Problem **5.58**:

(a) It does not matter if the rotating object on figure 5.65 is a donut, or a ring, or any other kind of a toroid, as long as it's thin compared to its length $\ell = 2\pi R$, and both the charge and the mass are uniformly distributed along the length.

For the rotation around the symmetry axis z as shown on the figure 5.65, the moment of inertial of the thin toroid is

$$\mathcal{I} = MR^2, \tag{1}$$

so its angular momentum is

$$L = \mathcal{I} \times \omega = MR^2 \times \omega \tag{2}$$

At the same time, rotating charged toroid carries electric current

$$I = \frac{Q_{\text{net}}}{\text{rotating period}} = \frac{Q}{2\pi/\omega} = \frac{Q \times \omega}{2\pi}, \tag{3}$$

and this current loop encloses area $A = \pi R^2$, hence magnetic dipole moment

$$m = IA = \frac{Q\omega}{2\pi} \times \pi R^2 = \frac{1}{2} QR^2 \times \omega \tag{4}$$

Comparing formulae. (2) for the angular momentum and (4) for the magnetic moment, we see that their ratio

$$g \stackrel{\text{def}}{=} \frac{m}{L} = \frac{Q}{2M} \tag{5}$$

does not depend on the toroid's radius or rotation frequency but only on the charge/mass ratio.

(b) Now consider a rotating solid ball. Or a rotating spherical shell, a rotating cylinder, a rotating cone, or any other rotationally-symmetric body rotating around its symmetry axis. Splitting the cross-section of such a body into infinitesimal squares, we decompose the whole body into a sum of thin rings — exactly as discussed in part (a). Moreover, of the whole body has uniform mass density and uniform charge density, — or at least the ration of the densities is uniform,

$$\frac{dQ/d\mathrm{Vol}}{dM/d\mathrm{Vol}} = \text{const}, \tag{6}$$

then every little rotating ring has the same gyromagnetic ratio

$$g = \frac{Q_{\text{ring}}}{2M_{\text{ring}}} = \frac{Q_{\text{body}}}{2M_{\text{body}}}. \tag{7}$$

Thus, for every little ring $m_{\text{ring}} = g \times L_{\text{ring}}$ for the same g, and therefore

$$m_{\text{body}} = \sum_{\text{rings}} m_{\text{ring}} = \sum_{\text{rings}} g \times L_{\text{ring}} = g \times \sum_{\text{rings}} L_{\text{ring}} = g \times L_{\text{body}}. \tag{8}$$

In other words, the whole rotating body has the same gyromagnetic ratio of the magnetic moment to angular momentum as the ring in part (a),

$$\frac{m}{L} = g = \frac{Q}{2M} \tag{9}$$

(c) In quantum mechanics, any angular momentum has an integer of a half-integer "magnitude" in units of Planck constant $\hbar$. For the electron in an atom, there are two sources of angular momentum — the orbital angular momentum $\mathbf{L} = \mathbf{r} \times m_e \mathbf{v}$ due to electron's motion around the atom's nucleus, and its *spin* $\mathbf{S}$ around its own center of mass. The orbital angular momentum has “magnitudes”[⋆]

$$|\mathbf{L}| \;=\; \hbar \times \text{an integer} \;=\; 0, \hbar, 2\hbar, 3\hbar, \ldots \tag{11}$$

while the electron's motion around the nucleus gives rise to the magnetic moment

$$\mathbf{m}_{\text{orbital}} \;=\; \frac{-e}{2m_e}\,\mathbf{L}, \tag{12}$$

with the gyromagnetic ratio precisely as in the classical eq. (5). The “magnitude” of the orbital magnetic moment is integer in units of the *Bohr magneton*

$$m_B \;=\; \frac{e\hbar}{2m_e} \quad \langle\!\langle\,\text{in MKSA units}\,\rangle\!\rangle \;=\; 9.1 \cdot 10^{-24}\ \text{A}\cdot\text{m}^2. \tag{13}$$

The electron's spin $\mathbf{S}$ does not have a classical explanation, it is purely quantum in origin,[†] and the spin's “magnitude” is completely fixed to the $\frac{1}{2}$ of the unit of orbital angular momentum, $|\mathbf{S}| = \frac{1}{2}\hbar$. Also, the spin does lead to a motion-independent magnetic moment of the electron, but the gyromagnetic ratio for the spin is twice as large as the classical ratio (5),

$$\mathbf{m}_{\text{spin}} \;=\; 2\frac{-e}{2m_e}\,\mathbf{S}\,. \tag{14}$$

Consequently, the “magnitude” of this intrinsic magnetic moment of the electron is

$$|\mathbf{m}_{\text{spin}}| \;=\; \frac{e\hbar}{2m_e} \;=\; m_B \;=\; 9.1 \cdot 10^{-24}\ \text{A}\cdot\text{m}^2 \tag{15}$$

instead of naively expected $|\mathbf{m}| = \frac{1}{2}m_B = \ 4.5 \cdot 10^{-24}\ \text{A}\cdot\text{m}^2$.

⋆ I say “magnitudes” rather than magnitudes because in quantum mechanics the magnitude of a vector whose components cannot be measured at the same time is ill-defined. The precise statement about the orbital angular momentum operator $\hat{\mathbf{L}}$ is that its square $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ has discrete spectrum of allowed values of the form

$$\hat{\mathbf{L}}^2 \;=\; \hbar^2 \times \ell(\ell+1) \quad \text{for integer } \ell = 0, 1, 2, 3, \ldots. \tag{10}$$

For the students who have not yet taken a quantum mechanics class, I an simplifying this quantum formula to “magnitude” $|\mathbf{L}| = \hbar \times$ an integer.

† The naive classical explanation in terms of a finite-size electron spinning on its axis require rotation speeds much faster than the speed of light, so this cannot be right!

The best explanation for the gyromagnetic moment for the electron's spin comes from the quantum field theory, in particular from the Quantum ElectroDynamics (QED). It turns out that

$$\frac{g_{\rm spin}}{\text{naive }(e/2m)} \;=\; 2 \tag{16}$$

is only a leading approximation. The leading correction to this ratio was calculated in 1948 by Schwinger, Feynman, and Tomonaga, and by now the ratio (16) has been both calculated theoretically and measured experimentally to the incredible precision of 13 significant figures,

$$\frac{g_{\rm spin}}{\text{naive }(e/2m)} \;=\; 2.002\,319\,304\,361\,5\,(5). \tag{17}$$

Problem **6.8**:

Since the magnetization of the cylinder is not uniform, $\mathbf{M}(\mathbf{r}) \neq$ const, we have bound currents not only on the cylinder's surface but also through its volume. Specifically, in cylindrical coordinates,

$$\begin{aligned}
\mathbf{M}(s,\phi,z) \;&=\; M_\phi(s)\,\hat{\phi} \;=\; ks^2\,\hat{\phi}\,, && (18)\\
\mathbf{J}_b(s,\phi,z) \;&=\; \nabla\times\mathbf{M} \;=\; \frac{1}{s}\,\frac{\partial(sM_\phi)}{\partial s}\,\hat{\mathbf{z}} \;=\; 3ks\,\hat{\mathbf{z}}\,, && (19)\\
\mathbf{K}(s=R,\phi,z) \;&=\; \mathbf{M}\times\mathbf{n} \;=\; (kR^2\,\hat{\phi})\times\hat{\mathbf{s}} \;=\; -kR^2\,\hat{\mathbf{z}}\,. && (20)
\end{aligned}$$

Note: both the volume current $\mathbf{J}_b$ and the surface current $\mathbf{K}_b$ flow along the cylinder's axis z, but in the opposite directions! In fact, the net bound current through the cylinder is zero:

$$\begin{aligned}
I_{\rm net} \;&=\; \iint\limits_{\text{crosssection}} J^z_b\,d^2A \;+\; \int\limits_{\text{perimeter}} K^z_b\,d\ell\\
&=\; \int\limits_0^R (3ks)\times 2\pi s\,ds \;-\; (kR^2)\times 2\pi R \qquad (21)\\
&=\; 2\pi kR^3 \;-\; 2\pi kR^3 \;=\; 0.
\end{aligned}$$

The magnetic field of these volume and surface bound currents follow from the symmetries and the Ampere's Law. The symmetries of the magnetized cylinder and the bound currents are similar to those of a long straight wire with a round cross-section, thus

$$\mathbf{B}(s,\phi,z) \;=\; B(s\text{ only})\,\hat{\phi}. \tag{22}$$

Hence, by the Ampere's Law for a circular loop centered on the z axis,

$$B(s) \;=\; \frac{\mu_0}{2\pi s}\times I[\,\text{through the loop of radius } s\,]. \tag{23}$$

And since the problem in question has no conduction currents, juts the bound currents due to magnetization, eq. (23) amounts to

$$B(s) \;=\; \frac{\mu_0}{2\pi s}\times I_b[\,\text{through the loop of radius } s\,]. \tag{24}$$

In particular, for $s > R$ the Ampere loop surrounds the whole cylinder and hence the net bound current — volume and surface. As we saw in eq. (21), this net bound current is zero, so *the magnetic field outside the magnetized cylinder is zero.*

On the other hand, for $s < R$ the loop runs inside the cylinder, so it surrounds part of the volime current but none of the surface current. Consequently,

$$I[\,\text{through the loop of radius } s\,] \;=\; \int\limits_0^s \bigl(J_b^z(s')=3ks'\bigr)\times 2\pi s'\,ds' \;=\; 2\pi ks^3, \tag{25}$$

and therefore

$$B(s) \;=\; \frac{\mu_0}{2\pi s}\times 2\pi ks^3 \;=\; \mu_0 ks^2. \tag{26}$$

Or in vector notations,

$$\mathbf{B}^{\text{inside}} \;=\; \mu_0 ks^2\,\hat{\phi} \;=\; \mu_0\mathbf{M}. \tag{27}$$

Comment:

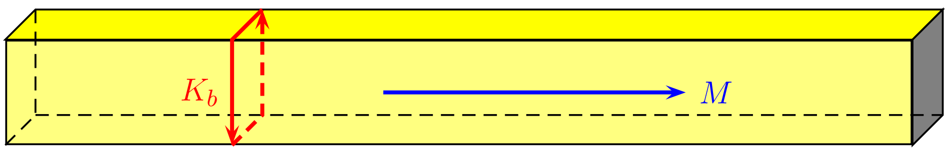
In this problem we found $\mathbf{B} = \mu_0 \mathbf{M}$, both inside and outside the cylinder. In terms of the magnetic intensity field $\mathbf{H}$, this means $\mathbf{H} \equiv 0$ everywhere, which agrees with the absence of any *free* currents anywhere in the system.

However, the absence of free currents does not guarantee that $\mathbf{H} \equiv 0$. Indeed, any permanent magnet has only bound currents but no free currents anywhere, yet it does create both $\mathbf{B} \neq 0$ and $\mathbf{H} \neq 0$. In particular, outside the magnet itself, $\mathbf{H} = \mu_0^{-1} \mathbf{B} \neq 0$.

In general, in the absence of any free currents but in presence of magnetized materials, we have $\nabla \times \mathbf{H} = 0$ but $\mathbf{H} \neq 0$. For example, see the magnetized ball example in my notes on magnetization and the **H** field. To make the **H** field vanish — just as it did in problem 6.8 — we need symmetries which make both the magnetization **M** and the **B** field parallel to to the outer surface of the magnet. (Or to any other interface between different magnetic materials, if any.) Without such symmetries, we would have $\mathbf{H} \neq 0$ and hence $\mathbf{B} \neq \mu_0 \mathbf{M}$.

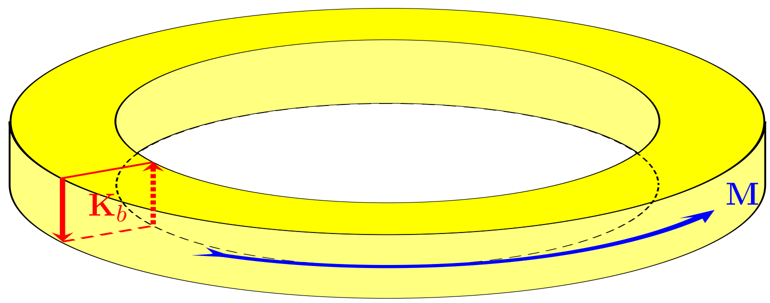
Problem **6.10**:

As a warm-up exercise, consider the uniformly magnetized iron rod bent into a complete toroid, without the gap. Uniform magnetisation means no volume bound current, but there is a surface bound current of density $\mathbf{K}_b = \mathbf{M} \times \mathbf{n}$. For the original (un-bent) rod, the magnetization **M** was in the long direction of the rod, so the bound current of magnitude $K_b = M$ went around the rod,



(28)

For the rod bent into the toroid shape, the magnetization **M** points in the toroidal direction while the surface current flows in the poloidal direction:



Such poloidal surface current is similar to the conduction current in the toroidal coil, so we get a similar magnetic field:

$$\text{inside the toroid } \mathbf{B} = \mu_0 \mathbf{n} \times \mathbf{K} = \mu_0 \mathbf{M}, \tag{29}$$

$$\text{outside the toroid } \mathbf{B} = 0. \tag{30}$$

Now let's make a small gap in the iron toroid. The two sides of the gap are $\perp$ to the magnetization $\mathbf{M}$ in the iron, so the normal vectors to those sides are parallel to the $\mathbf{M}$ and hence $\mathbf{M} \times \mathbf{n} = 0$. Consequently, there are no bound surface currents along either side of the gap.

On the other hand, the gap disturbs the poloidal bound current on the outer surface of the toroid: This current flows around the rod's perimeter along its whole length, except in the gap. Thus, we may view the gapped toroid as a superposition of a gapless iron toroid, and a narrow square ring around the gap which carries the counter-current $\mathbf{K}' = -\mathbf{K}_b$ in the opposite direction from bound current around the toroid. The net counter-current around the square loop is

$$I_{\text{counter}} = K_b \times w = M \times w, \tag{31}$$

and the magnetic field of this counter-current inside the gap is of the order

$$B[I_{\rm counter}] \;\sim\; \frac{\mu_0 I_{\rm square}}{a} \;\sim\; \mu_0 M \times \frac{w}{a}\,. \tag{32}$$

Note: this field is not uniform, but this is its typical magnitude (except very close to the outer surface of the gap), and in the $w \ll a$ limit this magnitude is negligible compared to the toroid's field $\mathbf{B} = \mu_0 \mathbf{M}$.

The bottom line is, in the $w \ll a \ll L$ limit, the magnetic field inside the gap is approximately the same as inside the magnetized iron, namely $\mathbf{B} = \mu_0 \mathbf{M}$.

<u>Problem **6.12**</u>:

(a) The magnetization $\mathbf{M}$ is parallel to the cylinder's axis but its magnitude depends on the radius s, so we have both volume and surface bound currents. In cylindrical coordinates

$$\begin{aligned} \mathbf{M}(s,\phi,z) \;&=\; ks\,\hat{\mathbf{z}}, & (33)\\ \mathbf{J}_b(s,\phi,z) \;&=\; \nabla\times\mathbf{M} \;=\; -\frac{\partial M_z}{\partial s}\,\hat{\phi} \;=\; -k\hat{\phi}, & (34)\\ \mathbf{K}(s=R,\phi,z) \;&=\; \mathbf{M}\times\mathbf{n} \;=\; kR\,\hat{\mathbf{z}}\times\hat{\mathbf{s}} \;=\; +\,kR\,\hat{\phi}. & (35) \end{aligned}$$

Note that the volume and the surface bound currents flow in opposite circular directions $\mp\hat{\phi}$: $\mathbf{J}_b$ flows clockwise while $\mathbf{K}_b$ flows counterclockwise.

By the symmetries of the bound currents, the magnetic field everywhere points in the vertical direction $\hat{\mathbf{z}}$ while its magntidude depends only on the cylindric radius s but not on z or ϕ,

$$\mathbf{B}(s,\phi,z) \;=\; B(s\,\text{only})\,\hat{\mathbf{z}}. \tag{36}$$

The magnitude $B(s)$ follows from the Ampere's Law:

$$\nabla\times\mathbf{B} \;=\; \mu_0\mathbf{J}_{\rm net} \;=\; \mu_0\mathbf{J}_b \;+\; \mu_0\mathbf{K}_n\delta(s-R). \tag{37}$$

For the field as in eq. (36),

$$\nabla\times\mathbf{B} \;=\; -\frac{dB}{ds}\,\hat{\phi}\,, \tag{38}$$

so for the currents at hand we must have

$$\frac{dB}{ds} = +\mu_0 k\Theta(R-s) - \mu_0 kR\delta(s-R), \tag{39}$$

where Θ is the step fucntion. Integrating eq. (39), we arrive at

$$B(s) = ks\Theta(R-s) + \text{const}, \tag{40}$$

where the integration constant is the magnetic field at infinity. Physically, this constant should be zero, hence

$$\mathbf{B} = ks\Theta(R-s)\hat{\phi} = \begin{cases} ks\hat{\phi} & \text{inside the cylinder,} \\ 0 & \text{outside the cylinder.} \end{cases} \tag{41}$$

Or in other words, $\mathbf{B} = \mu_0\mathbf{M}$, both inside and outside the magnetized cylinder.

(b) In terms of the **H** field, the solution is much simpler. The symmetries on the magentized cylinder and its magnetization include rotation around the z axis, translation along the z axis, and the mirror reflection $z \to -z$. (Note that **M** is the axial vector, so $M_z \to +M_z$ under this reflection.) In light of these symmetries, the **H** and the **B** magnetic fields both inside and outside the magnetized cylinder must point in the $\hat{\mathbf{z}}$ direction while their magnitudes depend only on s, thus

$$\mathbf{B}(s,\phi,z) = B(s)\,\hat{\mathbf{z}}, \quad \mathbf{H}(s,\phi,z) = H(s)\,\hat{\mathbf{z}}. \tag{42}$$

Moreover, there are no free currents anywhere in the system, so by the Ampere Law $\nabla\times\mathbf{H} = 0$. For the H field as in eq. (42), this means

$$\frac{dH}{ds} = 0 \quad\Longrightarrow\quad H(s) = \text{const}, \tag{43}$$

and since we need $\mathbf{H} \to 0$ at $s \to \infty$, it follows that $\mathbf{H} \equiv 0$ everywhere in the system!

Consequently, $\mathbf{B} = \mu_0\mathbf{M}$, both inside and outside the magnetized cylinder.

Problem **6.13**:

In the absence of free currents, the magnetic fields **B**, **H**, and **M** act very similarly to the electric fields **D**, **E**, and **P** in the absence of free charges: Indeed,

$$\langle\!\langle \text{ when } \mathbf{J}_{\text{free}} = 0 \,\rangle\!\rangle \qquad \nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{H} = 0, \qquad \mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}, \tag{44}$$

$$\langle\!\langle \text{ when } \rho_{\text{free}} = 0 \,\rangle\!\rangle \qquad \nabla\cdot\mathbf{D} = 0, \qquad \nabla\times\mathbf{E} = 0, \qquad \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}. \tag{45}$$

Consequently, finding the magnetic field in a small cavity in a magnetized material is exactly similar to finding the electric field in a small cavity inside a dielectric, which you should have done back in homework set#8, problem **4.16**. Quoting from the solutions to homework#8, we have:

In a spherical cavity,

$$\mathbf{E} = \frac{2}{3}\mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{D}_0\,, \tag{46}$$

$$\mathbf{D} = \frac{2\epsilon_0}{3}\mathbf{E}_0 + \frac{1}{3}\mathbf{D}_0\,. \tag{47}$$

In a needle-shaped cavity,

$$\mathbf{E} = \mathbf{E}_0\,, \tag{48}$$

$$\mathbf{D} = \epsilon_0\mathbf{E}_0\,. \tag{49}$$

In a wafer-shaped cavity,

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D}_0\,, \tag{50}$$

$$\mathbf{D} = \mathbf{D}_0\,. \tag{51}$$

Applying eqs. (44)–(45) as a dictionary to these old results, we immediately obtain the answers to the present problem abound the magnetic fields:

In a spherical cavity,

$$\mathbf{H} = \frac{2}{3}\mathbf{H}_0 + \frac{1}{3\mu_0}\mathbf{B}_0 = \mathbf{H}_0 + \frac{1}{3}\mathbf{M}_0\,, \tag{52}$$

$$\mathbf{B} = \frac{2\mu_0}{3}\mathbf{H}_0 + \frac{1}{3}\mathbf{B}_0 = \mathbf{B}_0 - \frac{2\mu_0}{3}\mathbf{M}_0\,. \tag{53}$$

In a needle-shaped cavity,

$$\mathbf{H} = \mathbf{H}_0 \,, \tag{54}$$

$$\mathbf{B} = \mu_0 \mathbf{H}_0 = \mathbf{B}_0 - \mu_0 \mathbf{M}_0 \,. \tag{55}$$

In a wafer-shaped cavity,

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}_0 = \mathbf{H}_0 + \mathbf{M}_0 \,, \tag{56}$$

$$\mathbf{B} = \mathbf{B}_0 \,. \tag{57}$$