Problem **1**:

(a) The $\ell = 0$ multipole moment — the net charge — obviously vanishes for the ring plus point charge system in question. The next multipole moment for $\ell = 1$ — the dipole moment — also happens to vanish. Indeed, the p_z component of the dipole moment vanishes because all charges lie in the XY plane and have $z = 0$. And at the same time, the axial symmetry of the system — invariance under rotations around the Z axis — guarantees that the p_x and the p_y components of the dipole moment happen to vanish.

In part (b) we shall see the system has a non-zero quadrupole moment, so that is the leading $\ell = 2$ multipole of the system.

(b–c) For any axially symmetric system, the quadrupole moment tensor has form

$$\mathcal{Q}_{i,j} \;=\; \mathcal{Q}_{z,z}\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & +1 \end{pmatrix}, \tag{1}$$

and the potential it generates has form

$$V_{\text{quadrupole}}(r,\theta,\phi) \;=\; \frac{\sum_{i,j}\mathcal{Q}_{i,j}\hat{r}_i\hat{r}_j}{4\pi\epsilon_0\, r^3} \;=\; \frac{\mathcal{Q}_{z,z}}{4\pi\epsilon_0}\,\frac{P_2(\cos\theta)}{r^3} \;=\; \frac{\mathcal{Q}_{z,z}}{4\pi\epsilon_0}\,\frac{3\cos^2\theta - 1}{2r^3}\,, \tag{2}$$

So all we need to calculate is the $\mathcal{Q}_{z,z}$ component of the quadrupole moment. In general,

$$\mathcal{Q}_{z,z} \;=\; \sum\int dQ\,\left(\tfrac{3}{2}z^2 \;-\; \tfrac{1}{2}r^2\right), \tag{3}$$

where the sum is over the charged bodies of the system, and the integral for each body is over its volume, area, or length, depending on the body's geometry. The system at hand comprises 2 bodies — the point charge and the ring, — thus

$$\mathcal{Q}_{z,z} \;=\; (-Q)\times\left(\tfrac{3}{2}z^2 \,-\, \tfrac{1}{2}r^2\right)_{\rm point} \,+\, \int\limits_{\rm ring}\left(\tfrac{3}{2}z^2 \,-\, \tfrac{1}{2}r^2\right)\times\frac{+Q}{2\pi}\,d\phi. \tag{4}$$

The first term here vanishes since the point charge lies at $z = r = 0$, while for the ring

$$\left(\tfrac{3}{2}z^2 \,-\, \tfrac{1}{2}r^2\right) \;=\; -\tfrac{1}{2}R^2 \;=\; \text{const}, \tag{5}$$

hence in the second term

$$\int\limits_{\rm ring}\left(\tfrac{3}{2}z^2 \,-\, \tfrac{1}{2}r^2\right)\times\frac{+Q}{2\pi}\,d\phi \;=\; -\frac{R^2}{2}\times\int\limits_0^{2\pi}\frac{+Q}{2\pi}\,d\phi \;=\; -\frac{R^2}{2}\times(+Q). \tag{6}$$

Altogether,

$$\mathcal{Q}_{z,z} \;=\; -\frac{QR^2}{2}\,, \tag{7}$$

which generates the leading-multipole potential

$$V(r,\theta,\phi) \;=\; -\frac{QR^2}{8\pi\epsilon_0}\,\frac{3\cos^2\theta-1}{2r^3}\,. \tag{8}$$

Problem **2**:

(a) First, in light of the cylindrical symmetry of the system — rotations around the Z axis and translations along that axis — all 3 vector field $\mathbf{E}$, $\mathbf{P}$, and $\mathbf{D}$ must point in the radial direction $\hat{\mathbf{s}}$ of the cylindrical coordinate system (s,ϕ,z), while their magnitudes should depend only on the cylindrical radius s,

$$\begin{aligned}
\mathbf{E}(s,\phi,z) \;&=\; E(s\,\text{only})\,\hat{\mathbf{s}}\,,\\
\mathbf{P}(s,\phi,z) \;&=\; P(s\,\text{only})\,\hat{\mathbf{s}}\,,\\
\mathbf{D}(s,\phi,z) \;&=\; D(s\,\text{only})\,\hat{\mathbf{s}}\,.
\end{aligned} \tag{9}$$

Second, thanks to this symmetry, the magnitude $D(s)$ easily obtains from the Gauss Law:

$$2\pi s \times L \times D(s) \;=\; Q^{\rm net}_{\rm free}[\text{inside } s] \;=\; \begin{cases} 0 & \text{for } s < a, \\ +Q & \text{for } a < s < b, \\ 0 & \text{for } s > b. \end{cases} \tag{10}$$

Thus, inside the inner tube, — or outside the outer tube, — we have $\mathbf{D} = 0$, and since the dielectric is linear this also means $\mathbf{E} = 0$ and $\mathbf{P} = 0$. However, we do have non-zero fields between the two tubes, for $a < s < b$. Specifically,

$$D(s) \;=\; \frac{Q}{2\pi L s}\,, \tag{11}$$

$$E(s) \;=\; \frac{D(s)}{\epsilon(s)\epsilon_0} \;=\; \frac{Q}{2\pi\epsilon_0\, Ls} \Big/ \frac{b^2}{s^2} \;=\; \frac{Qs}{2\pi\epsilon_0\, Lb^2}\,, \tag{12}$$

$$P(s) \;=\; D(s) \;-\; \epsilon_0 E(s) \;=\; \frac{Q}{2\pi L}\left(\frac{1}{s} \;-\; \frac{s}{b^2}\right). \tag{13}$$

(b) The voltage between the tubes obtains by integrating the radial component of the $\mathbf{E}$ field:

$$V \;=\; \int\limits_a^b E(s)\,ds \;=\; \int\limits_a^b \frac{Qs}{2\pi\epsilon_0 Lb^2}\,ds \;=\; \frac{Q}{2\pi\epsilon_0 Lb^2}\int\limits_a^b s\,ds \;=\; \frac{Q}{2\pi\epsilon_0 Lb^2} \times \frac{b^2-a^2}{2}\,. \tag{14}$$

Consequently,

$$\frac{1}{C} \;=\; \frac{V}{Q} \;=\; \frac{1}{4\pi\epsilon_0 L} \times \frac{b^2-a^2}{b^2} \tag{15}$$

and hence the capacitance

$$C \;=\; 4\pi\epsilon_0 L \times \frac{b^2}{b^2-a^2}\,. \tag{16}$$

Finally, the net electrostatic energy of this capacitor is

$$U \;=\; \frac{Q^2}{2C} \;=\; \frac{Q^2}{8\pi\epsilon_0 L} \times \frac{b^2-a^2}{b^2}\,. \tag{17}$$

⋆ The bound charge density on the outer surface of the dielectric (at $s = b$) is

$$\begin{aligned}\sigma_b(\text{outer}) &= \mathbf{n}_{\text{outer}} \cdot \mathbf{P}(s = b) \;=\; +\hat{\mathbf{z}} \cdot \mathbf{P}(s = b) \;=\; +P(s = b) \\ &= \frac{Q}{2\pi L}\left(\frac{1}{b} - \frac{b}{b^2}\right) \;=\; 0.\end{aligned} \tag{18}$$

The bound charge density on the inner surface of the dielectric (at $s = a$) is

$$\begin{aligned}\sigma_b(\text{inner}) &= \mathbf{n}_{\text{inner}} \cdot \mathbf{P}(s = a) \;=\; -\hat{\mathbf{z}} \cdot \mathbf{P}(s = a) \;=\; -P(s = a) \\ &= -\frac{Q}{2\pi L}\left(\frac{1}{a} - \frac{a}{b^2}\right) \;=\; -\frac{Q(b^2 - a^2)}{2\pi L a b^2}\,.\end{aligned} \tag{19}$$

The volume charge density in the bulk of the dielectric is

$$\begin{aligned}\rho_b &= -\nabla \cdot (\mathbf{P} \;=\; P(s)\hat{\mathbf{s}}) \;=\; -\left(\frac{dP}{ds} + \frac{P}{s}\right) \\ &= -\frac{Q}{2\pi L}\left(\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{b^2}\right) + \frac{1}{s}\left(\frac{1}{s} - \frac{s}{b^2}\right)\right) \\ &= -\frac{Q}{2\pi L}\left(-\frac{1}{s^2} - \frac{1}{b^2} + \frac{1}{s^2} - \frac{1}{b^2}\right) \\ &= +\frac{2Q}{2\pi L b^2}\,.\end{aligned} \tag{20}$$

Finally, let's calculate the net bound charge

$$Q_b^{\text{net}} \;=\; Q_b(\text{outer}) \;+\; Q_b(\text{inner}) \;+\; Q_b(\text{bulk}), \tag{21}$$

where

$$Q_b(\text{outer}) \;=\; 2\pi L b \times \sigma_b(\text{outer}) \;=\; 2\pi L b \times 0 \;=\; 0, \tag{22}$$

$$Q_b(\text{inner}) \;=\; 2\pi L a \times \sigma_b(\text{inner}) \;=\; 2\pi L a \times \frac{-Q(b^2 - a^2)}{2\pi L a b^2} \;=\; -Q\,\frac{b^2 - a^2}{b^2}\,, \tag{23}$$

while

$$
\begin{aligned}
Q_b(\text{bulk}) &= \iiint \rho_b \, d^3\text{Vol} \;=\; \int_a^b \rho_b(s) \times 2\pi L s \, ds \\
&= \int_a^b \frac{2Q}{2\pi L b^2} \times 2\pi L s \, ds \;=\; \frac{Q}{b^2} \int_a^b 2s \, ds \\
&= \frac{Q}{b^2}(b^2 - a^2)\,.
\end{aligned} \tag{24}
$$

Altogether, the net bound charge (21) of the dielectric amounts to

$$
Q_b^{\text{net}} \;=\; 0 \;-\; Q\frac{b^2 - a^2}{b^2} \;+\; Q\frac{b^2 - a^2}{b^2} \;=\; 0. \tag{25}
$$

Problem **3**:

(a) The magnetic force on the second wire stems from the magnetic field of the first wire. For an infinite straight wire along the z axis, the magnetic field is

$$
\mathbf{B}_1 \;=\; \frac{\mu_0 I_1}{2\pi} \frac{\hat{\phi}}{s}\,, \tag{26}
$$

or in Cartesian coordinates

$$
\mathbf{B}_1(x,y,z) \;=\; \frac{\mu_0 I_1}{2\pi} \frac{x\,\hat{\mathbf{y}} \;-\; y\,\hat{\mathbf{x}}}{x^2 + y^2}\,. \tag{27}
$$

The force of this field on an element $d\ell$ of the second wire is

$$
d\mathbf{F} \;= I_2 \, d\vec{\ell} \times \mathbf{B}_1 \tag{28}
$$

where the vector $d\vec{\ell}$ points along the wire's direction. In light of eq. (E.2) for the wire,

$$
d\vec{\ell} \;=\; d\ell\,(\sin\theta\,\hat{\mathbf{y}} \;+\; \cos\theta\,\hat{\mathbf{z}}), \tag{29}
$$

hence

$$
\begin{aligned}
d\mathbf{F} \;&=\; I_2\, d\vec{\ell} \times \mathbf{B}_1 \\
&=\; I_2\, d\ell\, (\sin\theta\,\hat{\mathbf{y}} \,+\, \cos\theta\,\hat{\mathbf{z}}) \times \frac{\mu_0 I_1}{2\pi}\,\frac{x\,\hat{\mathbf{y}} \,-\, y\,\hat{\mathbf{x}}}{x^2+y^2} \\
&=\; \frac{\mu_0 I_1 I_2}{2\pi(x^2+y^2)}\, d\ell \left(\begin{array}{c} x\sin\theta(\hat{\mathbf{y}}\times\hat{\mathbf{y}} \,=\, 0) \,+\, x\cos\theta(\hat{\mathbf{z}}\times\hat{\mathbf{y}} \,=\, -\hat{\mathbf{x}}) \\ -\,y\sin\theta(\hat{\mathbf{y}}\times\hat{\mathbf{x}} \,=\, -\hat{\mathbf{z}}) \,-\, y\cos\theta(\hat{\mathbf{z}}\times\hat{\mathbf{x}} \,=\, +\hat{\mathbf{y}}) \end{array} \right).
\end{aligned}
\tag{30}
$$

In this formula, x and y are coordinates of the appropriate point of the second wire. According to eq. (E.2),

$$
x \;=\; a \;\langle\!\langle\,\text{fixed}\,\rangle\!\rangle, \quad y \;=\; \sin\theta\times\ell,
\tag{31}
$$

hence

$$
d\mathbf{F} \;=\; \frac{\mu_0 I_1 I_2}{2\pi(a^2 \,+\, \ell^2\sin^2\theta)}\, d\ell \Big(-a\cos\theta\,\hat{\mathbf{x}} \,+\, \ell\sin\theta\,(\sin\theta\,\hat{\mathbf{z}} - \cos\theta\,\hat{\mathbf{y}}) \Big).
\tag{32}
$$

(b) Now let's integrate the force elements (32) to get the net force on the second wire:

$$
\begin{aligned}
\mathbf{F}_{\text{net}} \;&=\; \int\limits_{\ell=-\infty}^{\ell=+\infty} d\mathbf{F} \;=\; \int\limits_{-\infty}^{+\infty} \frac{\mu_0 I_1 I_2}{2\pi(a^2 \,+\, \ell^2\sin^2\theta)} \Big(-a\cos\theta\,\hat{\mathbf{x}} \,+\, \ell\sin\theta(\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{y}}) \Big)\, d\ell \\
&=\; \frac{\mu_0 I_1 I_2}{2\pi}\,(-a\cos\theta\,\hat{\mathbf{x}}) \int\limits_{-\infty}^{+\infty} \frac{d\ell}{a^2 \,+\, \sin^2\theta\times\ell^2} \\
&\qquad +\; \frac{\mu_0 I_1 I_2}{2\pi}\,\sin\theta(\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{y}}) \int\limits_{-\infty}^{+\infty} \frac{\ell\, d\ell}{a^2 \,+\, \sin^2\theta\times\ell^2} \\
&\qquad \langle\!\langle\,\text{using integrals (E.3)}\,\rangle\!\rangle \\
&=\; \frac{\mu_0 I_1 I_2}{2\pi} * (-a\cos\theta\,\hat{\mathbf{x}}) * \frac{\pi}{a\,\sin\theta} \;+\; \frac{\mu_0 I_1 I_2}{2\pi} * \sin\theta(\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{y}}) * 0 \\
&=\; \frac{\mu_0 I_1 I_2}{2\tan\theta}\,(-\hat{\mathbf{x}}).
\end{aligned}
\tag{33}
$$

Note: when $\theta = 0$ or $\theta = 180°$, the force between the two wires become infinite. Indeed, for $\theta = 0$ or $\theta = 180°$, the two wires are parallel, so the force per unit length does not diminish with $\ell \to \pm\infty$, hence infinite net force.

As to the direction of the force (33), it's $-\hat{\mathbf{x}}$ for $\theta < 90^\circ$ and $+\hat{\mathbf{x}}$ for $\theta > 90^\circ$. In both cases, the force is along the shortest distance between the two wires; it is attractive for $\theta < 90^\circ$ and repulsive for $\theta > 90^\circ$.

Problem **4**:

(a) A steady current density must have zero divergence. Let's verify that for the current density (E.4):

$$\nabla\cdot\mathbf{J} \;=\; \frac{\partial J_x}{\partial x} \;+\; \frac{\partial J_y}{\partial y} \;+\; \frac{\partial J_z}{\partial z} \;=\; 0 \;+\; 0 \;+\; 0 \;=\; 0, \tag{34}$$

where the first two terms vanish because the current does not depend on x or y, and the third term vanish because the current does not have a z component, $J_z = 0$. Thus, the current density (E.4) indeed has zero divergence, which allows it to be steady. (We cannot prove its steadiness from the eq. (E.4) alone, but at least it can be steady.)

(b) First, the translational symmetries of the current (E.4) in the x and y directions make the vector potential independent of the x and y coordinates, thus

$$\mathbf{A}(x,y,z) \;=\; \mathbf{A}(z\text{ only}). \tag{35}$$

Second, the current (E.4) is symmetric under 180° rotation around the x axis, so on that axis — and hence for $z=0$ but any x and y — the vector potential should point in the $\hat{\mathbf{x}}$ direction,

$$\mathbf{A}(z=0) \;=\; A_0\hat{\mathbf{x}} \tag{36}$$

for some constant A_0. Finally, the current (E.4) has helical symmetry — that is, it's invariant under simultaneous translations in the z directions and rotations around the z axis,

$$\mathbf{J}(z+\Delta z) \;=\; \text{Rotate}[\text{angle}=k\Delta z,\text{axis}=\hat{\mathbf{z}}]\big(\mathbf{J}(z)\big). \tag{37}$$

In particular,

$$\mathbf{J}(z) \;=\; \text{Rotate}[\text{angle}=kz,\text{axis}=\hat{\mathbf{z}}]\big(\mathbf{J}(0)\big). \tag{38}$$

The vector potential should have a similar symmetry, hence

$$\mathbf{A}(z) = \text{Rotate}[\text{angle} = kz, \text{axis} = \hat{\mathbf{z}}]\big(\mathbf{A}(0) = A_0\hat{\mathbf{x}}\big) = A_0\left(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\right). \tag{39}$$

Quod erat demonstrandum.

(c) In general, the Ampere's Law for the **B** field translates to the equation

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \tag{40}$$

for the vector potential **A**. But the vector potential (E.5) has obviously zero divergence — for the same reasons the current (E.4) has zero divergence — so eq. (40) becomes the Poisson equation

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}. \tag{41}$$

Let's verify this Poisson equation for the current (E.4) and the potential (E.5).

Since the potential (E.5) depends only on the z coordinate, its Laplacian is simply its second derivative,

$$\nabla^2\mathbf{A}(z) = \frac{d^2}{dz^2}\,\mathbf{A}(z). \tag{42}$$

Calculating this derivative components we have

$$\nabla^2 J_x = \frac{d^2}{dz^2}\big(a\,\cos(kz)\big) = -A_0k^2\,\cos(kz), \tag{43}$$

$$\nabla^2 J_y = \frac{d^2}{dz^2}\big(a\,\sin(kz)\big) = -A_0k^2\,\sin(kz), \tag{44}$$

$$\nabla^2 J_z = 0, \tag{45}$$

thus in vector notations

$$\nabla^2\mathbf{J}(z) = -A_0k^2\big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\big). \tag{46}$$

Comparing this formula to the current density (E.4), we see that indeed

$$\nabla^2\mathbf{A}(z) = -\mu_0\mathbf{J}(z) \quad \text{provided} \quad A_0k^2 = \mu_0J_0. \tag{47}$$

This nails down the overall coefficient A_0 of the vector potential (E.5),

$$A_0 \;=\; \frac{\mu_0 J_0}{k^2}\,. \tag{48}$$

(d) Given the vector potential $\mathbf{A}(\mathbf{r})$, the magnetic field obtains as its curl $\mathbf{B} = \nabla\times\mathbf{A}$. For the potential (E.5) which depends only on the z coordinate, the curl simplifies to

$$\begin{aligned}
B_x &= -\frac{dA_y}{dz} = -\frac{d\big(A_0\,\sin(kz)\big)}{dz} = -A_0k\,\cos(kz), &(49)\\
B_y &= +\frac{dA_x}{dz} = +\frac{d\big(A_0\,\cos(kz)\big)}{dz} = -A_0k\,\sin(kz), &(50)\\
B_z &= 0, &(51)
\end{aligned}$$

or in vector notations

$$\mathbf{B}(z) \;=\; -A_0k\big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\big) \;=\; -\frac{\mu_0 J_0}{k}\,\big(\cos(kz)\,\hat{\mathbf{x}} + \sin(kz)\,\hat{\mathbf{y}}\big). \tag{52}$$