1)(1-\cos\theta_2)\\ &=\;\frac{1}{16}\int_0^{2\pi} d\theta_2\,(1\,+\,2\cos\theta_2\,-\,3\cos^2\theta_2)\\ &=\;\frac{1}{16}\left((2\pi)\,+\,0\,-\,3\times\frac{2\pi}{2}\right)\;=\;-\frac{\pi}{16}\,,\end{aligned}\tag{104}$$

while the radial integral evaluates to

$$\int_0^R r^2\times\frac{kR(R-2r)}{r^2}\times r^2\,dr \;=\; kR\int_0^R (Rr^2-2r^3)\,dr \;=\; kR\times\left(R\times\frac{R^3}{3}\,-\,2\times\frac{R^4}{4}\right)\;=\;-\frac{kR^5}{6}\,.\tag{105}$$

Altogether, the quadrupole moment comes out to

$$\mathcal{M}_2 \;=\; \frac{-kR^5}{6}\times\frac{-\pi}{16}\times 2\pi \;=\; +\frac{\pi^2}{48}\,kR^5\;\neq\;0.$$

Note: unlike the net charge and the dipole moment, the quadrupole moment of the charged ball in question does *not* vanish. Therefore, it is this quadrupole moment which dominates the electric potential at large distances:

$$V(r,\theta)\;\approx\;V_{\text{quadrupole}}\;=\;\frac{\mathcal{M}_2}{4\pi\epsilon_0}\times\frac{P_2(\cos\ell)}{r^3}\;=\;\frac{\pi kR^5}{192\,\epsilon_0}\times\frac{3\cos^2\theta-1}{2r^3}\,.\tag{106}$$

Along the z axis where $\cos\theta=\pm1$, this potential becomes

$$V\;\approx\;\frac{\pi kR^5}{192\,\epsilon_0}\times\frac{1}{r^3}\,.\tag{107}$$