Problem **4.26**:

Inside the conducting ball — for $r < a$ — there is no electric field. Outside that ball, there are no free charges, hence by spherical symmetry of the problem the displacement field is simply

$$\mathbf{D} = \frac{Q\,\hat{\mathbf{r}}}{4\pi r^2} \quad \text{for any } r > a, \tag{1}$$

both inside the dielectric shell ($a < r < b$) and outside it ($r > b$). The electric tension field follows from this displacement field as

$$\mathbf{E} = \begin{cases} \dfrac{\mathbf{D}}{\epsilon_0} & \text{in air,} \\ \dfrac{\mathbf{D}}{\epsilon\epsilon_0} & \text{in the dielectric,} \end{cases} \tag{2}$$

hence

$$\mathbf{E} = \begin{cases} 0 & \text{for } r < a, \\ \dfrac{Q\,\hat{\mathbf{r}}}{4\pi\epsilon\epsilon_0\, r^2} & \text{for } a < r < b, \\ \dfrac{Q\,\hat{\mathbf{r}}}{4\pi\epsilon_0\, r^2} & \text{for } r > b. \end{cases} \tag{3}$$

At this point, there are two ways to calculate the electrostatic energy of this system. One way is based on the formula

$$U = \frac{1}{2}\iiint \rho_{\text{free}}\, V\, d^3\text{Vol}. \tag{4}$$

Note that this formula involves the free charge density ρ_{free} rather than the net charge density because we only count the work needed to move around the free charges. The bound charges move themselves as we change the electric field, and we do not need to perform any mechanical work to move them.

For the system at hand, the only free charges are on the surface of the metal ball at $r=a$, hence

$$U \;=\; \frac{Q}{2}\times V(a) \tag{5}$$

so all we need to calculate is the potential of that ball (relative to the infinity). Thus,

$$\begin{aligned}
V(a) \;&=\; \int\limits_a^b E_r\,dr \;+\int\limits_b^\infty E_r\,dr \\
&=\; \int\limits_a^b \frac{Q}{4\pi\epsilon\epsilon_0\,r^2}\,dr \;+\int\limits_b^\infty \frac{Q}{4\pi\epsilon_0\,r^2}\,dr \\
&=\; \frac{Q}{4\pi\epsilon\epsilon_0}\times\left(\frac{1}{a}-\frac{1}{b}\right) \;+\; \frac{Q}{4\pi\epsilon_0}\times\frac{1}{b} \\
&=\; \frac{Q}{4\pi\epsilon_0}\times\left(\frac{1}{\epsilon a}+\frac{\epsilon-1}{\epsilon b}\right),
\end{aligned} \tag{6}$$

and consequently

$$U \;=\; \frac{Q^2}{8\pi\epsilon_0}\times\left(\frac{1}{\epsilon a}+\frac{\epsilon-1}{\epsilon b}\right). \tag{7}$$

The other way to calculate the energy is based on the textbook equation (4.58),

$$U \;=\; \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}\cdot\mathbf{D}\,d^3\text{Vol}. \tag{8}$$

For the problem at hand, we have

$$\mathbf{E}\cdot\mathbf{D} \;=\; \frac{Q^2}{16\pi^2\epsilon_0}\times\frac{1}{r^4}\times\begin{cases} 0 & \text{for } r<a,\\ (1/\epsilon) & \text{for } a<r<b,\\ 1 & \text{for } r>b.\end{cases} \tag{9}$$

Consequently,

$$\begin{aligned}
U &= \frac{Q^2}{32\pi^2\epsilon_0}\left(\frac{1}{\epsilon}\int\limits_a^b \frac{1}{r^4}\times 4\pi r^2\,dr\;+\int\limits_b^\infty \frac{1}{r^4}\times 4\pi r^2\,dr\right)\\
&= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{\epsilon}\int\limits_a^b \frac{dr}{r^2}\;+\int\limits_b^\infty \frac{dr}{r^2}\right)\\
&= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{\epsilon}\times\left(\frac{1}{a}-\frac{1}{b}\right)\;+\;\frac{1}{b}\right)\\
&= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{\epsilon a}\;+\;\frac{\epsilon-1}{\epsilon b}\right).
\end{aligned} \tag{10}$$

By inspection of eqs. (7) and (10), both methods yield the same electrostatic energy.

Problem **4.28**:

The two coaxial tubes on textbook figure **4.32** act as plates of a capacitor, which pulls in a dielectric — in this case, the oil from the reservoir below the cylinders. The force of this pull obtains from the textbook equation (4.67),

$$F = \frac{\Delta V^2}{2}\times\frac{dC}{dh} \tag{11}$$

where ΔV is the voltage between the plates while dC/dh is the derivative of the capacitance C with respect to the dielectric oil level h.

To find this derivative, we must first understand a coaxial capacitor which is completely filled with a dielectric. In such a capacitor, the charges $\pm Q$ are uniformly spread out over the two coaxial tubes, hence by the Gauss Law and the cylindrical symmetry, the displacement field between the tubes is

$$\mathbf{D} = \frac{Q/L}{2\pi}\frac{\hat{\mathbf{s}}}{s} \tag{12}$$

where s is the radial coordinate and L is the length of the capacitor tubes. The electric tension field $\mathbf{E}$ follows from this displacement field as

$$\mathbf{E} \;=\; \frac{\mathbf{D}}{\epsilon\epsilon_0} \;=\; \frac{Q/L}{2\pi\epsilon\epsilon_0}\,\frac{\hat{\mathbf{s}}}{s}\,, \tag{13}$$

so integrating this field between the tubes we get the voltage

$$\Delta V \;=\; \int_a^b E_s\,ds \;=\; \frac{Q/L}{2\pi\epsilon\epsilon_0} \times \int_a^b \frac{ds}{s} \;=\; \frac{Q/L}{2\pi\epsilon\epsilon_0} \times \ln\frac{b}{a}\,. \tag{14}$$

Consequently, the capacitance of this capacitor is

$$C_{\text{filled}} \;=\; \frac{Q}{\Delta V} \;=\; \frac{L \times 2\pi\epsilon\epsilon_0}{\ln(b/a)}\,. \tag{15}$$

Next, consider a capacitor of the same geometry but without the dielectric. Treating the air between the tubes as just another dielectric with $\epsilon \approx 1$, we immediately obtain the capacitance

$$C_{\text{empty}} \;=\; \frac{Q}{\Delta V} \;=\; \frac{L \times 2\pi\epsilon_0}{\ln(b/a)}\,. \tag{16}$$

Finally, consider a coaxial capacitor where part of the tube's length L is filled with the dielectric oil while the rest is filled with air. We may treat this system as two parallel capacitors: one has length h and is filled with oil, while the other has length $L-h$ and filled with air, but both have the same radii a and b. consequently,

$$C_1 \;=\; h \times \epsilon \times \frac{2\pi\epsilon_0}{\ln(b/a)}\,, \tag{17}$$

$$C_2 \;=\; (L-h) \times 1 \times \frac{2\pi\epsilon_0}{\ln(b/a)}\,, \tag{18}$$

$$\begin{aligned} C_{\text{net}} \;&=\; C_1 \;+\; C_2 \\ &=\; \Big(L \;+\; (\epsilon - 1) \times h\Big) \times \frac{2\pi\epsilon_0}{\ln(b/a)}\,. \end{aligned} \tag{19}$$

We see that the capacitance grows linearly with the oil-filled length h, specifically

$$\frac{dC}{dh} \;=\; (\epsilon-1)\times\frac{2\pi\epsilon_0}{\ln(b/a)} \;=\; \frac{2\pi\chi\epsilon_0}{\ln(b/a)} \tag{20}$$

where $\chi=\epsilon-1$ is the electric susceptibility of the oil. Consequently, the force pulling the oil into the capacitor is

$$F \;=\; \frac{\Delta V^2}{2}\times\frac{2\pi\chi\epsilon_0}{\ln(b/a)}\,. \tag{21}$$

When the tubes are set up vertically as in figure **4.32**, the electric force (21) pulls the oil up until the weight of the oil column cancels the electric force. Specifically, a column of height h has volume

$$\mathcal{V} \;=\; \pi(b^2-a^2)\times h \tag{22}$$

and hence weight

$$W \;=\; g\rho\times\mathcal{V} \;=\; g\rho\times\pi(b^2-a^2)\times h \tag{23}$$

where ρ is the mass density of the oil. (Please do not confuse it with a charge density, there are no volume charge densities in this problem.) In equilibrium, we must have $W=F$, thus

$$g\rho\times\pi(b^2-a^2)\times h \;=\; \frac{\Delta V^2}{2}\times\frac{2\pi\chi\epsilon_0}{\ln(b/a)}\,, \tag{24}$$

and hence the height of the oil column

$$h \;=\; \frac{\Delta V^2\,\chi\epsilon_0}{g\rho(b^2-a^2)\ln(b/a)}\,. \tag{25}$$

For a numeric example, take the coaxial capacitor with $a=10.0$ mm and $b=11.0$ mm, the transformer oil with density $\rho=882\ \mathrm{kg/m^3}$ and susceptibility $\chi=1.34$, and the voltage

$\Delta V = 12.0$ kV. For this example,

$$\begin{aligned}
\Delta V^2 \times \epsilon_0 \times \chi &= (1.20 \cdot 10^4 \text{ V})^2 \times (8.85 \cdot 10^{-12} \text{ F/m}) \times 1.34 \;=\; 1.71 \cdot 10^{-3} \text{ J/m},\\
g\rho &= 9.80 \text{ N/kg} \times 882 \text{ kg/m}^3 \;=\; 8.64 \cdot 10^3 \text{ N/m}^3,\\
\frac{\Delta V^2 \times \epsilon_0 \chi}{g\rho} &= \frac{1.71 \cdot 10^{-3} \text{ J/m}}{8.64 \cdot 10^3 \text{ N/m}^3} \;=\; 1.98 \cdot 10^{-7} \text{ m}^3 \;=\; 198 \text{ mm}^3,\\
(b^2 - a^2) \times \ln(b/a) &= (11.0^2 - 10.0^2) \text{ mm}^2 \times \ln(1.10/1.00) \;=\; 2.0 \text{ mm}^2,\\
h &= \frac{198 \text{ mm}^3}{2.0 \text{ mm}^2} \;=\; 99 \text{ mm} \;\approx\; 1\underline{0} \text{ cm},
\end{aligned} \tag{26}$$

thus the oil between the tubes would rise to the height of 10 cm, about 4 inches.

Non-textbook problem **I**:

The external electric field induces a dipole moment $\mathbf{p}$ in the dielectric ball, and in a non-uniform but slowly varying electric field there is a net force on that dipole moment, namely

$$\mathbf{F}_{\text{net}} \;=\; (\mathbf{p} \cdot \nabla)\mathbf{E}_{\text{ext}}\,. \tag{27}$$

See my notes on electric dipoles for the derivation of this formula.

In my notes on dielectric boundary problems I have calculated the dipole moment induced in a uniform linear dielectric ball in a uniform external electric field, namely

$$\mathbf{p} \;=\; 4\pi\epsilon_0\, R^3\, \frac{\epsilon - 1}{\epsilon + 2}\, \mathbf{E}_{\text{ext}}\,; \tag{28}$$

for the sake of notational compactness, let's rewrite this formula as

$$\mathbf{p} \;=\; 2C\mathbf{E}_{\text{ext}} \quad \text{where} \quad C \;=\; 2\pi\epsilon_0\, R^3\, \frac{\epsilon - 1}{\epsilon + 2}\,. \tag{29}$$

In the problem at hand, the external electric field is non-uniform, but it's slowly varying, so over the distance scale of a few ball diameters we may neglect its variation and approximate the $\mathbf{E}_{\text{ext}}$ as uniform. Consequently, the induced dipole moment of the ball may be approximated as

$$\mathbf{p} \;\approx\; 2C\mathbf{E}_{\rm ext}(\text{@ the ball's location}). \tag{30}$$

Combining eq. (30) for the induced dipole moment with eq. (27) for the electric force on a small dipole, we immediately obtain

$$\mathbf{F} \;=\; 2C(\mathbf{E}_{\rm ext}\cdot\nabla)\mathbf{E}_{\rm ext}\,. \tag{31}$$

Rewriting this formula as

$$\mathbf{F} \;=\; C\nabla\bigl(\mathbf{E}_{\rm ext}^2\bigr) \tag{32}$$

is a matter of simple vector calculus. Indeed, according to the product formula (5) inside the textbook's front cover, for any two vector fields $\mathbf{A}$ and $\mathbf{B}$, the gradient of their dot product obtains as

$$\nabla(\mathbf{A}\cdot\mathbf{B}) \;=\; \mathbf{A}\times(\nabla\times\mathbf{B}) \;+\; \mathbf{B}\times(\nabla\times\mathbf{A}) \;+\; (\mathbf{A}\cdot\nabla)\mathbf{B} \;+\; (\mathbf{B}\cdot\nabla)\mathbf{A}. \tag{33}$$

In particular, for $\mathbf{A}=\mathbf{B}=\mathbf{E}_{\rm ext}$ we have

$$\nabla\bigl(\mathbf{E}_{\rm ext}^2\bigr) \;=\; 2\mathbf{E}_{\rm ext}\times\bigl(\nabla\times\mathbf{E}_{\rm ext}\bigr) \;+\; 2\bigl(\mathbf{E}_{\rm ext}\cdot\nabla\bigr)\mathbf{E}_{\rm ext}\,. \tag{34}$$

Moreover, $\mathbf{E}_{\rm ext}$ is an electrostatic field, so its curl is zero, $\nabla\times\mathbf{E}_{\rm ext}=0$; this kills the first term on the above RHS and leaves us with

$$\nabla\bigl(\mathbf{E}_{\rm ext}^2\bigr) \;=\; 2\bigl(\mathbf{E}_{\rm ext}\cdot\nabla\bigr)\mathbf{E}_{\rm ext}\,. \tag{35}$$

And that's why eq. (31) for the force on the induced dipole moment is mathematically equivalent to eq. (32).

Non-textbook problem **II**:

(a) The Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ does not do any mechanical work on a moving charged particle, so the magnetic field does not contribute to its potential energy. On the other hand, the electric force $\mathbf{F} = q\mathbf{E}$ does perform mechanical work $W = q\Delta V$, so it leads to the potential energy $U_{\text{pot}} = qV(x, y, z)$. Altogether, the net energy of a non-relativistic charged particle is

$$U_{\text{net}} = U_{\text{kin}} + U_{\text{pot}} = \tfrac{1}{2}m\mathbf{v}^2 + qV(x, y, z). \tag{36}$$

(b) Now consider the specific example of the cycloid motion from the textbook example **5.2**: Uniform electric and magnetic fields $\perp$ to each other, and a particle starting from rest. Following the textbook figure **5.7**, we take $\mathbf{E} = E\,\hat{\mathbf{z}}$ while $\mathbf{B} = B\,\hat{\mathbf{x}}$, while the particle moves in the yz plane according to

$$\begin{aligned} y(t) &= R \times \big(\omega t - \sin(\omega t)\big), \\ z(t) &= R \times \big(1 - \cos(\omega t)\big), \\ v_y(t) &= R\omega \times \big(1 - \cos(\omega t)\big), \\ v_z(t) &= R\omega \times \sin(\omega t), \end{aligned} \tag{37}$$

where

$$\omega = \frac{qB}{m} \quad \text{and} \quad R = \frac{mE}{qB^2} \quad \Longrightarrow \quad R\omega = \frac{E}{B}\,. \tag{38}$$

If you mark a point on the rim of a wheel of radius R and let it roll without slipping on the y axis at speed ωR, then the motion of the marked point would be precisely as in eqs. (37) for the charged particle in question.

The kinetic energy of the motion (37) is

$$\begin{aligned} U_{\text{kin}} &= \tfrac{1}{2}m\big(v_y^2 + v_z^2\big) = \tfrac{1}{2}m\Big(R^2\omega^2\big(1 - \cos(\omega t)\big)^2 + R^2\omega^2\sin^2(\omega t)\Big) \\ &= \tfrac{1}{2}m\,R^2\omega^2 \times \big(1 - 2\cos(\omega t) + \cos^2(\omega t) + \sin^2(\omega t) = 2 - 2\cos(\omega t)\big) \\ &= \frac{mE^2}{B^2} \times \big(1 - \cos(\omega t)\big). \end{aligned} \tag{39}$$

At the same time, the potential energy follows from the electric potential

$$V(x,y,z) \;=\; -E\times z, \tag{40}$$

hence

$$U_{\text{pot}} \;=\; -Eq\times z(t) \;=\; -Eq\times R\times\big(1\,-\,\cos(\omega t)\big). \tag{41}$$

where

$$Eq\times R \;=\; \frac{mE^2}{B^2}\,. \tag{42}$$

Altogether,

$$\begin{aligned}
U_{\text{pot}}(t) \;&=\; -\frac{ME^2}{B^2}\times\big(1\,-\,\cos(\omega t)\big),\\
U_{\text{kin}}(t) \;&=\; +\frac{ME^2}{B^2}\times\big(1\,-\,\cos(\omega t)\big),\\
U_{\text{net}}(t) \;&=\; U_{\text{pot}}(t)\;+\;U_{\text{kin}}(t)\\
&=\; 0 \quad \text{at all times } t,
\end{aligned} \tag{43}$$

so the net energy of the moving particle is indeed conserved.

Problem **5.3**:
(a) The net force on an electron in a beam is

$$\mathbf{F} \;=\; q_e\Big(\mathbf{E}\,+\,\mathbf{v}\times\mathbf{B}\Big). \tag{44}$$

In J. J. Thomson's apparatus, the **E** and the **B** fields were both $\perp$ to the electron beam and also $\perp$ to each other, so without loss of generality let us assume

$$\mathbf{E} \;=\; E\,\hat{\mathbf{z}}\,, \quad \mathbf{B} \;=\; B\,\hat{\mathbf{x}}\,, \quad \mathbf{v} \;=\; v\,\hat{\mathbf{y}}, \tag{45}$$

and consequently

$$\mathbf{F} \;=\; q_e\Big(E\,-\,v\times B\Big)\hat{\mathbf{z}}. \tag{46}$$

Thus, for generic values of the electric and magnetic fields, the electron beam is deflected up or down by the combined electric and magnetic forces. However, these two forces act in opposite directions, so for

$$E = v \times B \tag{47}$$

these forces cancel each other and there is no deflection of the beam. In Thomson's experiment, the electric and magnetic fields were tuned to achieve such cancellation, and from their values one could easily obtain the electron beam's velocity as

$$v = \frac{E}{B}. \tag{48}$$

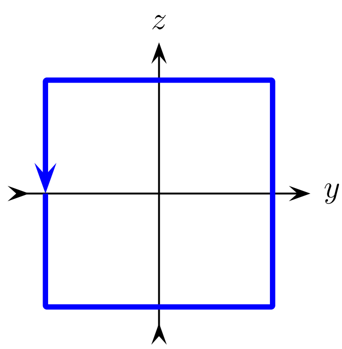
(b) Once the electric field is turned off, the un-canceled magnetic force $\mathbf{F} = (-e)\mathbf{v} \times \mathbf{B}$ makes the electrons move in a circle of radius

$$R = \frac{v^2}{F/m_e} = \frac{m_e v}{eB} \tag{49}$$

Given the magnetic field B and the electrons' velocity v found in part (a), the electron's charge/mass ratio obtains from the above radius as

$$\frac{e}{m_e} = \frac{RB}{v},. \tag{50}$$

<u>Problem **5.4**</u>:



(51)

Let's calculate the magnetic force on each segment of this loop. The top segment of the loop is located at constant $z = +(a/2)$, so the magnetic field over the top segment is uniform

$\mathbf{B}^{\rm top} = k(a/2)\hat{\mathbf{x}}$. The current through the top segment flows left — *i.e.*, in the $-\hat{\mathbf{y}}$ direction, — so the net force on the top segment is

$$\mathbf{F}^{\rm top} \;=\; I\mathbf{L}^{\rm top}\times\mathbf{B}^{\rm top} \;=\; \bigl(-Ia\,\hat{\mathbf{y}}\bigr)\times\bigl(+k(a/2)\,\hat{\mathbf{x}}\bigr) \;=\; -\tfrac{1}{2}kIa^2\,\bigl(\hat{\mathbf{y}}\times\hat{\mathbf{x}}\,=\,-\hat{\mathbf{z}}\bigr) \;=\; +\tfrac{1}{2}kIa^2\,\hat{\mathbf{z}}. \tag{52}$$

Likewise, the bottom segment is located at $z=-(a/2)$ where the magnetic field is uniform $\mathbf{B}^{\rm bot} = -k(a/2)\hat{\mathbf{x}}$ — which is opposite from the magnetic field over the top segment. But the current in the bottom segment also flows in the direction opposite from the top segment — to the right, in the $+\hat{\mathbf{y}}$ direction, — so the net force on the bottom segment is the same as on the top segment. Indeed,

$$\mathbf{F}^{\rm bot} \;=\; I\mathbf{L}^{\rm bot}\times\mathbf{B}^{\rm bot} \;=\; \bigl(+Ia\,\hat{\mathbf{y}}\bigr)\times\bigl(-k(a/2)\,\hat{\mathbf{x}}\bigr) \;=\; -\tfrac{1}{2}kIa^2\,\bigl(\hat{\mathbf{y}}\times\hat{\mathbf{x}}\,=\,-\hat{\mathbf{z}}\bigr) \;=\; +\tfrac{1}{2}kIa^2\,\hat{\mathbf{z}}. \tag{53}$$

Now consider the right segment of the loop. The current flows through this segment in the $+\hat{\mathbf{z}}$ direction while the magnetic field points along the x axis, but the value of B is not uniform: Instead, $B_x = kz$ for z ranging from $-a/2$ to $+a/2$. Consequently, the net force on the right segment is

$$\begin{aligned}
\mathbf{F}^{\rm right} &= \int\bigl(I\,d\mathbf{L}\bigr)\times\mathbf{B} \;=\; \int_{-a/2}^{+a/2}\bigl(I\,\hat{\mathbf{y}}\,dz\bigr)\times\bigl(kz\,\hat{\mathbf{x}}\bigr)\\
&= Ik\,\bigl(\hat{\mathbf{y}}\times\hat{\mathbf{x}}\bigr)\int_{-a/2}^{+a/2} z\,dz\\
&= 0 \quad \langle\!\langle\,\text{because the integral vanishes.}\,\rangle\!\rangle
\end{aligned} \tag{54}$$

Similarly, the net force on the left segment of the loop vanishes for the same reason.

Altogether, the net force on the loop is

$$\mathbf{F}^{\rm net} \;=\; \mathbf{F}^{\rm top} \,+\, \mathbf{F}^{\rm bot} \,+\, \mathbf{F}^{\rm right} \,+\, \mathbf{F}^{\rm left} \;=\; +\tfrac{1}{2}kIa^2\,\hat{\mathbf{z}} \,+\, \tfrac{1}{2}kIa^2\,\hat{\mathbf{z}} \,+\, 0 \,+\, 0 \;=\; kIa^2\,\hat{\mathbf{z}}. \tag{55}$$

Problem **5.10**:

Let me use the coordinate system where the straight wire at the bottom of figure 5.24(a) or 5.24(b) runs along the x axis while the square or the triangular loop lays within the xy plane. In that plane, on the $y>0$ side of the wire, its magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{z}}}{y}. \qquad (56)$$

This magnetic field exerts forces on the segments of the current-carrying loop, and our task is to calculate the net force on the loop,

$$\mathbf{F}^{\text{net}} = \oint\limits_{\text{loop}} (I\,d\mathbf{L})\times\mathbf{B}. \qquad (57)$$

(a) Let's start with the left side of the square loop. The current through this side flows in $+\hat{\mathbf{y}}$ direction, hence

$$\big(I\,d\mathbf{L}\big)\times\mathbf{B} = \big(+I\,dy\,\hat{\mathbf{y}}\big)\times\left(\frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{z}}}{y}\right) = \frac{\mu_0 I^2}{2\pi}\frac{dy}{y}\big(\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}\big), \qquad (58)$$

so the net force on the left side is

$$\mathbf{F}^{\text{left}} = \frac{\mu_0 I^2}{2\pi}\times\hat{\mathbf{x}}\times\int\limits_s^{s+a}\frac{dy}{y} = \frac{\mu_0 I^2}{2\pi}\times\hat{\mathbf{x}}\times\ln\frac{s+a}{s}. \qquad (59)$$

Likewise, for the right side of the loop, the current flows in the $-\hat{\mathbf{y}}$ direction, hence

$$\big(I\,d\mathbf{L}\big)\times\mathbf{B} = \big(-I\,dy\,\hat{\mathbf{y}}\big)\times\left(\frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{z}}}{y}\right) = \frac{\mu_0 I^2}{2\pi}\frac{dy}{y}\big(-\hat{\mathbf{y}}\times\hat{\mathbf{z}} = -\hat{\mathbf{x}}\big), \qquad (60)$$

and therefore

$$\mathbf{F}^{\text{right}} = \frac{\mu_0 I^2}{2\pi}\times(-\hat{\mathbf{x}})\times\int\limits_s^{s+a}\frac{dy}{y} = \frac{\mu_0 I^2}{2\pi}\times(-\hat{\mathbf{x}})\times\ln\frac{s+a}{s} = -\mathbf{F}^{\text{left}}. \qquad (61)$$

Thus, the forces on the left and right side of the square loop cancel each other.

Now consider the horizontal top and bottom sides. For the top side, the current flows in the $+\hat{\mathbf{x}}$ direction while the magnetic field is uniform

$$\mathbf{B}^{\rm top} = \frac{\mu_0 I}{2\pi}\,\frac{\hat{\mathbf{z}}}{s+a}\,. \tag{62}$$

Consequently, the net force on the top side is simply

$$\mathbf{F}^{\rm top} = \bigl(+Ia\,\hat{\mathbf{x}}\bigr)\times\left(\frac{\mu_0 I}{2\pi}\,\frac{\hat{\mathbf{z}}}{s+a}\right) = \frac{\mu_0 I^2}{2\pi}\times\frac{a}{s+a}\times(\hat{\mathbf{x}}\times\hat{\mathbf{z}}) = \frac{\mu_0 I^2}{2\pi}\times\frac{a}{s+a}\times(-\hat{\mathbf{y}}). \tag{63}$$

Likewise, in the bottom side the current flows in the $-\hat{\mathbf{x}}$ direction while the magnetic field is uniform

$$\mathbf{B}^{\rm bot} = \frac{\mu_0 I}{2\pi}\,\frac{\hat{\mathbf{z}}}{s}\,. \tag{64}$$

Consequently, the net force on the top side is

$$\mathbf{F}^{\rm bot} = \bigl(-Ia\,\hat{\mathbf{x}}\bigr)\times\left(\frac{\mu_0 I}{2\pi}\,\frac{\hat{\mathbf{z}}}{s}\right) = \frac{\mu_0 I^2}{2\pi}\times\frac{a}{s}\times(-\hat{\mathbf{x}}\times\hat{\mathbf{z}}) = \frac{\mu_0 I^2}{2\pi}\times\frac{a}{s}\times(+\hat{\mathbf{y}}). \tag{65}$$

Altogether, the net force on the square loop is

$$\begin{aligned}\mathbf{F}^{\rm net} &= \mathbf{F}^{\rm bot} + \mathbf{F}^{\rm top} + \cancel{\mathbf{F}^{\rm left}} + \cancel{\mathbf{F}^{\rm right}}\\ &= \frac{\mu_0 I^2}{2\pi}\times\left(\frac{a}{s} - \frac{a}{s+a}\right)\times\hat{\mathbf{y}} = \frac{\mu_0 I^2}{2\pi}\times\frac{a^2}{s(s+a)}\times\hat{\mathbf{y}}.\end{aligned} \tag{66}$$

Note the $+\hat{\mathbf{y}}$ direction of this net force — away from the wire.

(b) Now consider the triangular loop on figure 4.24(b). The bottom side of this loop is similar to the bottom of the square loop — hence a similar formula for the net force on the bottom,

$$\mathbf{F}^{\rm bot} = \frac{\mu_0 I^2}{2\pi}\times\frac{a}{s}\times(+\hat{\mathbf{y}}), \tag{67}$$

but the other two sides are more complicated. Let's start with the left side of the triangle, where the current flows in the direction

$$\cos(60^\circ)\,\hat{\mathbf{x}}\,+\,\sin(60^\circ)\,\hat{\mathbf{y}}\,. \tag{68}$$

In other words,

$$I d\mathbf{L} \;=\; I\,dL\Big(\cos(60^\circ)\,\hat{\mathbf{x}}\,+\,\sin(60^\circ)\,\hat{\mathbf{y}}\Big) \;=\; I\,dy\left(\hat{\mathbf{y}}\,+\,\frac{\hat{\mathbf{x}}}{\tan(60^\circ)=\sqrt{3}}\right),$$

and hence

$$\big(I\,d\mathbf{L}\big)\times\mathbf{B} \;=\; I\,dy\left(\hat{\mathbf{y}}\,+\,\frac{\hat{\mathbf{x}}}{\sqrt{3}}\right)\times\frac{\mu_0 I}{2\pi}\,\frac{\hat{\mathbf{z}}}{y} \;=\; \frac{\mu_0 I^2}{2\pi}\,\frac{dy}{y}\left(\hat{\mathbf{x}}\,-\,\frac{\hat{\mathbf{y}}}{\sqrt{3}}\right). \tag{69}$$

It remains to integrate this expression from $Y_{\min}=s$ to $y_{\max}=s+\frac{\sqrt{3}}{2}a$:

$$\int\limits_{y_{\min}}^{y_{\max}}\frac{dy}{y} \;=\; \ln\frac{y_{\max}}{y_{\min}} \;=\; \ln\frac{s+\frac{\sqrt{3}}{2}a}{s}\,, \tag{70}$$

hence

$$\mathbf{F}^{\text{left}} \;=\; \frac{\mu_0 I^2}{2\pi}\times\ln\frac{s+\frac{\sqrt{3}}{2}a}{s}\times\left(\hat{\mathbf{x}}\,-\,\frac{\hat{\mathbf{y}}}{\sqrt{3}}\right). \tag{71}$$

As to the right side of the triangle, the current there flows in the direction

$$\cos(60^\circ)\,\hat{\mathbf{x}}\,-\,\sin(60^\circ)\,\hat{\mathbf{y}}\,, \tag{72}$$

hence

$$\big(I\,d\mathbf{L}\big)\times\mathbf{B} \;=\; I\,dy\left(-\hat{\mathbf{y}}\,+\,\frac{\hat{\mathbf{x}}}{\sqrt{3}}\right)\times\frac{\mu_0 I}{2\pi}\,\frac{\hat{\mathbf{z}}}{y} \;=\; \frac{\mu_0 I^2}{2\pi}\,\frac{dy}{y}\left(-\hat{\mathbf{x}}\,-\,\frac{\hat{\mathbf{y}}}{\sqrt{3}}\right), \tag{73}$$

and therefore the net force on the right side

$$\mathbf{F}^{\text{left}} \;=\; \frac{\mu_0 I^2}{2\pi}\times\ln\frac{s+\frac{\sqrt{3}}{2}a}{s}\times\left(-\hat{\mathbf{x}}\,-\,\frac{\hat{\mathbf{y}}}{\sqrt{3}}\right). \tag{74}$$

Combining the two slanted sides of the triangle, we get

$$\mathbf{F}^{\text{left}} + \mathbf{F}^{\text{right}} = \frac{\mu_0 I^2}{2\pi} \times \ln\frac{s + \sqrt{\frac{3}{4}}a}{s} \times \left(0\,\hat{\mathbf{x}} - 2\,\frac{\hat{\mathbf{y}}}{\sqrt{3}}\right), \tag{75}$$

hence adding the force on the bottom side of the triangle we obtain the net force

$$\mathbf{F}^{\text{net}} = \frac{\mu_0 I^2}{2\pi}\left(\frac{a}{s} - \frac{2}{\sqrt{3}}\ln\frac{s + \frac{\sqrt{3}}{2}a}{s}\right)\hat{\mathbf{y}}. \tag{76}$$

Note: The expression in $(\cdots)$ here is always positive for any positive a and s, so the net force on the loop is away from the wire.