Problem **7.2**:

(a) Suppose at some time t after the switch is closed the capacitor has charge $Q(t)$. Then at that moment of time, the voltage on the capacitor is

$$V(t) = \frac{Q(t)}{C}, \tag{1}$$

and the same voltage also applies to the resistor R. Consequently, the current through the resistor is

$$I(t) = \frac{V(t)}{R} = \frac{Q(t)}{RC}. \tag{2}$$

The same current also flows through the capacitor itself, or rather it flows from the positive plate of the capacitor through the wires and the resistor all the way to the negative plate of the capacitor. This current makes the capacitor discharge at the rate $I(t)$, thus its charge diminishes with time according to

$$\frac{dQ}{dt} = -I(t) = -\frac{Q(t)}{RC} \tag{3}$$

Solving this first-order differential equation gives us exponentially-decreasing charge

$$Q(t) = Q_0 \times \exp(-t/\tau) \tag{4}$$

where Q_0 is the original charge at time $t = 0$ when the switch was closed, and

$$\tau = R \times C \tag{5}$$

is the *time constant* of the RC circuit: After time $t = \tau$, the charge has diminished to $Q = \exp(-1) \times Q_0 \approx 0.37Q_0$.

The voltage on the capacitor and the current through the resistor also decrease exponentially with time,

$$V(t) \;=\; V_0 \times \exp(-t/\tau), \quad I(t) \;=\; I_0 \times \exp(-t\tau), \tag{6}$$

for the same time constant τ as the charge $Q(t)$, but

$$V_0 \;=\; \frac{Q_0}{C} \quad \text{and} \quad I_0 \;=\; \frac{V_0}{R} \;=\; \frac{Q_0}{RC}\,. \tag{7}$$

(b) The instantaneous electric power dissipated by the resistor is $P = I^2R$. For the exponentially decreasing current (6),

$$P(t) \;=\; RI^2(t) \;=\; RI_0^2 \times \exp(-2t/\tau), \tag{8}$$

so integrating this power over time we obtain the net dissipated energy as

$$W \;=\; \int_0^\infty P(t)\,dt \;=\; RI_0^2 \times \int_0^\infty \exp(-2t/\tau)\,dt \;=\; RI_0^2 \times \frac{\tau}{2}\,. \tag{9}$$

In light of eq. (5) for the time constant, this energy is

$$W \;=\; RI_0^2 \times \frac{RC}{2} \;=\; \frac{C}{2} \times (RI_0)^2 \;=\; \frac{C}{2} \times V_0^2\,. \tag{10}$$

On the other hand, the energy stored in the capacitor before it began to discharge through the resistor is

$$U_0 \;=\; \frac{C}{2} \times V_0^2\,, \tag{11}$$

so we see that $W = U_0$: The net energy dissipated by the resistor is precisely the energy originally stored in the capacitor. *Quod erat demonstrandum.*

(c) Now consider the circuit on textbook figure 7.5(b). For simplicity, let's neglect the internal resistance of the battery, so the voltage V_0 on the battery stays constant at all times. Let's also assume that the capacitor was completely discharged by the time $t = 0$ when the switch was closed.

Suppose at some time $t > 0$ the capacitor has voltage $V(t)$ between its plates, so its charge is $Q(t) = C \times V(t)$. At the same time, the voltage between the resistor's plates is

$$V_R(t) \;=\; V_0 \;-\; V(t), \tag{12}$$

so the current flowing through the resistor — and hence through the whole circuit — is

$$I(t) \;=\; \frac{V_R(t)}{R} \;=\; \frac{V_0 - V(t)}{R}\,. \tag{13}$$

From the capacitor's point of view, the direction of this current is from the negative plate to the positive plate, through the resistor and the battery, so this current charges the capacitor. Thus,

$$\frac{dQ}{dt} \;=\; +I(t) \;=\; +\frac{V_0 - V(t)}{R}\,, \tag{14}$$

so the voltage $V(t)$ between the capacitor's plates obeys

$$\frac{dV}{dt} \;=\; \frac{1}{C} \times \frac{dQ}{dt} \;=\; \frac{V_0 - V(t)}{RC}\,. \tag{15}$$

The solution to this differential equation with the initial condition $V(0) = 0$ is

$$V(t) \;=\; V_0 \times \bigl(1 \;-\; \exp(-t/\tau)\bigr) \tag{16}$$

for the time constant

$$\tau \;=\; R \times C, \tag{17}$$

exactly as in eq. (5) in part (a).

As to the time dependence of the capacitor charge $Q(t)$ and the charging current $I(t)$, they follow from the time-dependent voltage (16):

$$Q(t) \;=\; CV_0 \times \bigl(1 \;-\; \exp(-t/\tau)\bigr), \quad I(t) \;=\; \frac{V(t)}{R} \;=\; \frac{V_0}{R} \times \exp(-t/\tau). \tag{18}$$

(d) The instantaneous power delivered by the battery is $P(t) = V_0 \times I(t)$, so for the exponentially decreasing current (18),

$$P(t) \;=\; V_0 \times \frac{V_0}{R} \times \exp(-t/\tau). \tag{19}$$

The net electrical work by the battery while the capacitor is charging is the time integral of this power, thus

$$W_{\rm net} \;=\; \int_0^\infty P(t)\,dt \;=\; \frac{V_0^2}{R} \times \int_0^\infty \exp(-t/\tau)\,dt \;=\; \frac{V_0^2}{R} \times \tau. \tag{20}$$

For the time constant $\tau = RC$ we have found in part (c), the bet work by the battery amounts to

$$W_{net} \;=\; \frac{V_0^2}{R} \times RC \;=\; C \times V_0^2\,. \tag{21}$$

Some of this electric work goes towards charging the capacitor, the rest is dissipated by the resistor. The capacitor's energy is

$$U(t) \;=\; \frac{C}{2} \times V^2(t) \tag{22}$$

where the voltage increases with time according to eq. (16). Asymptotically, the capacitor voltage approaches the battery voltage V_0, so its energy gets to

$$U_{\rm net} \;=\; \frac{C}{2} \times V_0^2\,. \tag{23}$$

Note that this energy is only a half of the electric work (21) delivered by the battery, so the other half of this work is dissipated by the resistor,

$$W_{\rm dissipated} \;=\; W_{\rm net} \;-\; U_{\rm net} \;=\; \frac{C}{2} \times V_0^2. \tag{24}$$

Note: this dissipated energy is exactly the same as in part (b) because the current $I(t)$ through the resistor has exactly the same form as in part (b), *cf.* eqs. (6) and (18).

Problem **7.5**:

The voltage on the battery $V = \mathcal{E} - rI$ is equal to the voltage $V = RI$ on the load,

$$\mathcal{E} \;-\; r \times I \;=\; R \times I, \tag{25}$$

hence the current through the circuit is

$$I \;=\; \frac{\mathcal{E}}{r + R}\,. \tag{26}$$

The power delivered to the load is

$$P \;=\; V \times I \;=\; R \times I^2 \;=\; \frac{R}{(r+R)^2} \times \mathcal{E}^2\,. \tag{27}$$

To maximize this power as a function of the load's resistance R we need $R = r$. Indeed,

$$\frac{dP}{dR} \;=\; \mathcal{E}^2 \times \left(\frac{1}{(r+R)^2} \;-\; \frac{2R}{(r+R)^3}\right) \;=\; \mathcal{E}^2 \times \frac{r - R}{(r+R)^3}\,, \tag{28}$$

thus the power increases with R for $R < r$ but decreases for $R > r$, with the maximum at $R = r$.

Problem **7.7**:

(a) For the magnetic field $\mathbf{B}$ pointing into the page and the metal bar moving to the right, the direction of the vector product $\mathbf{v} \times \mathbf{B}$ is up-the-page. Consequently, the Lorentz force $\mathbf{F} = (-e)\mathbf{v} \times \mathbf{B}$ pushes the electrons in the moving bar down-the-page, which makes the current flow in the opposite direction, up-the-page. That is, the current flows up through the moving bar; through the rest of the circuit it flows counter-clockwise. In particular, in the resistor at the left end of the circuit, the current flows down.

Now consider the magnitude of this current. The EMF induced in the moving bar is

$$\mathcal{E} \;=\; \vec{\ell} \cdot (\mathbf{v} \times \mathbf{B}) \;=\; \ell v B. \tag{29}$$

I assume the resistance R of the resistor is much larger that the resistance of the moving bar or the connecting wires, so we may approximate the net resistance of the circuit as R.

Consequently, the current through the circuit due to the EMF (29) induced in the moving bar is

$$I = \frac{\mathcal{E}}{R_{\text{net}}} \approx \frac{\mathcal{E}}{R} = \frac{\ell v B}{R}\,. \tag{30}$$

(b) The magnetic force on the bar due to the current (30) is

$$\mathbf{F} = I\vec{\ell} \times \mathbf{B}\,. \tag{31}$$

For the current flowing up-the-page through the bar and the magnetic field direction into-the-page, *the direction of this force is to the left, opposite to the bar's velocity*, while its magnitude is

$$F = I \times \ell B = \frac{\ell^2 B^2}{R} \times v. \tag{32}$$

(c) By Newton's second law,

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}\,. \tag{33}$$

As we saw in part (b), the force on the moving bar is in the opposite direction to the bar's velocity, while the force's magnitude (32) is proportional to the speed. Therefore,

$$\frac{dv}{dt} = -\frac{F}{m} = -\frac{\ell^2 B^2}{mR} \times v. \tag{34}$$

The solution of this differential equation is an exponentially decreasing velocity,

$$v(t) = v_0 \times \exp\left(-\frac{t}{\tau}\right) \quad \text{for} \quad \tau = \frac{mR}{\ell^2 B^2}\,. \tag{35}$$

(d) As the bar slows down, its kinetic energy is dissipated as the Joule heat in the resistor. Indeed, the current (30) due to EMF induced in the moving bar dissipates power

$$P = I^2 R = \frac{\ell^2 B^2 v^2}{R}, \tag{36}$$

and this power is precisely the rate at which the moving bar looses its kinetic energy,

$$-\frac{dU_{\text{kin}}}{dt} = -\mathbf{v}\cdot\mathbf{F} = +v\times F = v^2\times\frac{\ell^2 B^2}{R} = P. \tag{37}$$

Consequently, the net energy dissipated while the bar slows down to a complete stop should equal to the initial kinetic energy of the bar.

Let's check this fact directly from the equation (36):

$$W = \int_0^\infty P(t)\,dt = \int_0^\infty \frac{\ell^2 B^2}{R}\times v^2(t)\,dt = \frac{\ell^2 B^2}{R}\times\int_0^\infty v_0^2\times e^{-2t/\tau}\,dt = \frac{\ell^2 B^2}{R}\times v_0^2\times\frac{\tau}{2}. \tag{38}$$

For the time constant (35) obtained in part (c), this formula yields

$$W = \frac{\ell^2 B^2}{R}\times\frac{v_0^2}{2}\times\left(\tau = \frac{mR}{\ell^2 B^2}\right) = \frac{v_0^2}{2}\times m, \tag{39}$$

which is precisely the initial kinetic energy of the particle. *Quod erat demonstrandum.*

Problem **7.8**:

(a) In the plane of the figure **7.18**, the magnetic field above the wire is directed into-the-page, while its magnitude depends only on the up-the-page coordinate y, specifically

$$B(y) = \frac{\mu_0 I}{2\pi y} \tag{40}$$

Consequently, the flux through the square loop is

$$\Phi = \int_0^a dx \int_s^{s+a} dy\,B(y) = a\times\int_s^{s+a} dy\,\frac{\mu_0 I}{2\pi y} = a\times\frac{\mu_0 I}{2\pi}\times\ln\frac{s+a}{s}. \tag{41}$$

(b) The EMF induced in a moving straight wire is $\mathcal{E} = \vec{\ell} \cdot (\mathbf{v} \times \mathbf{B})$. For the magnetic field pointing into-the-page while the loop moves up-the-page, the direction of the vector product $\mathbf{v} \times \mathbf{B}$ is left. Consequently, there is no EMF induced in the vertical wires, while there is leftward EMF induced in the horizontal wires. Relative to the loop, the EMF in the bottom wire is clockwise while the EMF in the top wire is counterclockwise.

The magnitudes of these EMFs are

$$\begin{aligned} \mathcal{E}^{\text{bottom wire}} &= a \times v \times B(y = s) = a \times v \times \frac{\mu_0 I}{2\pi s}, \\ \mathcal{E}^{\text{top wire}} &= a \times v \times B(y = s + a) = a \times v \times \frac{\mu_0 I}{2\pi (s + a)}, \end{aligned} \tag{42}$$

with a larger EMF in the bottom wire. Consequently, the net EMF induced in the square loop is clockwise, so *the current in the loop flows clockwise.*

As to the magnitude of the net EMF,

$$\mathcal{E}^{\text{net}} = \mathcal{E}^{\text{bottom wire}} - \mathcal{E}^{\text{top wire}} = \frac{a v \mu_0 a}{2\pi} \left(\frac{1}{s} - \frac{1}{s + a} \right). \tag{43}$$

Let's compare this magnitude with the Faraday's law,

$$\mathcal{E}^{\text{net}} = -\frac{d\Phi}{dt}, \tag{44}$$

The magnetic flux (41) from part (a) changes with time because s changes when the loop moves up,

$$\frac{ds}{dt} = v. \tag{45}$$

Consequently,

$$\frac{d}{dt} \ln \frac{s + a}{s} = v \times \frac{d}{ds} \ln \frac{s + a}{s} = v \times \left(\frac{1}{s + a} - \frac{1}{a} \right), \tag{46}$$

and therefore

$$-\frac{d\Phi}{dt} = -\frac{a \mu_0 I}{2\pi} \frac{d}{dt} \ln \frac{s + a}{s} = +\frac{a \mu_0 I}{2\pi} \times \left(\frac{1}{s} - \frac{1}{s + a} \right) \times v, \tag{47}$$

in perfect agreement with the net EMF (43) induced in the loop.

(c) When the loop moves to the right — parallel to the current-carrying wire — the magnetic flux through the loop does not change, so we expect zero net EMF induced in the loop.

To see how this works, consider the EMFs $\mathcal{E} = \vec{\ell} \times (\mathbf{v} \times \mathbf{B})$ induced in each side of the square loop. For $\mathbf{B}$ pointing into-the-page while the velocity vector $\mathbf{v}$ points right, the vector product $\mathbf{v} \times \mathbf{B}$ points up-the-page. Consequently, there is not EMF induced in the horizontal top or bottom wires, while there is upward EMF induced in the vertical left or right wires. By symmetry, the upward EMFs in the left and the right wires are equal, but their directions relative to the loop are opposite: clockwise in the left wire but counterclockwise in the right wire. Thus, the EMFs induced in the two vertical wire precisely cancel each other, and the net EMF in the loop is zero.

Problem **7.12**:

Inside a long solenoid, the magnetic field is uniform and points along the solenoid's axis. For a circular loop of radius $a/2$ completely inside the solenoid and $\perp$ to its axis, the magnetic flux is

$$\Phi = B \times \text{Area} = B \times \pi(a/2)^2. \tag{48}$$

When the magnetic field changes with time, the flux also changes,

$$\Phi(t) = B(t) \times \pi(a/2)^2, \tag{49}$$

which leads to the EMF induced in the loop,

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{dB}{dt} \times \pi(a/2)^2. \tag{50}$$

For the harmonically oscillating magnetic field $B(t) = B_0 \times \cos(\omega t)$, this EMF is

$$\mathcal{E} = -\pi(a/2)^2 \times B_0 \times \frac{d\cos(\omega t)}{dt} = +\pi(a/2)^2 \times B_0 \times \omega \times \sin(\omega t). \tag{51}$$

Consequently, there is a harmonically oscillating current in the loop,

$$I(t) = \frac{\mathcal{E}(t)}{R} = \frac{\pi(a/2)^2 B_0 \omega}{R} \times \sin(\omega t). \tag{52}$$

Problem **7.17**:

(a) First, let's find the magnetic flux through the loop as a function of the current I in the solenoid. The magnetic field inside the solenoid (without an iron core) is

$$\mathbf{B} = \mu_0 I n\,\hat{\mathbf{z}} \tag{53}$$

while outside the solenoid the magnetic field is negligible. Consequently, the magnetic flux through any loop surrounding the solenoid is simply the net flux through the solenoid itself,

$$\Phi = \mathbf{B}\cdot\mathbf{a}_{\text{solenoid}} = \mu_0 I n \times \pi a^2. \tag{54}$$

When the current through the solenoid changes, this flux changes at the rate

$$\frac{d\Phi}{dt} = \frac{dI}{dt} \times \mu_0 n(\pi a^2), \tag{55}$$

which induces the EMF in the loop surrounding the solenoid,

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{dI}{dt} \times \mu_0 n(\pi a^2). \tag{56}$$

Consequently, there is a current in the loop

$$I_{\text{loop}} = \frac{\mathcal{E}}{R} = -\frac{dI}{dt} \times \frac{\mu_0 n(\pi a^2)}{R}. \tag{57}$$

The '$-$' sign here reflects the Lenz rule: the current in the loop oppose the changes of the current in the solenoid. Thus, if the current in the solenoid is clockwise (as shown on figure 7.28) and increasing, then the current in the loop is counterclockwise.

(b) Now suppose instead of changing the current in the solenoid, we simply pull the loop off the solenoid and move it far away. Moving the loop leads to the motional EMF, which also obtains from the Faraday's law,

$$\mathcal{E} \;=\; -\frac{d\Phi}{dt}\,, \tag{58}$$

whatever the reason for changing of the magnetic flux. Consequently, the current in the loop is

$$I_{\rm loop}(t) \;=\; \frac{\mathcal{E}(t)}{R} \;=\; -\frac{1}{R}\times\frac{d\Phi}{dt}\,, \tag{59}$$

Integrating this current over time, we obtain the net charge which has flown through the loop,

$$Q_{\rm loop} \;=\; \int I_{\rm loop}\,dt \;=\; -\frac{1}{R}\int\frac{d\Phi}{dt}\,dt \;=\; -\frac{1}{R}\times\Delta\Phi \;=\; \frac{\Phi_{\rm init}-\Phi_{\rm final}}{R} \tag{60}$$

In particular, when the loop is removed from the solenoid to a far-away place without a magnetic field, we have $\Phi_{\rm final}=0$ and therefore

$$Q_{\rm loop} \;=\; \frac{\Phi_{\rm init}}{R}\,. \tag{61}$$

If the loop initially surrounds the solenoid as in figure 7.28, then the initial flux through the loop is the flux in the solenoid as in eq. (54), hence

$$Q_{\rm loop} \;=\; \frac{\mu_0 In(\pi a^2)}{R}\,. \tag{62}$$

For example, for a solenoid of radius $a = 1$ cm and density $n = 1000$ loops/m carrying current $I = 1$ A, and the loop having net resistance $R = 4\ \Omega$ (which mostly comes from the resistor), the net charge flowing through the loop as it is removed is about $Q = 10^{-7}$ C.

Problem **7.19**:

The mathematical relation between the induced electric field and the time-dependent magnetic field is very similar to the relation between the magnetics field itself and the electric current density. Indeed, compare the equations

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t}\,, \quad \nabla\cdot\mathbf{E} \;=\; 0 \quad \langle\!\langle\,\text{assuming } \rho=0\,\rangle\!\rangle \tag{63}$$

and

$$\nabla\times\mathbf{B} \;=\; \mu_0\mathbf{J}, \quad \nabla\cdot\mathbf{B} \;=\; 0. \tag{64}$$

Similar equations call for similar solutions, hence the Biot–Savart–Laplace-like formula for the induced electric field:

$$\mathbf{E}(r,t) \;=\; -\frac{1}{4\pi}\iiint\frac{\partial\mathbf{B}(\mathbf{r}',t)}{\partial t}\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,d^3\mathrm{Vol}'. \tag{65}$$

For the toroidal coil in question, there is strong magnetic field inside the coil but no field outside the coil. For a thin toroid (compared to its radius), we may integrate over the cross-section while treating the $\mathbf{r}-\mathbf{r}'$ as approximately constant, thus

$$\iint\limits_{\substack{\text{cross}\\\text{section}}}\frac{\partial\mathbf{B}}{\partial t} \;=\; \frac{d\Phi}{dt}\,\hat{\phi} \tag{66}$$

where Φ is the net magnetic flux through the toroid, and hence

$$\mathbf{E}(\mathbf{r}) \;=\; -\frac{d\Phi}{dt}\times\frac{1}{4\pi}\oint\limits_{\text{length}} d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,. \tag{67}$$

The integral here is precisely the same integral as for calculating the magnetic field of a circular current loop, *cf.* the textbook example 5.6 (page 227). For generic points $\mathbf{r}$, this integral is a mess of elliptic functions, but for the $\mathbf{r}$ lying on the symmetry axis of the toroid, the integral simplifies to

$$\oint_{\text{length}} d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \;=\; \frac{2\pi a^2}{(a^2+z^2)^{3/2}}\,\hat{\mathbf{z}}\,, \tag{68}$$

exactly as in the textbook equation (5.41). For the present problem, eq. (68) means that *on the toroid's axis*, the electric field is

$$\mathbf{E}(0,0,z) \;=\; -\frac{d\Phi}{dt} \times \frac{a^2}{2(a^2+z^2)^{3/2}}\,\hat{\mathbf{z}} \tag{69}$$

Finally, the magnetic field inside the toroid is explained in the textbook example 5.10 (pages 238–239),

$$\mathbf{B}^{\text{inside}} \;=\; \frac{\mu_0 IN}{2\pi s}\,\hat{\phi}\,, \tag{5.60}$$

For a thin toroid with $w \ll a$ this field is approximately uniform (inside the toroid), so the flux through the toroid is simply

$$\Phi \;=\; hw \times B_\phi \;=\; \mu_0 IN \times \frac{hw}{2\pi a}\,. \tag{70}$$

While the geometry of the toroid should be time-independent, the current I may change with time, which leads to a time-dependent magnetic flux, changing at the rate

$$\frac{d\Phi}{dt} \;=\; \frac{\mu_0 Nhw}{2\pi a} \times \frac{dI}{dt}\,. \tag{71}$$

Plugging this formula into eq. (69) for the electric field, we finally arrive at

$$\mathbf{E}(0,0,z) \;=\; (-\hat{\mathbf{z}}) \times \frac{dI}{dt} \times \frac{\mu_0 Nhwa}{4\pi(a^2+z^2)^{3/2}}\,. \tag{72}$$

Problem **7.51**:

First, let's find the magnetic field of the moving wire. Since the wire is moving at a constant velocity v that is much less than the speed of light c, we may use the quasi-static approximation: The magnetic field $\mathbf{B}(\mathbf{r},t)$ is approximately the same as the field of the stationary wire located wherever the wire happens to be at time t. In particular, at $t=0$ when the wire runs along the z axis, we have

$$\mathbf{B}(t=0,\mathbf{r}) \;=\; \frac{\mu_0 I}{2\pi}\,\frac{\hat{\phi}}{s} \;=\; \frac{\mu_0 I}{2\pi}\,\frac{x\,\hat{\mathbf{y}}\,-\,y\,\hat{\mathbf{x}}}{x^2\,+\,y^2}\,. \tag{73}$$

For future convenience, let's express this field in terms of the vector potential,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \quad\text{for}\quad \mathbf{A}(t=0,\mathbf{r}) \;=\; \frac{\mu_0 I}{2\pi}\,\ln(s)\,\hat{\mathbf{z}} \;=\; \frac{\mu_0 I}{4\pi}\,\ln(x^2+y^2)\,\hat{\mathbf{z}}\,. \tag{74}$$

At other times $t\neq 0$, we have similar formulae for the magnetic field and the vector potential in terms of the *relative coordinates* $x - X_{\text{wire}}(t) = x$ and $y - Y_{\text{wire}}(t) = y - vt$, thus

$$\mathbf{B}(t,\mathbf{r}) \;=\; \frac{\mu_0 I}{2\pi}\,\frac{x\,\hat{\mathbf{y}}\,-\,(y-vt)\,\hat{\mathbf{x}}}{x^2\,+\,(y-vt)^2}\,, \tag{75}$$

$$\mathbf{A}(t,\mathbf{r}) \;=\; \frac{\mu_0 I}{4\pi}\,\ln\bigl(x^2+(y-vt)^2\bigr)\,\hat{\mathbf{z}}\,. \tag{76}$$

Note: in a coordinate frame moving with the wire, the magnetic field (73) and the vector potential (74) are time-independent. But in the lab frame, the magnetic field at any fixed point (x,y,z) changes with time because the distance to the moving wire changes with time, thus

$$\left(\frac{\partial\mathbf{B}(t;x,y,z)}{\partial t}\right)_{\text{fixed lab frame }(x,y,z)} \;\neq\; 0. \tag{77}$$

Consequently, in the lab frame there is electric field induced by the time-dependent magnetic field.

The induced electric field $\mathbf{E}(t, \mathbf{r})$ obtains from solving the Induction Law equation as well as the Gauss Law for $\rho \equiv 0$,

$$\nabla \times \mathbf{E}(t, \mathbf{r}) \;=\; -\frac{\partial \mathbf{B}(t, \mathbf{r})}{\partial t}, \quad \nabla \cdot \mathbf{E}(t, \mathbf{r}) \;=\; 0. \tag{78}$$

The simplest way to solve these is in terms of the vector potential $\mathbf{A}(t, \mathbf{r})$ and the scalar potential $V(t, \mathbf{r})$,

$$\mathbf{E}(t, \mathbf{r}) \;=\; -\frac{\partial \mathbf{A}(t, \mathbf{r})}{\partial t} \;-\; \nabla V(t, \mathbf{r}). \tag{79}$$

The vector potential here should be exactly as in eq. (76), so that $\nabla \times \mathbf{A}(\mathbf{r}, t)$ is the time-dependent magnetic field; this will make the electric field (77) obey the Induction Law for any $V(t, \mathbf{r})$.

As to the scalar potential, it should take care of the Gauss Law by obeying the Poisson equation

$$\nabla^2 V(t, \mathbf{r}) \;=\; -\frac{\partial}{\partial t}\big(\nabla \cdot \mathbf{A}(t, \mathbf{r})\big). \tag{80}$$

Fortunately, the vector potential (76) at hand has zero divergence —

$$\nabla \cdot \mathbf{A} \;=\; \frac{\partial A_z}{\partial z} \;=\; 0 \tag{81}$$

— hence the scalar potential should obey $\nabla^2 V = 0$, and the simplest solution to this Laplace equations is simply $V(x, y, x, t) \equiv 0$.

Consequently,

$$\begin{aligned}
\mathbf{E}_{\text{induced}}(t, \mathbf{r}) \;&=\; -\frac{\partial}{\partial t}\,\mathbf{A}(t, \mathbf{r}) \\
&=\; -\frac{\partial}{\partial t}\left(\frac{\mu_0 I}{4\pi}\,\ln\!\big(x^2 + (y - vt)^2\big)\,\hat{\mathbf{z}}\right) \\
&=\; -\frac{\mu_0 I}{4\pi}\,\frac{\partial \ln\!\big(x^2 + (y - vt)^2\big)}{\partial t}\,\hat{\mathbf{z}} \\
&=\; +\frac{\mu_0 I}{4\pi}\,\frac{2v(y - vt)}{x^2 + (y - vt)^2}\,\hat{\mathbf{z}},
\end{aligned} \tag{82}$$

In particular, at time $t=0$ when the wire runs along the z axis,

$$\mathbf{E}(t=0;x,y,z) \;=\; \frac{\mu_0 I v}{2\pi}\,\frac{y\,\hat{\mathbf{z}}}{x^2+y^2}\,, \tag{83}$$

or in cylindrical coordinates

$$\mathbf{E}(t=0;s,\phi,z) \;=\; \frac{\mu_0 I v}{2\pi}\,\frac{\sin\phi}{s}\,\hat{\mathbf{z}}\,. \tag{84}$$

Problem **7.22**:

(a) Let's assume the little loop is much smaller than the big loop or the distance between the loops, $a \ll b, z$, so the magnetic field of the big loop is approximately uniform over the little loop,

$$\mathbf{B}_{\rm big}(\mathbf{r}\in\text{little loop}) \;\approx\; \text{const.} \tag{85}$$

Evaluating this approximately uniform field at the little loop's center — which happens to lie on the big loop's axis — we find

$$\mathbf{B}_{\rm big}(\mathbf{r}\in\text{little loop}) \;\approx\; \mathbf{B}_{\rm big}(0,0,z) \;=\; \frac{\mu_0 I_{\rm big}}{2}\,\frac{b^2}{(b^2+z^2)^{3/2}}\,\hat{\mathbf{z}}\,, \tag{86}$$

where the second equality here is the textbook equation (5.41) for the magnetic field of a circular loop. Consequently, the magnetic flux (of the big loop field's) through the little loop is

$$\Phi_{\rm little} \;\approx\; \mathbf{a}_{\rm little}\cdot\mathbf{B}_{\rm big}(0,0,z) \;=\; \pi a^2\times B^z_{\rm big}(0,0,z) \;=\; \frac{\mu_0 I_{\rm big}}{2}\times\pi a^2\times\frac{b^2}{(b^2+z^2)^{3/2}}\,. \tag{87}$$

(b) Now consider the magnetic flux of the little loop through the big loop. Again, we assume that the little loop is very small compared to the distance to the big loop, and this allows us to use the dipole approximation,

$$\mathbf{B}_{\text{little}}(\mathbf{r}') \;=\; \frac{\mu_0}{4\pi}\,\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}}')\hat{\mathbf{r}}' \,-\, \mathbf{m}}{r'^3} \tag{88}$$

where

$$\mathbf{m} \;=\; (\pi a^2) I_{\text{little}}\,\hat{\hat{\mathbf{z}}} \tag{89}$$

is the magnetic moment of the little loop, and the radius vector $\mathbf{r}'$ is taken relative to the little loop's center. In cylindrical coordinates, the magnetic field (88) becomes

$$\mathbf{B}_{\text{little}}(s,\phi,z) \;=\; \frac{\mu_0(\pi a^2)I_{\text{little}}}{4\pi}\left(\frac{2z'^2-s^2}{(z'^2+s^2)^{5/2}}\,\hat{\mathbf{z}} \,+\, \frac{3z's}{(z'^2+s^2)^{5/2}}\,\hat{\mathbf{s}}\right) \tag{90}$$

where $z'=z-z^{\text{little loop}}$.

Now let's calculate the flux of this magnetic field through the big loop. Spanning the big loop with a flat disk of radius b in the xy plane — hence at constant $z=0$ while $z'=-z^{\text{little loop}}$, — we have

$$\begin{aligned}
\Phi_{\text{big}} &= \iint\limits_{\text{disk}} B^z_{\text{little}}(s,\phi)\,d^2A \;=\; \int_0^b 2\pi s\,ds \times \frac{\mu_0(\pi a^2)I_{\text{little}}}{4\pi}\,\frac{2z'^2-s^2}{(z'^2+s^2)^{5/2}} \\
&= \frac{\mu_0(\pi a^2)I_{\text{little}}}{2} \times \int_0^b \frac{(2z'^2-s^2)\,s\,ds}{(z'^2+s^2)^{5/2}} \\
&\qquad \langle\!\langle \text{ changing variables from } s \text{ to } \nu=s^2+z'^2 \,\rangle\!\rangle \\
&= \frac{\mu_0(\pi a^2)I_{\text{little}}}{2} \times \int_{z'^2}^{b^2+z'^2} \frac{(3z'^2-\nu)\times\frac{1}{2}d\nu}{\nu^{5/2}} \\
&= \frac{\mu_0(\pi a^2)I_{\text{little}}}{2} \times \int_{z'^2}^{b^2+z'^2} d\left(-\frac{z'^2}{\nu^{3/2}} \,+\, \frac{1}{\nu^{1/2}}\right) \\
&= \frac{\mu_0(\pi a^2)I_{\text{little}}}{2} \times \left(\left(\frac{1}{\sqrt{b^2+z'^2}} \,-\, \frac{z'^2}{(b^2+z'^2)^{3/2}}\right) \,-\, \left(\frac{1}{|z'|} \,-\, \frac{z'^2}{|z'|^3}\right)\right) \\
&= \frac{\mu_0(\pi a^2)I_{\text{little}}}{2} \times \frac{b^2}{(b^2+z'^2)^{3/2}}
\end{aligned} \tag{91}$$

Altogether,

$$\Phi_{\rm big} \;=\; \frac{\mu_0 I_{\rm little}}{2}\times(\pi a^2)\times\frac{b^2}{(b^2+z_{\rm little}^2)^{3/2}}\,. \tag{92}$$

Alternative solution:

In the dipole approximation to the little loop's magnetic field, the vector potential is

$$\mathbf{A}_{\rm little}(\mathbf{r}') \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}\times\hat{\mathbf{r}}'}{r'^2}\,, \tag{93}$$

or in cylindrical coordinates

$$\mathbf{A}_{\rm little}(\mathbf{r}') \;=\; \frac{\mu_0(m=\pi a^2 I_{\rm little})}{4\pi}\,\frac{s'}{(s'^2+z'^2)^{3/2}}\,\hat{\phi}'. \tag{94}$$

In terms of this vector potential, the magnetic flux through the big loop is

$$\Phi_{\rm big} \;=\; \oint\limits_{\rm big\,loop} \mathbf{A}_{\rm little}(\mathbf{r}')\cdot d\mathbf{r}'. \tag{95}$$

Geometrically, the big loop is located at fixed $s'=b$ and fixed $z'=-\Delta z=-z^{\rm littleloop}$ (relative to the little loop), while ϕ' span the full circle from 0 to 2π. Consequently,

$$\mathbf{A}_{\rm little}(\mathbf{r}')\cdot d\mathbf{r}' \;=\; \frac{\mu_0(\pi a^2)I_{\rm little}}{4\pi}\times\frac{b}{b^2+z_\ell^2)^{3/2}}\times b\,d\phi\,, \tag{96}$$

so the magnetic flux (95) is simply

$$\Phi_{\rm big} \;=\; \frac{\mu_0(\pi a^2)I_{\rm little}}{4\pi}\times\frac{b}{b^2+z_\ell^2)^{3/2}}\times 2\pi b \;=\; \frac{\mu_0 I_{\rm little}}{2}\times(\pi a^2)\times\frac{b^2}{(b^2+z_{\rm little}^2)^{3/2}}\,. \tag{92}$$

(c) In terms of mutual inductances, eq. (87) from part(a) becomes

$$\begin{aligned}
\Phi_{\text{little}} &= M_{l,b} \times I_{\text{big}} && (97) \\
\text{where} \quad M_{l,b} &= \frac{\mu_0}{2} \times \pi a^2 \times \frac{b^2}{(b^2 + \Delta z^2)^{3/2}} \,. && (98)
\end{aligned}$$

Similarly, eq. (92) from part (b) becomes

$$\begin{aligned}
\Phi_{\text{big}} &= M_{b,l} \times I_{\text{little}} && (99) \\
\text{where} \quad M_{b.l} &= \frac{\mu_0}{2} \times \pi a^2 \times \frac{b^2}{(b^2 + \Delta z^2)^{3/2}} \,. && (100)
\end{aligned}$$

By inspection of eqs (98) and (100), the two mutual inductances are equal,

$$M_{l,b} = M_{b,l} \,. \qquad (101)$$

Quod erat demonstrandum.

Problem **7.29**:

The geometry of the coil is shown on textbook figure **7.34** (page 327): rectangular cross-section $(b - a) \times h$, where a is the inner radius of the toroid and b is the outer radius. Assuming the toroid is uniformly and densely wound, the magnetic field vanishes anywhere outside the coil, while inside the coil

$$\mathbf{B}^{\text{inside}} = \frac{\mu_0 N I}{2\pi s}\, \hat{\phi} \,. \qquad (102)$$

According to the textbook equation (7.35), the net energy stored in this magnetic field is

$$\begin{aligned}
U &= \frac{1}{2\mu_0} \iiint\limits_{\substack{\text{whole}\\\text{space}}} \mathbf{B}^2\, d^3\text{Vol} \\
&= \frac{1}{2\mu_0} \iiint\limits_{\text{toroid}} \left(\frac{\mu_0 NI}{2\pi s}\right)^2 d^3\text{Vol} \\
&= \frac{1}{2\mu_0} \times \frac{\mu_0^2 N^2 I^2}{4\pi^2} \times \int_0^h dz \int_a^b \frac{2\pi s\, ds}{s^2} \\
&= \frac{\mu_0 N^2 I^2}{8\pi^2} \times h \times 2\pi \ln\frac{b}{a} \\
&= \frac{I^2}{2} \times \frac{\mu_0 N^2 h \ln(b/a)}{2\pi}\,.
\end{aligned} \tag{103}$$

On the other hand, treating the toroid as an inductor with self-inductance

$$L = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi} \tag{7.28}$$

(see the textbook example **7.11** (page 325) for the calculation), we expect the net magnetic energy stored in this inductor to be

$$U = \frac{I^2 L}{2} = \frac{I^2}{2} \times \frac{\mu_0 N^2 h \ln(b/a)}{2\pi}\,. \tag{104}$$

By inspection, this is precisely the same energy as in eq. (103).

Problem **7.31**:

(a) Once the switch has been in position A for a long time, the current reaches a steady value

$$I_0 = \frac{\mathcal{E}_0}{R}\,. \tag{105}$$

When at time $t_0 = 0$ the switch is moved to position B, the current through the inductor cannot suddenly change, thus $I(t = 0) = I_0$.

But after that, the current starts changing, which generates the EMF in the inductor

$$\mathcal{E}(t) \;=\; -L\frac{dI}{dt} \tag{106}$$

and hence the voltage on the resistor

$$V_R \;=\; -V_L \;=\; +\mathcal{E} \;=\; -L\frac{dI}{dt}\,. \tag{107}$$

At the same time, the voltage on the resistor is related to the current through it by the Ohm's law,

$$V_R(t) \;+\; R\times I(t), \tag{108}$$

hence the differential equation

$$-L\frac{dI}{dt} \;=\; R\times I \tag{109}$$

for the time dependence of the current $I(t)$. The solution of this equation is obviously

$$I(t) \;=\; I_0\times e^{-t/\tau} \tag{110}$$

where the time constant τ obtains as

$$\frac{L}{\tau} \;=\; R \quad\Longrightarrow\quad \tau \;=\; \frac{L}{R}\,. \tag{111}$$

(b) The electric power going to the resistor is

$$P(t) \;=\; I^2(t)\times R, \tag{112}$$

so the net energy delivered to the resistor after the switch is thrown is

$$W_{\rm net} \;=\; \int\limits_0^\infty RI^2(t)\,dt. \tag{113}$$

For the current changing with time according to eq. (110),

$$RI^2(t) \;=\; RI_0^2 \times e^{-2t/\tau}, \tag{114}$$

hence

$$W_{\text{net}} \;=\; \int_0^\infty RI_0^2 \times e^{-2t/\tau}\,dt \;=\; RI_0^2 \times \frac{\tau}{2}\,. \tag{115}$$

Or in light of eq. (111) for the time constant τ,

$$W_{\text{net}} \;=\; \tfrac{1}{2}L \times I_0^2\,. \tag{116}$$

(c) The magnetic energy stored in the inductor at the time the switch was thrown was

$$U \;=\; \tfrac{1}{2}L \times I_0^2\,. \tag{117}$$

By inspection, this is precisely the net electric energy (116) delivered to the resistor after the switch was thrown.