Problem **1**(a):

A *static* electric field must have zero curl, $\nabla\times\mathbf{E}=0$ everywhere in space. Let's check this conditions for the fields (1–3). Using

$$\nabla\times\mathbf{E} \;=\; (\nabla E_x)\times\hat{\mathbf{x}} \;+\; (\nabla E_y)\times\hat{\mathbf{y}} \;+\; (\nabla E_z)\times\hat{\mathbf{z}} \tag{1}$$

we find:

$$\begin{aligned}
\nabla\times\mathbf{E}_a \;&=\; [2\alpha y\hat{\mathbf{x}} \,+\, 2\alpha(x+y)\hat{\mathbf{y}}]\times\hat{\mathbf{x}} \;+\; [2\alpha(x+y)\hat{\mathbf{x}} \,+\, 2\alpha\hat{\mathbf{y}}]\times\hat{\mathbf{y}} \;-\; [6\alpha z\hat{\mathbf{z}}]\times\hat{\mathbf{z}}\\
&=\; 2\alpha y(\hat{\mathbf{x}}\times\hat{\mathbf{x}}) \;+\; 2\alpha(x+y)(\hat{\mathbf{y}}\times\hat{\mathbf{x}}+\hat{\mathbf{x}}\times\hat{\mathbf{y}}) \;+\; 2\alpha x(\hat{\mathbf{y}}\times\hat{\mathbf{y}}) \;-\; 6\alpha z(\hat{\mathbf{z}}\times\hat{\mathbf{z}})\\
&=\; 0 \,+\, 0 \,+\, 0 \,-\, 0 \;=\; 0,
\end{aligned}\tag{2}$$

$$\begin{aligned}
\nabla\times\mathbf{E}_2 \;&=\; [2\alpha xy\hat{\mathbf{x}} \,+\, \alpha(x^2+3y^2)\hat{\mathbf{y}}]\times\hat{\mathbf{x}} \;-\; [2\alpha xy\hat{\mathbf{y}} \,+\, \alpha(3x^2+y^2)]\hat{\mathbf{x}}]\times\hat{\mathbf{y}}\\
&\qquad\langle\!\langle\,\text{using } \hat{\mathbf{x}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}}\times\hat{\mathbf{y}}=0 \text{ and } \hat{\mathbf{x}}\times\hat{\mathbf{y}}=-\hat{\mathbf{y}}\times\hat{\mathbf{x}}=\hat{\mathbf{z}}\,\rangle\!\rangle\\
&=\; 0 \;-\; \alpha(x^2+3y^2)\hat{\mathbf{z}} \;-\; 0 \;-\; \alpha(y^2+3x^2)\hat{\mathbf{z}}\\
&=\; -4\alpha(x^2+y^2)\hat{\mathbf{z}} \;\neq\; 0,
\end{aligned}\tag{3}$$

$$\begin{aligned}
\nabla\times\mathbf{E}_3 \;&=\; \alpha(y\hat{\mathbf{z}}+z\hat{\mathbf{y}})\times\hat{\mathbf{x}} \;+\; \alpha(x\hat{\mathbf{z}}+z\hat{\mathbf{x}})\times\hat{\mathbf{y}} \;+\; \alpha(x\hat{\mathbf{y}}+y\hat{\mathbf{x}})\times\hat{\mathbf{z}}\\
&=\; \alpha x(\hat{\mathbf{z}}\times\hat{\mathbf{y}}+\hat{\mathbf{y}}\times\hat{\mathbf{z}}) \;+\; \alpha y(\hat{\mathbf{z}}\times\hat{\mathbf{x}}+\hat{\mathbf{x}}\times\hat{\mathbf{z}}) \;+\; \alpha z(\hat{\mathbf{y}}\times\hat{\mathbf{x}}+\hat{\mathbf{x}}\times\hat{\mathbf{y}})\\
&=\; 0 \,+\, 0 \,+\, 0 \;=\; 0.
\end{aligned}\tag{4}$$

Thus, we see that the $\mathbf{E}_1$ and the $\mathbf{E}_3$ fields have zero curls and therefore are allowed as electrostatic fields, but the $\mathbf{E}_2$ field is forbidden.

Problem **1**(b):

For the allowed fields $\mathbf{E}_1$ and $\mathbf{E}_3$ the electric charge densities ρ obtain from the Gauss Law $\rho=\epsilon_0\nabla\cdot\mathbf{E}$. Thus,

$$\begin{aligned}
\rho_1 \;&=\; \epsilon_0\alpha\left(\frac{\partial(2xy+y^2)}{\partial x} \;+\; \frac{\partial(2xy+y^2)}{\partial y} \;+\; \frac{\partial(-3z^2)}{\partial z}\right)\\
&=\; \epsilon_0\alpha\Bigl(2y \,+\, 2x \,-\, 6z\Bigr),
\end{aligned}\tag{5}$$

$$
\begin{aligned}
\rho_3 \;&=\; \epsilon_0\alpha\left(\frac{\partial(yz)}{\partial x} \;+\; \frac{\partial(xz)}{\partial y} \;+\; \frac{\partial(xy)}{\partial z}\right) \\
&=\; \epsilon_0\alpha(0+0+0) \;=\; 0.
\end{aligned}
\tag{6}
$$

Finally, it is easy to see that

$$
\mathbf{E}_1 \;=\; \alpha\nabla\big(xy(x+y)) \;-\; z^3\big), \tag{7}
$$

$$
\mathbf{E}_3 \;=\; \alpha\nabla(xyz), \tag{8}
$$

which immediately gives us the potentials for the two allowed electrostatic fields:

$$
V_1(x,y,z) \;=\; \alpha\big(z^3 \;-\; xy(x+y)\big), \tag{9}
$$

$$
V_3(x,y,z) \;=\; -\alpha xyz. \tag{10}
$$

Problem **2**(a):

The induced charges on the grounded outer shell must completely screen all the charges inside it from the outside world. Hence, its net charge — on that shell and everything inside it — must vanish; for the problem at hand, this means

$$
Q_1 \;+\; Q_2 \;+\; Q_3 \;=\; 0 \quad\Longrightarrow\quad Q_3 \;=\; -Q_1 \;-\; Q_2\,. \tag{11}
$$

Or in terms of the charges per unit length,

$$
\lambda_3 \;=\; -\lambda_1 \;-\; \lambda_2\,. \tag{12}
$$

Problem **2**(b):

By the Gauss Law for cylindrically symmetric systems,

$$\mathbf{E}(s,\phi,z) \;=\; \frac{Q[\text{inside } s]}{2\pi\epsilon_0 L}\,\frac{\hat{\mathbf{s}}}{s}\,. \tag{13}$$

Thus, for the 3-shell system at hand,

$$\begin{aligned}
&\text{for } 0\le s<R_1, & \mathbf{E} &= 0, & (14)\\
&\text{for } R_1<s<R_2, & \mathbf{E} &= \frac{Q_1}{2\pi\epsilon_0 L}\,\frac{\hat{\mathbf{s}}}{s}\,, & (15)\\
&\text{for } R_2<s<R_3, & \mathbf{E} &= \frac{Q_1+Q_2}{2\pi\epsilon_0 L}\,\frac{\hat{\mathbf{s}}}{s}\,, & (16)\\
&\text{for } R_3<s<\infty, & \mathbf{E} &= \frac{Q_1+Q_2+Q_3}{2\pi\epsilon_0 L}\,\frac{\hat{\mathbf{s}}}{s}\\
&& &= 0, \quad \text{because } Q_1+Q_2+Q_3=0,\ \textit{cf.}\ \text{part(a)}. & (17)
\end{aligned}$$

Problem **2**(c):

Using

$$\int_a^b \frac{ds}{s} \;=\; \ln\frac{b}{a} \tag{18}$$

and the electric fields (15) and (16) between the shells, we immediately find the potential differences

$$V_1 - V_2 \;=\; \int_{\text{shell}_1}^{\text{shell}_2} \mathbf{E}\cdot d\mathbf{r} \;=\; \int_{R_1}^{R^2} \frac{Q_1}{2\pi\epsilon_0 L}\,\frac{ds}{s} \;=\; \frac{Q_1}{2\pi\epsilon_0 L}\times\ln\frac{R_2}{R_1}\,, \tag{19}$$

and likewise

$$V_2 - V_3 \;=\; \frac{Q_1+Q_2}{2\pi\epsilon_0 L}\times\ln\frac{R_3}{R_2}\,. \tag{20}$$

Furthermore, the outer shell is grounded, so its potential must be zero,

$$V_3 \;=\; 0. \qquad\qquad (\text{Exam}:4)$$

Consequently, the middle shell's potential is

$$V_2 \;=\; (V_2 - V_3) \;+\; (V_3 = 0) \;=\; \frac{Q_1 + Q_2}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_2}\,, \tag{Exam : 5}$$

while the inner shell's potential is

$$\begin{aligned} V_1 \;&=\; V_2 \;+\; (V_1 - V_2) \\ &=\; \frac{Q_1 + Q_2}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_2} \;+\; \frac{Q_1}{2\pi\epsilon_0 L} \times \ln\frac{R_2}{R_1} \\ &=\; \frac{Q_1}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_1} \;+\; \frac{Q_2}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_2}\,. \end{aligned} \tag{Exam : 6}$$

Problem **2**(d):

When the inner shall is grounded, its charge changes from Q_1 to some Q_1', while the charge on the still-grounded outer shell also changes $Q_3 \to Q_3'$ so as to keep zero net charge of the 3 shells,

$$Q_1' \;+\; Q_2 \;+\; Q_3' \;=\; 0. \tag{21}$$

Consequently, the potential for the inner shell changes from V_3 from eq. (Exam:6) to

$$V_1' \;=\; \frac{Q_1'}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_1} \;+\; \frac{Q_2}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_2}\,. \tag{22}$$

But for the grounded inner shell, this potential must vanish, $V_1' = 0$, hence

$$\frac{Q_1'}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_1} \;+\; \frac{Q_2}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_2} \;=\; 0. \tag{23}$$

Solving this equation for the new Q_1' charge of the inner shell, we find

$$Q_1' \;=\; -Q_2 \times \frac{\ln(R_3/R_2)}{\ln(R_3/R_1)}\,. \tag{24}$$

Therefore, the new charge of the outer shell is

$$Q_3' \;=\; -Q_1' \;-\; Q_2 \;= Q_2 \times \left(\frac{\ln(R_3/R_2)}{\ln(R_3/R_1)} \;-\; 1\right) \;=\; -Q_2 \times \frac{\ln(R_2/R_1)}{\ln(R_3/R_1)}\,. \tag{25}$$

Finally, the new potential of the isolated middle shell follows from eq. (Exam:5) for the new charges,

$$V_2 = \frac{Q_1' + Q_2}{2\pi\epsilon_0 L} \times \ln\frac{R_3}{R_2}$$
$$= \frac{Q_2}{2\pi\epsilon_0 L} \times \left(1 - \frac{\ln(R_3/R_2)}{\ln(R_3/R_1)}\right) \times \ln\frac{R_3}{R_2} \tag{26}$$
$$= \frac{Q_2}{2\pi\epsilon_0 L} \times \frac{\ln(R_2/R_1)\,\ln(R_3/R_2)}{\ln(R_3/R_1)}\,.$$

Problem **3**(a):

The electric field is (minus) the gradient of the potential (Exam:7), thus

$$\text{for } r < R, \qquad \mathbf{E}(\mathbf{r}) = -\frac{V_0}{R}\,\hat{\mathbf{r}}, \tag{27}$$

$$\text{for } r > R, \qquad \mathbf{E}(\mathbf{r}) = +\frac{V_0 R}{r^2}\,\hat{\mathbf{r}}. \tag{28}$$

Note that the potential (Exam:7) is continuous but the electric field is discontinuous at $r = R$.

Now consider the electric charges and their distribution. Since the potential is finite everywhere in space, there are no point or line charges. However, discontinuity of the electric field at the spherical surface $r = R$ implies surface charges on that sphere. Specifically, the surface charge density is

$$\sigma = \epsilon_0\,\mathrm{disc}[E_r]@(r=R) = \epsilon_0(E_r(r=R+0) - E_r(r=R-0))$$
$$= \epsilon_0\left(\frac{+V_0}{R} - \frac{-V_0}{R}\right) = +\frac{2V_0\epsilon}{R}\,. \tag{29}$$

As to the volume charge density, it obtains from the Gauss Law $\rho = \epsilon_0\nabla\cdot\mathbf{E}$, which for a spherically symmetric electric field becomes

$$\rho(r) = \epsilon_0\left(\frac{dE_r}{dr} + \frac{2E_r}{r}\right). \tag{30}$$

For the electric field at hand, this gives us

$$\text{for } r < R, \qquad \rho = -\frac{2\epsilon_0 V_0}{Rr}\,, \tag{31}$$

$$\text{for } r > R, \qquad \rho = 0. \tag{32}$$

Note that despite the $1/r$ singularity of the volume charge density at the center, the net electric charge inside the sphere is finite, specifically

$$Q_{\text{inside}} = \int_0^R \rho(r) \times 4\pi r^2\,dr = -\frac{2\epsilon_0 V_0}{R} \times \left(\int_0^R \frac{4\pi r^2\,dr}{r} = 2\pi R^2 \right) = -4\pi\epsilon_0 V_0 R. \tag{33}$$

Unlike the example discussed in the review session, this volume charge does not cancel the surface charge on the sphere

$$Q_{\text{surface}} = \sigma \times 4\pi R^2 = +8\pi\epsilon_0 V_0 R, \tag{34}$$

and since there are no charges outside the sphere, the net charge is

$$Q_{\text{net}} = Q_{\text{inside}} + Q_{\text{sphere}} + \left(Q_{\text{outside}} = 0\right) = +4\pi\epsilon_0 V_0 R. \tag{35}$$

By inspection, this is precisely the net charge responsible for the Coulomb potential $V = (V_0 R)/r$ outside the sphere.

Nota bene: In the exam I did not ask you to calculate the net charges, so if you did not it would not lower your grade.

Problem **3**(b):

There are at least 3 methods to calculate the net electrostatic energy, and for the purpose of this exam the students may use any one of these methods, whichever they like. But in these solutions, I will work out two most commonly used methods.

Method 1: Given $\rho(r)$ and σ at the boundary sphere calculated in part (a), the energy obtains as

$$
\begin{aligned}
U &= \frac{1}{2}\int V\,dQ \;=\; \frac{1}{2}\int\limits_0^R \rho(r)\times V(r)\times 4\pi r^2\,dr \;+\; \frac{1}{2}\sigma\times V(R)\times 4\pi R^2 \\
&= \frac{1}{2}\int\limits_0^R \frac{(-2\epsilon_0 V_0)}{Rr}\times\frac{V_0 r}{R}\times 4\pi r^2\,dr \;+\; \frac{1}{2}\,\frac{(+2\epsilon_0 V_0)}{R}\times V_0\times 4\pi R^2 \\
&= -\frac{4\pi\epsilon_0 V_0^2}{R^2}\times\left(\int\limits_0^R r^2\,dr \;=\; \frac{R^3}{3}\right) + \;4\pi\epsilon_0 V_0^2 R \\
&= +\frac{8\pi}{3}\,\epsilon_0 V_0^2 R.
\end{aligned}
\tag{36}
$$

Method II: Given the electric field everywhere, the energy obtains as

$$
\begin{aligned}
U &= \frac{\epsilon_0}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} \mathbf{E}^2\,d^3\mathrm{Vol} \;=\; \frac{\epsilon_0}{2}\int\limits_0^R E^2\times 4\pi r^2\,dr \;+\; \frac{\epsilon_0}{2}\int\limits_R^\infty E^2\times 4\pi r^2\,dr \\
&= \frac{\epsilon_0}{2}\int\limits_0^R \left(\frac{V_0}{R}\right)^2 4\pi r^2\,dr \;+\; \frac{\epsilon_0}{2}\int\limits_R^\infty \left(\frac{V_0 R}{r^2}\right)^2 4\pi r^2\,dr \\
&= 2\pi\epsilon_0 V_0^2\left(\frac{1}{R^2}\int\limits_0^R r^2\,dr \;+\; R^2\int\limits_R^\infty \frac{dr}{r^2}\right) \\
&= 2\pi\epsilon_0 V_0^2\left(\frac{1}{R^2}\times\frac{R^3}{3} \;+\; R^2\times\frac{1}{R} \;=\; \frac{4R}{3}\right) \\
&= \frac{8\pi}{3}\,\epsilon_0 V_0^2 R.
\end{aligned}
\tag{37}
$$

Nota bene: Any one of these two methods is sufficient for the exam, I do not expect the students to do both in the limited time they have.

Problem **4**:

Using the separation of variables method, we start by looking for potentials of the form

$$V(x,y,z) \;=\; f(x)\times g(y)\times h(z) \tag{38}$$

which obey the Laplace equation and the $V=0$ boundary conditions on the 4 sides and the bottom of the cubic box, but do not impose any boundary condition at the top. As explained in class, the $f(x)$, $g(y)$, and $h(z)$ functions obey

$$f''(x) \;+\; A_1 f(x) \;=\; 0 \quad\text{and}\quad f(0) \;=\; f(a) \;=\; 0, \tag{39}$$

$$g''(y) \;+\; A_2 g(y) \;=\; 0 \quad\text{and}\quad g(0) \;=\; g(a) \;=\; 0, \tag{40}$$

$$h''(z) \;+\; A_3 h(z) \;=\; 0 \quad\text{and}\quad h(0) \;=\; 0, \tag{41}$$

for some constants A_1, A_2, A_3 which add up to zero,

$$A_1 \;+\; A_2 \;+\; A_3 \;=\; 0. \tag{42}$$

The solutions to these equations have form

$$f(x) \;=\; \sin\frac{m\pi x}{a} \quad\text{for some integer } m,\quad A_1 \;=\; +\left(\frac{m\pi}{a}\right)^2, \tag{43}$$

$$g(y) \;=\; \sin\frac{n\pi y}{a} \quad\text{for some integer } n,\quad A_2 \;=\; +\left(\frac{n\pi}{a}\right)^2, \tag{44}$$

$$h(z) \;=\; \sinh(\kappa_{m,n} z), \tag{45}$$

$$\text{where } \kappa_{m,n} \;=\; \sqrt{-A_3 = A_1 + A_2} \;=\; \frac{\pi}{a}\sqrt{m^2+n^2}\,. \tag{46}$$

Altogether, we have an infinite series (or rather a double series in m and in n) of solutions of the separated form (38), so the generic solution is

$$V(x,y,z) \;=\sum_{m,n} C_{m,n}\sin(m\pi x/a)\sin(n\pi y/a)\sinh(\kappa_{m,n} z) \tag{47}$$

for some constant coefficients $C_{m,n}$. The specific values of these coefficients follow from the boundary condition at the top side of the cubic box:

$$V(x,y;z=a) \;=\; \sum_{m,n} C_{m,n}\sinh(\kappa_{m,n}a)\times\sin(m\pi x/a)\sin(n\pi y/a) \;=\; \text{given } V_b(x,y). \qquad (48)$$

In general, this calls for the double Fourier transform of the boundary potential,

$$C_{m,n}\times\sinh(\kappa_{m,n}a) \;=\; \frac{4}{a^2}\iint\limits_0^a dx\,dy\,V_b(x,y)\times\sin(m\pi x/a)\sin(n\pi y/a). \qquad (49)$$

However, for the problem at hand the boundary potential

$$V_b(x,y) \;=\; V_0\,\sin(3\pi x/a)\sin(4\pi y/a) \qquad (50)$$

look precisely like a single term ($m=3$, $n=4$) in the series (48), so there is no need for the Fourier transform. Instead, we simply let

$$C_{3,4}\times\sinh(\kappa_{3,4}a) \;=\; V_0, \quad \text{all other } C_{m,n} \;=\; 0. \qquad (51)$$

Moreover, for the cubic box

$$\kappa_{3,4} \;=\; \frac{\pi}{a}\sqrt{3^2+4^2} \;=\; \frac{\pi}{a}\times 5, \qquad (52)$$

so

$$C_{3,4} \;=\; \frac{V_0}{\sinh(5\pi)} \quad \text{while all other } C_{m,n} \;=\; 0. \qquad (53)$$

Thus altogether,

$$V(x,y,z) \;=\; \frac{V_0}{\sinh(5\pi)}\times\sin(3\pi x/a)\times\sin(4\pi y/a)\times\sinh(5\pi z/a). \qquad (54)$$

Problem $\mathbf{4}(\star)$:

The surface charge density $\sigma(x,y)$ on the conducting bottom square of the cubic box obtains from the electric field immediately above the bottom,

$$\sigma(x,y) \;=\; \epsilon_0 E_z(x,y,z\to 0) \;=\; -\epsilon_0 \left|\frac{\partial V}{\partial z}\right|_{z\to+0}. \tag{55}$$

For the potential (54),

$$\frac{\partial V}{\partial z} \;=\; \frac{V_0}{\sinh(5\pi)} \times \sin(3\pi x/a) \times \sin(4\pi y/a) \times (5\pi/a)\cosh(5\pi z/a), \tag{56}$$

where

$$\cosh(5\pi z/a) \;\to\; 1 \quad \text{for } z\to 0. \tag{57}$$

Consequently,

$$\sigma(x,y) \;=\; -\frac{5\pi}{\sinh(5\pi)} \times \frac{\epsilon_0 V_0}{a} \times \sin(3\pi x/a) \times \sin(4\pi y/a) \tag{58}$$

where

$$\frac{5\pi}{\sinh(5\pi)} \;\approx\; 2.37\cdot 10^{-6}. \tag{59}$$