Problem **4.10**:

(a) According to the textbook equations (4.11) and (4.12), the bound charges on the surface of a dielectric and in its bulk are

$$\sigma_b = +\mathbf{P}\cdot\mathbf{n}, \tag{4.11}$$
$$\rho_b = -\nabla\cdot\mathbf{P}. \tag{5.12}$$

For the problem at hand $\mathbf{P} = k\mathbf{r}$, hence the bound charges in the bulk of the dielectric ball have density

$$\rho_b = -k(\nabla\cdot\mathbf{r}) = -3k, \tag{1}$$

while the bound charges on the surface of the ball have density

$$\sigma_b = +k\mathbf{r}\cdot\mathbf{n} = +kR. \tag{2}$$

Note that the net bound charge of the dielectric should be zero. Indeed,

$$Q_b^{\rm net} = 4\pi R^2\times\sigma_b + \frac{4\pi R^3}{3}\times\rho_b = +4\pi R^3 k - 4\pi R^3 k = 0. \tag{3}$$

(b) I assume there are no free charges inside or outside the dielectric ball, just the bound charges (1) and (2). These charges are spherically symmetric, and the net charge of the dielectric ball is zero, hence *outside the ball, the electric field is zero.*

On the other hand, inside the ball, the electric field follows from the Gauss Law:

$$E(r) = \frac{Q[\text{inside radius } r]}{4\pi\epsilon_0\, r^2}\,. \tag{4}$$

For $r < R$, the Gaussian surface — the sphere of radius r — includes the bulk bound charges of density ρ_b but not the surface bound charges. Thus

$$Q[\text{inside radius } r] \;=\; \frac{4\pi r^3}{3} \times \rho_b \;=\; -4\pi r^3 k, \tag{5}$$

and hence *the electric field inside the dielectric ball is*

$$E(r) \;=\; -\frac{kr}{\epsilon_0}\,, \tag{6}$$

where the $-$ sign indicates the direction towards the center (for $k > 0$).

Note: both inside and outside the ball, $\mathbf{E} = -(1/\epsilon_0)\mathbf{P}$, so the displacement field $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ vanishes. This is entirely appropriate for a system which has bound charges but no free charges.

Problem **4.13**:

There are no free charges inside or outside the cylinder in question, but the dielectric polarization $\mathbf{P}$ inside the cylinder gives rise to the bound charges. Due to uniformity of the polarization vector $\mathbf{P} = P\hat{\mathbf{x}}$ inside the cylinder there are no volume bound charges,

$$\rho_b \;=\; -\nabla\cdot\mathbf{P} \;=\; 0, \tag{7}$$

but there are non-zero bound charges on the cylinder's surface,

$$\sigma_b \;=\; \mathbf{P}\cdot\mathbf{n} \;=\; P\hat{\mathbf{x}}\cdot\hat{\mathbf{s}} \;=\; P\cos\phi. \tag{8}$$

Let's calculate the electric field due to these bound charges on the cylinder's surface. We start with the general formula for the potential inside and outside the cylinder in the separation-of variables method,

$$V(s,\phi) \;=\; \sum_{m=0}^{\infty}\Big(A_m\cos(m\phi) \,+\, B_m\sin(m\phi)\Big) \times \begin{cases} \left(\frac{s}{R}\right)^m & \text{for } s < R, \\ \left(\frac{R}{s}\right)^m & \text{for } s > R, \end{cases} \tag{9}$$

where the A_m and B_m are some constants, to be determined by matching the discontinuity of the radial electric field to the given surface charge density:

$$\sigma(\phi) \;=\; \epsilon_0\,\mathrm{disc}(E_s\,@s=R) \;=\; \frac{2\epsilon_0}{R}\sum_{m=0}^{\infty} m\Big(A_m\cos(m\phi) \,+\, B_m\sin(m\phi)\Big). \tag{10}$$

For the problem at hand, we immediately see that

$$A_1 \;=\; \frac{RP}{2\epsilon_0}\,, \quad \text{all other } A_m \;=\; 0, \quad \text{all } B_m \;=\; 0. \tag{11}$$

Consequently, inside the cylinder

$$V(s,\phi) \;=\; \frac{RP}{2\epsilon_0}\times\frac{s\,\cos\phi}{R} \;=\; \frac{P}{2\epsilon_0}\times s\,\cos\phi \;=\; \frac{P}{2\epsilon_0}\times x \tag{12}$$

and hence the electric field

$$\mathbf{E} \;=\; -\nabla V \;=\; -\frac{P}{2\epsilon_0}\,\hat{\mathbf{x}} \;=\; -\frac{\mathbf{P}}{2\epsilon_0}\,. \tag{13}$$

Note uniformity of this electric field inside the cylinder.

At the same time, outside the cylinder the potential is

$$V(s,\phi) \;=\; \frac{RP}{2\epsilon_0}\times\frac{R\,\cos\phi}{s} \;=\; \frac{R^2P}{2\epsilon_0}\times\frac{\cos\phi}{s} \;=\; \frac{R^2P}{2\epsilon_0}\times\frac{x}{x^2+y^2}\,. \tag{14}$$

Taking the gradient of this potential, we obtain the electric field

$$\begin{aligned}
\mathbf{E} \;=\; -\nabla V \;&=\; -\frac{R^2P}{2\epsilon_0}\left(\frac{\hat{\mathbf{x}}}{s^2} \,-\, \frac{2x}{s^3}\,\hat{\mathbf{s}}\right)\\
&=\; \frac{R^2}{2\epsilon_0\,s^2}\left(-P\hat{\mathbf{x}} \,+\, \frac{2xP}{s}\,\hat{\mathbf{s}}\right)\\
&=\; \frac{R^2}{2\epsilon_0\,s^2}\Big(-\mathbf{P} \,+\, 2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}}\Big).
\end{aligned} \tag{15}$$

Quod erat demonstrandum.

Problem **4.15**:

(a) The polarized spherical shell in question has both volume and surface bound charges. In particular, the volume bound charges have density

$$\rho_b \;=\; -\nabla\cdot\mathbf{P} \;=\; -\nabla\cdot\left(\frac{k\hat{\mathbf{r}}}{r}\right) \;=\; -\frac{\partial}{\partial r}\left(\frac{k}{r}\right) \;-\; \frac{2}{r}\left(\frac{k}{r}\right) \;=\; +\frac{k}{r^2} \;-\; \frac{2k}{r^2} \;=\; -\frac{k}{r^2}\,, \tag{16}$$

the bound charges on the outer surface have density

$$\sigma_b^{(b)} \;=\; \mathbf{P}\cdot\mathbf{n} \;=\; \frac{k\hat{\mathbf{r}}}{b}\cdot(+\hat{\mathbf{r}}) \;=\; +\frac{k}{b}\,, \tag{17}$$

and the bound charges on the inner surface have density

$$\sigma_b^{(a)} \;=\; \mathbf{P}\cdot\mathbf{n} \;=\; \frac{k\hat{\mathbf{r}}}{a}\cdot(-\hat{\mathbf{r}}) \;=\; -\frac{k}{a}\,. \tag{18}$$

To cross-check these formula, let's calculate the net bound charge and make sure it vanishes:

$$Q_b^{\text{net}} \;=\; \int\limits_a^b \rho_b(r)\times 4\pi r^2\,dr \;+\; \sigma_b^{(a)}\times 4\pi a^2 \;+\; \sigma_b^{(b)}\times 4\pi b^2 \;=\; -4\pi k(b-a) \;-\; 4\pi ka \;+\; 4\pi kb \;=\; 0. \tag{19}$$

The electric field of all these charges follows from the spherical symmetry and the Gauss Law:

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{Q(r)\hat{\mathbf{r}}}{4\pi\epsilon_0\,r^2} \tag{20}$$

where $Q(r)$ is the net electric charge — free or bound — inside the sphere of radius r. Since there are no free charges in the system but only the bound charges, we have:

- For $r < a$ there are no charges inside r, thus $Q(r) = 0$.
- For $r > a$, $Q(r) = Q_b^{\text{net}} = 0$.

- For r within the shell, $a < r < b$, $Q(r)$ includes the inner surface and part of the shell's volume, thus

$$Q(r) \;=\; 4\pi a^2 \times \sigma_b^{(a)} \;+\; \int\limits_b^r \rho_b(r') \times 4\pi r'^2\, dr' \;=\; -4\pi ka \;-\; 4\pi k(r-a) \;=\; -4\pi kr. \tag{21}$$

Consequently, the electric field vanishes outside the shell or inside the cavity, while inside the dielectric

$$\mathbf{E} \;=\; \frac{-4\pi kr}{4\pi\epsilon_0\, r^2}\,\hat{\mathbf{r}} \;=\; -\frac{k\hat{\mathbf{r}}}{\epsilon_0\, r} \;=\; -\frac{1}{\epsilon_0}\,\mathbf{P}. \tag{22}$$

(b) Now consider the displacement field $\mathbf{D} = \epsilon_0\mathbf{E}+\mathbf{P}$. By spherical symmetry and the Gauss Law,

$$\mathbf{D}(\mathbf{r}) \;=\; \frac{Q_f(r)\,\hat{\mathbf{r}}}{4\pi r^2} \tag{23}$$

where $Q_f(r)$ is the net *free charge* inside the sphere of radius r. But the system in question has no free charges at all, only the bound charges, thus $Q_f(r) \equiv 0$. Consequently, the displacement field $\mathbf{D}$ vanishes throughout the system — inside the cavity, within the dielectric, outside the shell, everywhere — and therefore

$$\mathbf{E}(\mathbf{r}) \;=\; -\frac{1}{\epsilon_0}\,\mathbf{P}(\mathbf{r}) \quad \text{at all } \mathbf{r}. \tag{24}$$

In particular, inside the cavity or outside the shell $\mathbf{E} \equiv 0$ while inside the dielectric

$$\mathbf{r}(\mathbf{r}) \;=\; -\frac{1}{\epsilon_0}\,\mathbf{P}(\mathbf{r}) \;=\; -\frac{k\hat{\mathbf{r}}}{\epsilon_0\, r}\,. \tag{25}$$

By inspection, the second method is much simpler than the first!

Problem **4.16**:

I assume that the polarization $\mathbf{P}$ of the dielectric is completely frozen and remains uniform even in the immediate vicinity of the cavity. Consequently, there are no bound charges in the bulk of the dielectric, but there are bound charges on the cavity's walls,

$$\sigma_b \;=\; -\mathbf{n}\cdot\mathbf{P}. \tag{26}$$

The minus sign in this formula stems from the direction of the normal unit vector $\mathbf{n}$ — outward from the cavity, which makes it into the dielectric rather than out from the dielectric.

(a) The bound charges on the cavity walls are given by eq. (26). For the spherical cavity, we have

$$\sigma_b(\theta) \;=\; -P\cos\theta \tag{27}$$

where we count the spherical angle θ from the direction of the polarization vector $\mathbf{P}$. The electric field caused by charges on a spherical surface is explained in §3.3 of the textbook, example 3.9, eqs. (3.78–87), and also in my notes on separation of variables, pages 33–36, eqs. (174–195). For the present purposes, we may identify the angular dependence of the bound charge density (27) as

$$\sigma_b(\theta) \;=\; k\times P_1(\cos\theta) \quad \text{for } k=-P, \tag{3.85}$$

exactly as in the textbook equation (3.85). Consequently, textbook equation (3.86) gives the potential inside the sphere due to the charges (3.85),

$$\begin{aligned} V^{\text{inside}}[\text{due to }\sigma_b](r,\theta) \;&=\; \frac{k}{3\epsilon_0}\times r\times P_1(\cos\theta) && (3.86)\\ &=\; -\frac{P}{3\epsilon_0}\times r\,\cos\theta, && (28)\\ \textit{i.e.,}\quad V^{\text{inside}}[\text{due to }\sigma_b](x,y,z) \;&=\; -\frac{P}{3\epsilon_0}\times z. && (29) \end{aligned}$$

Consequently, the electric field inside the cavity due to bound charges on its walls is uniform

$$\mathbf{E}^{\text{inside}}[\text{due to } \sigma_b](x,y,z) \;=\; +\frac{P}{3\epsilon_0}\,\hat{\mathbf{z}} \;=\; \frac{\mathbf{P}}{3\epsilon_0}\,. \tag{30}$$

Finally, adding this field due to bound charges to the external field $\mathbf{E}_0$, we arrive at the net electric field inside the cavity,

$$\mathbf{E}^{\text{net}}_{\text{inside}} \;=\; \mathbf{E}_0 \;+\; \frac{1}{3\epsilon_0}\,\mathbf{P}. \tag{31}$$

Or in terms of the displacement field $\mathbf{D}_0$ outside the cavity,

$$\mathbf{E}^{\text{net}}_{\text{inside}} \;=\; \frac{1}{\epsilon_0}\left(\mathbf{D}_0 \,-\, \frac{2}{3}\,\mathbf{P}\right) \;=\; \frac{2}{3}\,\mathbf{E}_0 \;+\; \frac{1}{3\epsilon_0}\,\mathbf{D}_0\,, \tag{32}$$

hence

$$\mathbf{D}^{\text{net}}_{\text{inside}} \;=\; \epsilon_0\mathbf{E}^{\text{net}}_{\text{inside}} \;=\; \frac{2}{3}\,\epsilon_0\mathbf{E}_0 \;+\; \frac{1}{3}\,\mathbf{D}_0\,. \tag{33}$$

(b) In a needle-shaped cavity running parallel to the polarization vector $\mathbf{P}$, the normal vector $\mathbf{n}$ to a wall is $\perp$ $\mathbf{P}$, hence the bound charge on the wall is zero,

$$\sigma_b \;=\; -\mathbf{n}\cdot\mathbf{P} \;=\; 0. \tag{34}$$

Or rather, $\sigma_b = 0$ over most of the cavity walls, except for the ends of the cavity where the walls are no longer parallel to $\mathbf{P}$. At those ends we have finite surface density σ_b, but since the ends have very small areas compared to the rest of the walls, the net charges at the ends are small and may be neglected.

Altogether, we have negligible surface charges on the cavity walls, and no volume bound charges outside the cavity, so the net electric field inside the cavity is simply the external electric field $\mathbf{E}_0$ from the outside,

$$\mathbf{E}^{\text{net}}_{\text{inside}} \;=\; \mathbf{E}_0\,. \tag{35}$$

As to the displacement field inside the cavity,

$$\mathbf{D}^{\text{net}}_{\text{inside}} \;=\; \epsilon_0\mathbf{E}^{\text{net}}_{\text{inside}} \;=\; \epsilon_0\mathbf{E}_0 \;\neq\; \mathbf{D}_0\,. \tag{36}$$

(c) To fix the notations, let the polarization vector $\mathbf{P}$ point vertically up while the wafer-shaped cavity is horizontal. The boundary of this cavity comprises two horizontal disks — one at the top and one at the bottom — and a vertical outer wall which of very low height. The disks are $\perp \mathbf{P}$ while the wall is $\parallel \mathbf{P}$, hence the bound charges on all these surfaces are

$$\sigma_b \;=\; \begin{cases} -P & \text{on the top disk,} \\ +P & \text{on the bottom disk,} \\ 0 & \text{on the cylindrical wall.} \end{cases} \tag{37}$$

The boundary charges on the disks act as plates in a parallel-plate capacitor — they create a uniform electric field

$$E^{\text{inside}}[\text{due to } \sigma_b] = \frac{\sigma_b}{\epsilon_0} \;=\; \frac{P}{\epsilon_0} \tag{38}$$

The direction of this field is from the positive bottom disk to the negative upper disk, *i.e.*, vertically up, which is the same direction as the polarization vector $\mathbf{P}$. Thus

$$\mathbf{E}^{\text{inside}}[\text{due to } \sigma_b] \;=\; \frac{\mathbf{P}}{\epsilon_0}\,, \tag{39}$$

and adding this field to the external field we obtain the net electric field inside the cavity,

$$\mathbf{E}_{\text{net}}^{\text{inside}} \;=\; \mathbf{E}_0 \;+\; \frac{\mathbf{P}}{\epsilon_0} \;=\; \frac{\mathbf{D}_0}{\epsilon_0}\,. \tag{40}$$

As to the displacement field inside the cavity,

$$\mathbf{D}_{\text{inside}}^{\text{net}} \;=\; \epsilon_0 \mathbf{E}_{\text{inside}}^{\text{net}} \;=\; \mathbf{D}_0 \;\equiv\; \mathbf{D}^{\text{outside}}. \tag{41}$$

Alternative solution to parts (b–c):

The electric field $\mathbf{E}$ and the displacement field $\mathbf{D}$ obey different Maxwell equations:

$$\nabla\cdot\mathbf{D} \;=\; \rho_{\text{free}} \quad\text{but}\quad \epsilon_0\,\nabla\cdot\mathbf{E} \;=\; \rho_{\text{free}} \;+\; \rho_{\text{bound}}\,, \tag{42}$$

while (for the static fields)

$$\nabla\times\mathbf{E} \;=\; 0 \quad\text{but}\quad \nabla\times\mathbf{D} \;\neq 0 \quad\text{is OK,} \tag{43}$$

hence

$$\mathbf{E} \;=\; -\nabla V \quad\text{but}\quad \mathbf{D} \;\neq\; \nabla(\text{anything}). \tag{44}$$

Consequently, at the boundary of two different dielectrics — or of a dielectric and the vacuum — we have different boundary conditions for the $\mathbf{E}$ and $\mathbf{D}$ fields:

$$\begin{aligned} \mathbf{E}_1^{\parallel} \;&=\; \mathbf{E}_2^{\parallel} \quad\text{but}\quad \mathbf{E}_1^{\perp} \;\neq\; \mathbf{E}_2^{\perp},\\ \mathbf{D}_1^{\perp} \;&=\; \mathbf{D}_2^{\perp} \quad\text{but}\quad \mathbf{D}_1^{\parallel} \;\neq\; \mathbf{D}_2^{\parallel}. \end{aligned} \tag{45}$$

Coming back to our problem, let's assume that in the dielectric, the electric field $\mathbf{E}_0$, the polarization $\mathbf{P}$ is parallel to the electric field $\mathbf{E}_0$, hence also $\mathbf{D}_0 \parallel \mathbf{E}_0$. For the needle-shaped cavity in part (b), this makes all the fields parallel to the dielectric-vacuum boundary, thus according to the boundary conditions (45),

$$\mathbf{E}^{\text{inside}} \;=\; \mathbf{E}^{\text{outside}} \;=\; \mathbf{E}_0 \tag{46}$$

while

$$\mathbf{D}^{\text{inside}} \;=\; \epsilon_0\mathbf{E}^{\text{inside}} \;=\; \epsilon_0\mathbf{E}_0 \;\neq\; \mathbf{D}_0\,. \tag{47}$$

On the other hand, for the wafer-shaped cavity in part (c), the fields are perpendicular to the boundary, hence

$$\mathbf{D}^{\text{inside}} \;=\; \mathbf{D}^{\text{outside}} \;=\; \mathbf{D}_0 \tag{48}$$

while

$$\mathbf{E}^{\text{inside}} \;=\; \frac{\mathbf{D}^{\text{inside}}}{\epsilon_0} \;=\; \frac{\mathbf{D}_0}{\epsilon_0} \;=\; \mathbf{E}_0 \;+\; \frac{\mathbf{P}}{\epsilon_0} \;\neq\; \mathbf{E}_0\,. \tag{49}$$

Problem **4.41**:

In a linear dielectric, a uniform macroscopic electric tension field $\mathbf{E}_m$ sets up polarization

$$\mathbf{P} = \chi\epsilon_0\mathbf{E}_m \tag{50}$$

Consider a small spherical cavity in the dielectric. As we saw in problem **4.16**(a), the electric field in the cavity is uniform but it's different from the macroscopic field $\mathbf{E}_m$. Specifically,

$$\mathbf{E}_{\text{cavity}} = \mathbf{E}_m + \frac{1}{3\epsilon_0}\mathbf{P} = \mathbf{E}_m + \frac{\chi}{3}\mathbf{E}_m = \left(1 + \frac{\chi}{3}\right)\mathbf{E}_m\,. \tag{51}$$

Now consider an atom or a non-polar molecule of polarizability α. In a dielectric gas or liquid, this atom/molecule is subject to electric field due to external sources as well as due to polarization of all the other atoms or molecules. so it develops an electric dipole moment $\mathbf{p} = \alpha\mathbf{E}$. Or more accurately,

$$\mathbf{p} = \alpha\mathbf{E}' \tag{52}$$

where $\mathbf{E}'$ is the net electric field from all the sources other than the atom's own electrons or nucleus. Over the atom's size, the microscopic field $\mathbf{E}'$ is approximately uniform, but it's different from the macroscopic field $\mathbf{E}_m$ since the latter is the space averaged microscopic field from all the sources, including the atom's own dipole moment (52).

Macroscopically, the difference between the $\mathbf{E}_m$ and the $\mathbf{E}'_m$ fields is excluding the atom is question from the rest of the dielectric, which is equivalent to making a small cavity around the atom in question. The proper size and shape of such a cavity is not obvious; in a crystal, it would depend on the geometry of the crystalline lattice. But in a gas or a liquid it makes more sense to keep the cavity rotationally symmetric, which calls for spherical geometry.

Thus, we identify the $\mathbf{E}'$ field in eq. (52) with the macroscopic field in a small spherical cavity surrounding the atom. According to eq. (51), this means

$$\mathbf{E}' = \left(1 + \frac{\chi}{3}\right)\mathbf{E}_m \tag{53}$$

and hence atom's dipole moment

$$\mathbf{p} \;=\; \alpha\left(1+\frac{\chi}{3}\right)\mathbf{E}_m\,. \tag{54}$$

The polarization $\mathbf{P}$ of the dielectric comes from the net dipole moment of all the atoms or molecules in a unit volume, hence

$$\mathbf{P} \;=\; n\mathbf{p} \;=\; n\alpha\left(1+\frac{\chi}{3}\right)\mathbf{E}_m \tag{55}$$

where n is the dielectric's density,

$$n \;=\; \frac{\#\text{atoms or molecules}}{\text{volume}}\,. \tag{56}$$

In terms of the dielectric's susceptibility χ, eq. (55) means

$$\chi\epsilon_0 \;=\; n\alpha\times\left(1+\frac{\chi}{3}\right). \tag{57}$$

This is a linear equation for the χ; the solution is the *Clausius–Mosotti formula*

$$\chi \;=\; \frac{(n\alpha/\epsilon_0)}{1\,-\,\frac{1}{3}(n\alpha/\epsilon_0)} \tag{58}$$

Quod erat demonstrandum.

PS: Experimentally, one can easily measure the dielectric constant $\epsilon=\chi+1$ of some material, while the atom number density n obtains as mass density divided by the atom/molecule's mass. Given these data, we may use the Clausius–Mosotti formula (58) to determine the atom/molecule's polarizability as

$$\alpha \;=\; \frac{3\epsilon_0}{n}\times\left(\frac{\chi}{\chi+3}\;=\;\frac{\epsilon-1}{\epsilon+2}\right). \tag{4.72}$$

Problem **4.19**:

(a) For the capacitor on textbook figure **4.25(a)**, the **E** and **D** fields are perpendicular to the boundaries between the dielectric and the air. Consequently, the displacement field **D** does not change at the those boundaries, which makes it uniform throughout the capacitor. In terms of the free charges $\pm Q$ on the capacitor plates,

$$D = \frac{Q}{A} \tag{59}$$

where A is the plates' area. Consequently, in the dielectric

$$E_{\text{dielectric}} = \frac{D}{\epsilon\epsilon_0} = \frac{Q}{\epsilon\epsilon_0 A}, \quad P_{\text{dielectric}} = (\chi = \epsilon - 1)\epsilon_0 \times E = \frac{\epsilon - 1}{\epsilon} \times \frac{Q}{A}, \tag{60}$$

while in the air gap (which we approximate as vacuum)

$$E_{\text{air}} \approx \frac{D}{\epsilon_0} = \frac{Q}{\epsilon_0 A}, \quad P_{\text{air}} \approx 0. \tag{61}$$

Note that the electric tension field **E** in the air is ϵ times stronger than in the dielectric.

The bound charges on the dielectric surfaces follow from the polarization **P**. Since its direction is $\perp$ to the boundary,

$$\sigma_b = \pm P = \pm\frac{\epsilon - 1}{\epsilon} \times \frac{Q}{A}, \tag{62}$$

where the sign is + for the surface facing the negative capacitor plate and − for the surface facing the positive plate.

Next, consider the voltage on the capacitor. Let d denote the distance between the capacitor plates. We are told that half of this distance is filled by the dielectric while the other half is filled with air. Consequently, the voltage follows from the electric tension fields E in the two media as

$$\Delta V = \frac{d}{2} \times E_{\text{dielectric}} + \frac{d}{2} \times E_{\text{air}} = \frac{d}{2} \times \frac{Q}{\epsilon\epsilon_0 A} + \frac{d}{2} \times \frac{Q}{\epsilon_0 A} = \frac{\epsilon + 1}{2\epsilon} \times \frac{dQ}{\epsilon_0 A}. \tag{63}$$

In terms of the capacitance $C = Q/\Delta V$, this means

$$C = \frac{2\epsilon}{\epsilon + 1} \times \frac{\epsilon_0 A}{d}. \tag{64}$$

(b) For the other capacitor on textbook figure **4.25(b)**, the boundary between the dielectric and the air is parallel to the **E** and the **D** fields. Consequently, it's the tension field **E** which does not change across the boundary. Moreover, the voltage between the plates is uniform throughout the plates' area, so the electric field **E** turns out to be uniform throughout the capacitor. In terms of the voltage,

$$E \;=\; \frac{\Delta V}{d} \;=\; \text{const.} \tag{65}$$

Given this electric tension field, the polarization and the displacement field follow by linearity of the dielectric:

$$P_{\text{dielectric}} \;=\; \chi\epsilon_0 E \;=\; (\epsilon-1)\epsilon_0 \times \frac{\Delta V}{d}\,, \quad D_{\text{dielectric}} \;=\; \epsilon\epsilon_0 E \;=\; \epsilon\epsilon_0 \times \frac{\Delta V}{d}\,, \tag{66}$$

while in the air

$$P_{\text{air}} \;\approx\; 0, \quad D_{\text{air}} \;\approx\; \epsilon_0 E \;=\; \epsilon_0 \times \frac{\Delta V}{d}\,. \tag{67}$$

Note that the displacement field D in the dielectric is ϵ times stronger than in the air.

The bound charges on the dielectric follow from the polarization field **P**. The direction of **P** follows the direction of the electric field **E** — from the positive plate to the negative plate — so it is parallel to the boundary between the dielectric and the air gap. Consequently, there are no bound charge on that surface of the dielectric. Instead, there are bound charges

$$\sigma_b \;=\; \pm P \;=\; \pm(\epsilon-1)\epsilon_0 \times \frac{\Delta V}{d} \tag{68}$$

on the surfaces touching the capacitor plates, with the + sign for the surface touching the negative plate and the − sign on the surface touching the positive plate.

Now consider the free charges on the capacitor plates themselves. The densities of these free charges is related to the displacement field **D** as

$$\sigma_{\text{free}} \;=\; \mathbf{D}\cdot\mathbf{n} \tag{69}$$

but since the **D** field happens to be $\perp$ to the plates, we have simply

$$\sigma_{\text{one plate}} \;=\; +D, \quad \sigma_{\text{other plate}} \;=\; -D. \tag{70}$$

But the D field in the capacitor in question is not uniform: It is ϵ times stronger in the dielectric that in the air gap. Consequently, the surface charge density σ is not uniform across the plates. Instead, the parts of the plates which touch the dielectric have ϵ times stronger charge density that the parts "hanging" in the air,

$$\sigma_{\text{dielectric}}^{\text{touching}} = \pm D_{\text{dielectric}} = \pm\epsilon \times \epsilon_0 \frac{\Delta V}{d}, \quad \text{while} \quad \sigma_{\text{in air}} = \pm D_{\text{air}} = \pm\epsilon_0 \frac{\Delta V}{d}. \tag{71}$$

Since the problem specifies that half of the plates touch the dielectric while the other half hangs in the air, the net charges on the plates are $\pm Q$ where

$$Q = \frac{A}{2} \times \sigma_{\text{dielectric}}^{\text{touching}} + \frac{A}{2} \times \sigma_{\text{in air}} = \frac{A}{2} \times \epsilon \times \epsilon_0 \frac{\Delta V}{d} + \frac{A}{2} \times \epsilon_0 \frac{\Delta V}{d} = \frac{\epsilon+1}{2} \times \frac{\epsilon_0 A}{D} \times \Delta V. \tag{72}$$

In terms of the capacity, this means

$$C = \frac{\epsilon+1}{2} \times \frac{\epsilon_0 A}{D}. \tag{73}$$

Note: the capacitors on figures 4.25(a) and 4.25(b) have different capacities!

Problem **4.36**:

Consider what happens to the electric tension field $\mathbf{E}$ and the displacement field $\mathbf{D}$ at the boundary between two dielectrics (or between a dielectric and the vacuum). Comparing the normal and the tangential components of the fields on the two sides of the boundary, we have

$$\mathbf{E}_1^{\|} = \mathbf{E}_2^{\|} \quad \text{and} \quad \mathbf{D}_1^{\perp} = \mathbf{D}_2^{\perp} \tag{74}$$

but

$$\mathbf{E}_1^{\perp} \neq \mathbf{E}_2^{\perp} \quad \text{and} \quad \mathbf{D}_1^{\|} \neq \mathbf{D}_2^{\|}. \tag{75}$$

For the linear dielectrics where $\mathbf{D}_1 = \epsilon_1\epsilon_0\mathbf{E}_1$ and $\mathbf{D}_2 = \epsilon_2\epsilon_0\mathbf{E}_2$, eq. (74) becomes

$$\mathbf{E}_1^{\|} = \mathbf{E}_2^{\|} \quad \text{but} \quad \epsilon_1\mathbf{E}_1^{\perp} = \epsilon_2\mathbf{E}_2^{\perp}. \tag{76}$$

Note that the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ lie in the same plane $\perp$ to the boundary. In that plane, they make angles

$$\theta_1 \;=\; \arctan\frac{E_1^{||}}{E_1^{\perp}} \quad\text{and}\quad \theta_2 \;=\; \arctan\frac{E_2^{||}}{E_2^{\perp}} \tag{77}$$

with the normal to the boundary, *cf.* textbook figure (4.34). Consequently,

$$\frac{\tan\theta_2}{\tan\theta_1} \;=\; \frac{E_2^{||}/E_2^{\perp}}{E_1^{||}/E_1^{\perp}} \;=\; \frac{E_2^{||}}{E_1^{||}}\times\frac{E_1^{\perp}}{E_2^{\perp}} \;=\; \langle\!\langle\,\text{in light of eq. (74)}\,\rangle\!\rangle \;=\; 1\times\frac{\epsilon_2}{\epsilon_1}\,, \tag{78}$$

Quod erat demonstrandum.

At small angles, eq. (78) looks like the Snell Law of optics, so the electric field crossing in and out of dielectrics should follow similar paths to the light rays crossing in and out of glass (or some other transparent material). Thus, a dielectric "lens" ought to focus the electric field lines.

But physically, the electric field lines do not act like the light rays. As long as they run through a uniform medium, the light rays are straight lines; but the electric field lines are usually quite curved even inside a uniform dielectric. Also, a light ray starts at the light source and then propagates in a definite direction until it gets somewhere — an eye, or a camera, or some other light detector, or simply some light-absorbing material — or goes away to ∞. We may block a light ray at some point, but it would not affect the path of the ray before the blocking point. On the other hand, an electric field line stretches between a positive charge and a negative charge, and it is equally affected by both of them. Also, one cannot block a field line, only distort it by bringing in more charges, but then the whole line will charge its geometry rather than just the "downstream" part of the line. Thus, *focusing the electric field lines does not make any physical sense.*

If we put a positive charge on one side of a dielectric "lens" and a negative charge on the other side and keep their distances from the lens just right, the electric field lines might look like the light rays between a source and its real image. But this resemblance is superficial and requires adjusting the locations of both charges — otherwise, the field lines would be curved rather than locally-straight.

Problem **4.37**:

There are no free charges inside or outside the sphere except for the pure dipole $\mathbf{p}$ at the sphere's center, so everywhere else the potential obeys the Laplace equation $\nabla^2 V = 0$. In spherical coordinates (r, θ, ϕ), the potential V does not depend on the ϕ coordinate due to axial symmetry of the system, while its dependence on the other two coordinates follows from the separation-of-variables method:

inside the sphere

$$V(r,\theta) = \sum_\ell \left(A_\ell \times r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \times P_\ell(\cos\theta), \tag{79}$$

outside the sphere

$$V(r,\theta) = \sum_\ell \left(C_\ell \times r^\ell + \frac{D_\ell}{r^{\ell+1}} \right) \times P_\ell(\cos\theta), \tag{80}$$

for some constant coefficients $A_\ell, B_\ell, C_\ell, D_\ell$.

The C_ℓ coefficients govern the asymptotic behavior of the potential far away from the sphere,

$$\text{for } r \to \infty, \quad V \approx \sum_\ell C_\ell \, r^\ell \, P_\ell(\cos\theta). \tag{81}$$

For the problem at hand, there is no external electric field so we should have $V \to 0$ for $r \to \infty$; this requires all the C_ℓ to vanish.

Likewise, the B_ℓ coefficients govern the potential near the center of the sphere,

$$\text{for } r \to 0, \quad V \approx \sum_\ell \frac{B_\ell}{r^{\ell+1}} \times P_\ell(\cos\theta). \tag{82}$$

Physically, they are related to the multipole moment of some charged source at the center. For the problem at hand, we have a pure dipole at the center, thus

$$\text{for } r \to 0, \quad V \approx \frac{1}{\epsilon} \times \frac{p}{4\pi\epsilon_0} \times \frac{\cos\theta}{r^2} \tag{83}$$

where the $1/\epsilon$ factor stems for the dipole being in the middle of a dielectric. In terms of the coefficients B_ℓ, this means

$$B_1 \;=\; \frac{p}{4\pi\epsilon_0} \times \frac{1}{\epsilon}\,, \quad \text{all other } B_\ell \;=\; 0. \tag{84}$$

The remaining coefficients A_ℓ and D_ℓ follow from the boundary conditions at the sphere's surface

$$\text{at } r = R \text{ and any } \theta, \quad V^{\text{outside}} \;=\; V^{\text{inside}} \quad \text{and} \quad E_r^{\text{outside}} \;=\; \epsilon \times E_r^{\text{inside}}. \tag{85}$$

As explained in my notes on dielectric boundaries (eqs. (10) and (13) on page 3), these boundary conditions require

$$\begin{aligned} A_\ell \times R^\ell \;+\; \frac{B_\ell}{R^{\ell+1}} \;&=\; C_\ell \times R^\ell \;+\; \frac{D_\ell}{R^{\ell+1}}\,, \\ -\epsilon\ell \times A_\ell \times R^\ell \;+\; \epsilon(\ell+1) \times \frac{B_\ell}{R^{\ell+1}} \;&=\; -\ell \times C_\ell \times R^\ell \;+\; (\ell+1) \times \frac{D_\ell}{R^{\ell+1}}\,. \end{aligned} \tag{86}$$

For all the $\ell \neq 1$ modes, we have $B_\ell = C_\ell = 0$, so solving eqs. (86) for the A_ℓ and D_ℓ we immediately get $A_\ell = D_\ell = 0$. For the remaining $\ell = 1$ mode we have $C_1 = 0$ but $B_1 \neq 0$, so eqs. (86) become

$$A_1 \times R^3 \;+\; B_1 \;=\; D_1\,, \quad -\epsilon \times A_1 \times R^3 \;+\; 2\epsilon \times B_1 \;=\; 2 \times D_1\,, \tag{87}$$

and their solution is

$$A_1 \;=\; \frac{2(\epsilon-1)}{\epsilon+2} \times \frac{B_1}{R^3} \;=\; \frac{2(\epsilon-1)}{\epsilon(\epsilon+2)} \times \frac{p}{4\pi\epsilon_0\, R^3}\,, \quad D_1 \;=\; \frac{3\epsilon}{\epsilon+2} \times B_1 \;=\; \frac{3}{\epsilon+2} \times \frac{p}{4\pi\epsilon_0}\,. \tag{88}$$

Altogether, the potential outside the dielectric sphere is

$$V(r,\theta) \;=\; D_1 \times \frac{P_1(\cos\theta)}{r^2} \;=\; \frac{3}{\epsilon+2} \times \frac{p}{4\pi\epsilon_0} \times \frac{\cos\theta}{r^2}\,. \tag{89}$$

It looks like the potential of a pure dipole moment

$$\mathbf{p}^{\text{net}} \;=\; \frac{3}{\epsilon+2} \times \mathbf{p}. \tag{90}$$

Due to bound charges at the sphere's center and also on the sphere's surface, this net dipole moment is different from the free dipole $\mathbf{p}$ at the center.

Finally, inside of the sphere the potential is

$$V(r,\theta) \;=\; \left(A_1\times r\,+\,\frac{B_1}{r^2}\right)\times\cos\theta \;=\; \frac{1}{\epsilon}\times\frac{p}{4\pi\epsilon_0}\times\frac{\cos\theta}{r^2}\;+\;\frac{2(\epsilon-1)}{\epsilon(\epsilon+2)}\times\frac{p}{4\pi\epsilon_0}\times\frac{r\cos\theta}{R^3}\,. \quad (91)$$

This potential is a superposition of a pure dipole $(1/\epsilon)\mathbf{p}$ and a uniform electric field

$$\mathbf{E} \;=\; -\frac{2(\epsilon-1)}{\epsilon(\epsilon+2)}\times\frac{\mathbf{p}}{4\pi\epsilon_0\,R^3}\,. \quad (92)$$

Problem **4.25**:

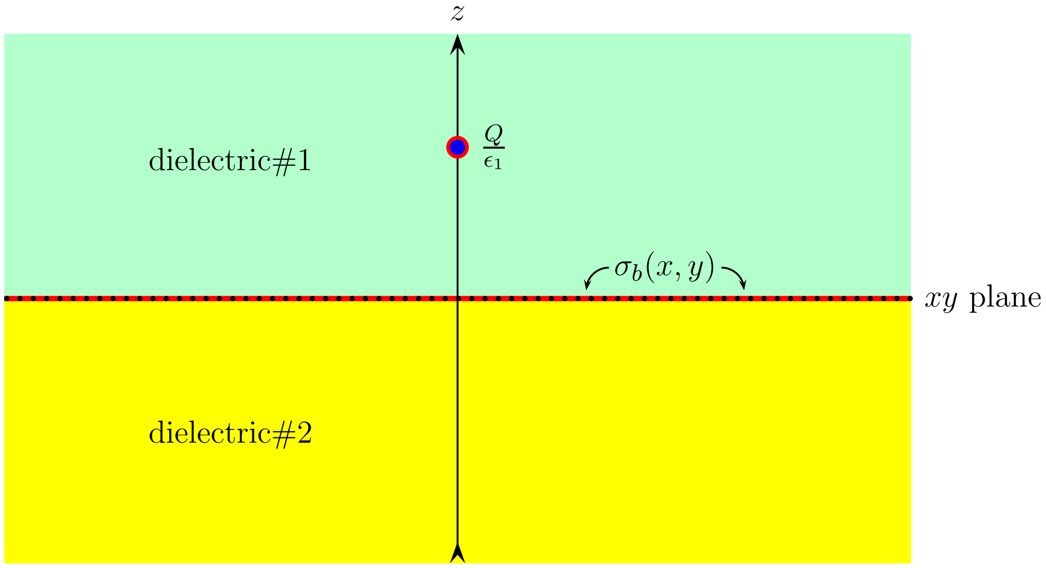
In this problem, the entire 3D space is filled up with two dielectrics: one dielectric with dielectric constant $\epsilon_1 = 1+\chi_1$ fills up the upper half of the space $z>0$, while the other dielectric with a different $\epsilon_2 = 1+\chi_2$ fill up the lower half $z<0$. There is a free point charge Q embedded in the upper dielectric at some point $(x,y,z)=(0,0,+a)$. There are no other *free* charges in the system, but there are bound charges due to dielectrics' polarization. Since both dielectrics are uniform, the bound charges in their middle — *i.e.*, at $z\neq 0$ — follow the free charges there. Specifically, there is a point-like bound charge

$$Q_b \;=\; -\frac{\epsilon_1-1}{\epsilon_1}\times Q \quad (93)$$

immediately surrounding the free charge Q, so the net charge at $(x,y,z)=(0,0,+a)$ is

$$Q^{\rm net} \;=\; Q\;+\;Q^{\rm bound} \;=\; +\frac{1}{\epsilon_1}\times Q. \quad (94)$$

There are no other bound charges in the middle of either dielectric. However, at $z=0$ where the two dielectrics border each other, there is some bound surface charge $\sigma_b(x,y)$. Here is the picture of all the charges in the system



(95)

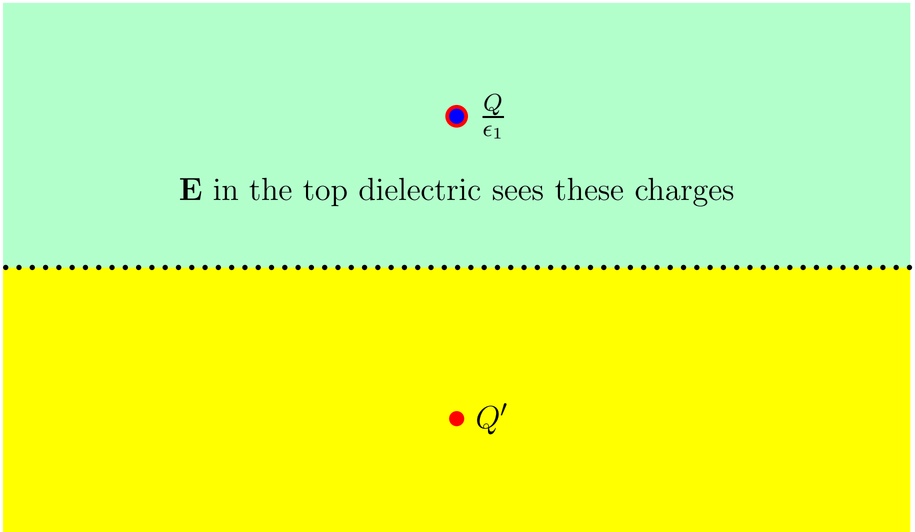
We do not know the surface charge density $\sigma_b(x,y)$ at the interface, so let me guess that it has the same general form as the charge density on the surface of a flat conductor, that is

$$\sigma_b(x,y) \;=\; \frac{Q'}{2\pi}\,\frac{a}{(x^2+y^2+a^2)^{3/2}} \tag{96}$$

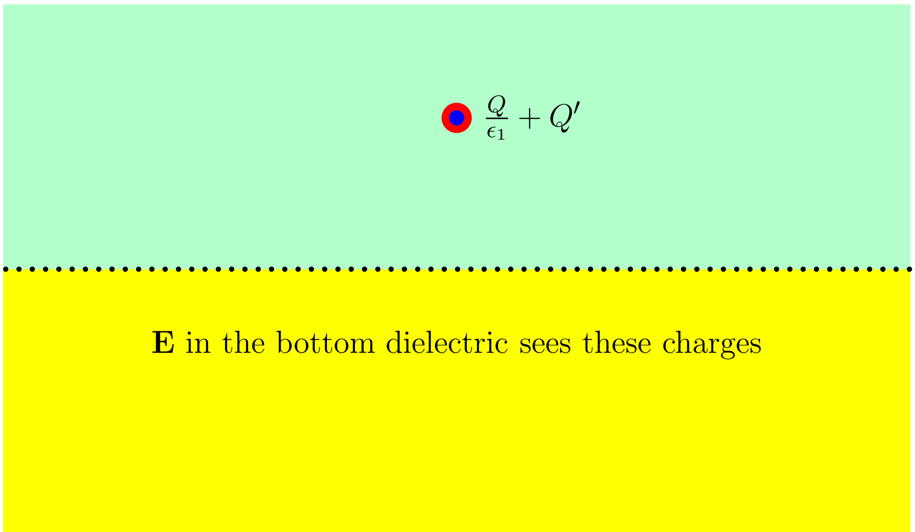
for some net charge Q' that we shall determine later. As we have learned when we studied the mirror image charges (textbook section §3.2), the electric field of the surface charges (96) mimics the electric field of a point charge located at the opposite side of the xy plane from where you measure the $\mathbf{E}$ field,

$$\begin{aligned}
\text{For } z>0,\quad \mathbf{E}[\sigma_b](x,y,z) \;&=\; \mathbf{E}[\text{point charge } Q' \text{ located at } (0,0,-a)](x,y,z),\\
\text{For } z<0,\quad \mathbf{E}[\sigma_b](x,y,z) \;&=\; \mathbf{E}[\text{point charge } Q' \text{ located at } (0,0,+a)](x,y,z),
\end{aligned} \tag{97}$$

Combining this field with the field of the point charge Q/ϵ_1 at $(0,0,+a)$, we get the following picture:



(98)



(99)

Now let's check the boundary conditions at the interface between the two dielectrics:

$$\text{at } z=0, \quad E_{1x} \;=\; E_{2x}\,, \quad E_{1y} \;=\; E_{2y}\,, \quad \text{but} \quad \epsilon_1 \times E_{1z} \;=\; \epsilon_2 \times E_{2z}\,. \tag{100}$$

For the problem at hand, the $\mathbf{E}_1$ and the $\mathbf{E}_2$ at the interface follow from the point charges (98) and (99), thus

at $z = 0$,

$$\begin{aligned}
\mathbf{E}_1 &= \frac{Q/\epsilon_1}{4\pi\epsilon_0}\,\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - a\hat{\mathbf{z}}}{(x^2+y^2+a^2)^{3/2}} \;+\; \frac{Q'}{4\pi\epsilon_0}\,\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + a\hat{\mathbf{z}}}{(x^2+y^2+a^2)^{3/2}} \\
&= \frac{(Q/\epsilon_1) + Q'}{4\pi\epsilon_0}\,\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{(x^2+y^2+a^2)^{3/2}} \;-\; \frac{(Q/\epsilon_1) - Q'}{4\pi\epsilon_0}\,\frac{a\hat{\mathbf{z}}}{(x^2+y^2+a^2)^{3/2}}\,,
\end{aligned} \tag{101}$$

$$\begin{aligned}
\mathbf{E}_2 &= \frac{(Q/\epsilon_1) + Q'}{4\pi\epsilon_0}\,\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - a\hat{\mathbf{z}}}{(x^2+y^2+a^2)^{3/2}} \\
&= \frac{(Q/\epsilon_1) + Q'}{4\pi\epsilon_0}\,\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{(x^2+y^2+a^2)^{3/2}} \;-\; \frac{(Q/\epsilon_1) + Q'}{4\pi\epsilon_0}\,\frac{a\hat{\mathbf{z}}}{(x^2+y^2+a^2)^{3/2}}\,.
\end{aligned} \tag{102}$$

We see that the x and y components of the fields on two sides of the interface match for any Q', but the z components are different. We want the z components to obey the third eq. (100), thus

$$\epsilon_1 \times \left(\frac{Q}{\epsilon_1} - Q'\right) \times \text{common factor} \;=\; \epsilon_2 \times \left(\frac{Q}{\epsilon_1} + Q'\right) \times \text{common factor}. \tag{103}$$

Simplifying the algebra here, we turn this equation into

$$Q \;-\; \frac{\epsilon_2}{\epsilon_1}\,Q \;=\; \epsilon_1\,Q' \;+\; \epsilon_2\,Q' \tag{104}$$

and hence

$$Q' \;=\; \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \times \frac{Q}{\epsilon_1}\,. \tag{105}$$

For this particular value of the Q' charge, all the boundary conditions are met, so the electric fields stemming from the apparent charges (98) and (99) are indeed the correct electric fields for the problem at hand. In terms of the potential $V(x, y, z)$,

In the top dielectric $(z > 0)$,

$$V(x,y,z) \;=\; \frac{Q/\epsilon_1}{4\pi\epsilon_0} \times \frac{1}{\sqrt{x^2+y^2+(z-a)^2}} \;+\; \frac{Q'}{4\pi\epsilon_0} \times \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\,, \tag{106}$$

In the bottom dielectric $(z < 0)$,

$$V(x,y,z) \;=\; \frac{(Q/\epsilon_1)+Q'}{4\pi\epsilon_0} \times \frac{1}{\sqrt{x^2+y^2+(z-a)^2}}\,, \tag{107}$$

where Q' is as in eq. (105) and

$$\frac{Q}{\epsilon_1} \;+\; Q' \;=\; \frac{2Q}{\epsilon_1+\epsilon_2}\,. \tag{108}$$