ELECTRIC MAGNETIC DUALITY

Consider the EM fields (or any other kind of $U(1)$ gauge fields) coupled to both electric and magnetic charges and currents. The classical Maxwell equations

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= J^\nu_{\text{electric}}\,, \\ \partial_\mu \tilde{F}^{\mu\nu} &= J^\nu_{\text{magnetic}}\,, \\ \text{where } \tilde{F}^{\mu\nu} &\stackrel{\text{def}}{=} \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \end{aligned} \tag{1}$$

— or in 3D terms

$$\begin{aligned} \nabla\cdot\mathbf{E} &= J^0_{\text{el}}\,, \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} - \mathbf{J}_{\text{mag}}\,, \\ \nabla\cdot\mathbf{B} &= J^0_{\text{mag}}\,, \\ \nabla\times\mathbf{B} &= +\frac{\partial\mathbf{E}}{\partial t} + \mathbf{J}_{\text{el}}\,, \end{aligned} \tag{2}$$

— allow us to swap the electric and the magnetic fields and currents with each other,

$$\begin{gathered} F'_{\mu\nu} = \tilde{F}_{\mu\nu} \iff \mathbf{E}' = +\mathbf{B}, \quad \mathbf{B}' = -\mathbf{E}, \\ \text{and also} \quad J'^\mu_{\text{el}} = +J^\mu_{\text{mag}}\,, \quad J'^\mu_{\text{mag}} = -J^\mu_{\text{el}}\,. \end{gathered} \tag{3}$$

Note however, that this *electric-magnetic duality* is a symmetry of the Maxwell equations but not of the classical Lagrangian, let alone of the quantum field theory. Indeed, even for the free theory with $J^\mu_{\text{el}} = J^\mu_{\text{mag}} = 0$, the original theory has $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, while the dual theory has

$$\widetilde{\mathcal{L}} = -\tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = +\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{for} \quad \tilde{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu\,. \tag{4}$$

Consequently, the Bianchi identity of one theory becomes the equation of motion of the other theory and vice verse. And as quantum field theories, the original and the dual theories exist in different Hilbert spaces, involve different operators not related by any symmetries, *etc.*; however, the two theories are *physically equivalent* to each other.

For the non-free EM theories — *i.e.*, in presence of electric and magnetic charges and currents, — the dual theory also has a different coupling constant

$$\alpha' = \frac{1}{\alpha}. \tag{5}$$

Indeed, the electric and the magnetic charges of a quantum theory are quantized in different units,

$$Q_{\text{el}} = n \times e, \quad M_{\text{mag}} = m \times \frac{4\pi}{e}, \quad 2n, m \in \mathbf{Z}, \tag{6}$$

so the electric Coulomb potential is

$$V(r) = \frac{Q_1 Q_2}{4\pi r} = \frac{n_1 n_2 e^2}{4\pi r} = \frac{\alpha}{r} \times n_1 n_2\,, \tag{7}$$

while the magnetic Coulomb potential is

$$V(r) = \frac{M_1 M_2}{4\pi r} = \frac{m_1 m_2 (4\pi/e)^2}{4\pi r} = \frac{1/\alpha}{r} \times m_1 m_2\,. \tag{8}$$

The EM duality swaps the electric and the magnetic charges, but to preserve the charge quantization it should act as

$$n' = m, \quad m' = -n \tag{9}$$

rather than

$$Q' = M, \quad M' = -Q, \tag{10}$$

which amounts to a charge rescaling. At the same time, to preserve the Coulomb potential, this charge rescaling must be compensated by changing the QED coupling constant according to

$$\alpha' = \frac{1}{\alpha}. \tag{5}$$

Thus, the electric-magnetic duality relates a weakly coupled theory with $\alpha \ll 1$ to a strongly coupled theory with $\alpha' \gg 1$ or vice verse.

In particular, in the weakly coupled theory like real-life QED, the electrically charged particles may appear elementary, like the real-life electrons. On the other hand, the magnetically charged particles arise as large semiclassical configurations of heavy fields. By large, I mean the configuration's geometric size R is much larger than its Compton wavelength $1/M$, thus $RM \gg 1$. Indeed, the 't Hooft–Polyakov 'hedgehog' monopole arising from the $SU(2)$ Higgsed down to $U(1)$ has geometric size $R \sim 1/M_W$ and mass $M \sim M_w/\alpha$, thus $RM \sim (1/\alpha) \gg 1$.

But in the strongly coupled regime of $\alpha \gg 1$, the electrons no longer look elementary; instead, they are surrounded with dense clouds of virtual photons, e^+e^- pairs, *etc.*, so they look like big fat composite objects with $MR \gg 1$. On the other hand the monopoles shrink in size (compared to their Compton wavelengths), and with luck may become approximately pointlike.

And of course, the best way to investigate the strongly coupled QED-like theory is to look at its EM-dual theory with a perturbatively dual coupling $\alpha' \ll 1$. In this dual theory, the electric charges — which are dual to the original magnetic charges — may indeed be approximately pointlike particles, while the magnetic charges — which are dual to the original electric charges — become big fat semiclassical configs of some heavy fields.

Thanks to the electric-magnetic duality, we understand both the weakly-coupled and the strongly-coupled regimes of QED-like abelian gauge theories. Unfortunately, the duality does not help with the mid-coupling regime of $\alpha \sim 1$, because in this regime both the original and the dual theories are too strongly coupled for the perturbation theory. Also, for $\alpha \sim 1$ neither electric nor magnetic charges look approximately point-like, so we may not treat them as elementary fields of some perturbative QFT.

Effect of the Theta Angle

In the QED-like theories without any magnetic charges,

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \;=\; -4\mathbf{E}\cdot\mathbf{B} \;=\; 2\partial_\mu\bigl(A_\nu\tilde{F}^{\mu\nu}\bigr) \tag{11}$$

is a total spacetime derivative, hence adding such term to the Lagrangian has no effect, perturbatively or even non-perturbatively. But in a theory which does have magnetic charges we have

$$\partial_\mu \tilde{F}^{\mu\nu} = J^\nu_{\rm mag} \neq 0 \quad \Longrightarrow \quad F_{\mu\nu}\tilde{F}^{\mu\nu} \neq \partial_\mu(\text{anything}), \tag{12}$$

so the Θ term in the Lagrangian does have a non-trivial effect, even classically. Specifically, Θ changes the electric charges of the magnetic monopoles or dyons.

To see how this works, let's write the EM Lagrangian in terms of the $U(1)$-normalized gauge fields and currents,

$$\mathcal{A}^\mu(x) = eA^\mu(x), \quad \mathcal{F}^{\mu\nu}(x) = eF^{\mu\nu}(x), \quad J^\mu[\text{electron}] = -\overline{\Psi}\gamma^\mu\Psi \quad \langle\!\langle \text{without } e \text{ factor} \rangle\!\rangle, \tag{13}$$

thus

$$\mathcal{L} = \frac{-1}{4e^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\Theta}{32\pi^2}\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu} - J_\mu\mathcal{A}^\mu, \tag{14}$$

or in 3D terms

$$\mathcal{L} = \frac{1}{2e^2}\big(\vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2\big) - \frac{\Theta}{8\pi^2}\,\vec{\mathcal{E}}\cdot\vec{\mathcal{B}} - J^0\mathcal{A}^0 + \mathbf{J}\cdot\vec{\mathcal{A}}. \tag{15}$$

For future convenience, let's define

$$\vec{\mathcal{C}} \stackrel{\rm def}{=} 4\pi\,\frac{\partial\mathcal{L}}{\partial\vec{\mathcal{E}}} = \frac{1}{\alpha}\,\vec{\mathcal{E}} - \frac{\Theta}{2\pi}\,\vec{\mathcal{B}}. \tag{16}$$

Then the Gauss Law obtains from the Lagrangian (15) as the time-independent equation

$$\frac{\partial\mathcal{L}}{\partial\mathcal{A}^0} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\mathcal{A}^0)} = 0, \tag{17}$$

hence

$$\nabla\cdot\vec{\mathcal{C}} = 4\pi J^0. \tag{18}$$

Now consider a point particle with electric charge $Q = n \times e$ and magnetic charge

$M = m \times (4\pi/e)$. In terms of the $U(1)$-normalized fields and charges, $Q = n \times e$ means

$$J^0(\mathbf{x}) \;=\; n\delta^{(3)}(\mathbf{x}) \quad \langle\!\langle\, \text{without the } e \text{ factor} \,\rangle\!\rangle, \tag{19}$$

hence by the Gauss Law

$$\nabla \cdot \vec{\mathcal{C}}(\mathbf{x}) \;=\; 4\pi n\delta^{(3)}(\mathbf{x}). \tag{20}$$

At the same time, the magnetic point charge M means

$$\nabla \cdot \vec{\mathcal{B}}(x) \;=\; e\nabla \cdot \mathbf{B}(\mathbf{x}) \;=\; eM\delta^{(3)}(\mathbf{x}) \;=\; 4\pi m\delta^{(3)}(\mathbf{x}), \tag{21}$$

hence by eq. (16)

$$\nabla \cdot \vec{\mathcal{E}} \;=\; \nabla \cdot \left(\alpha\vec{\mathcal{C}} + \frac{\alpha\Theta}{2\pi}\vec{\mathcal{B}}\right) \;=\; 4\pi\alpha\left(n + \frac{\Theta}{2\pi}m\right)\delta^{(3)}(\mathbf{x}). \tag{22}$$

Or in terms of the canonically normalized electric field $\mathbf{E} = (1/e)\vec{\mathcal{E}}$,

$$\nabla \cdot \mathbf{E} \;=\; e\left(n + \frac{\Theta}{2\pi}m\right)\delta^{(3)}(\mathbf{x}). \tag{23}$$

In other words, the point particle in question has *apparent electric charge*

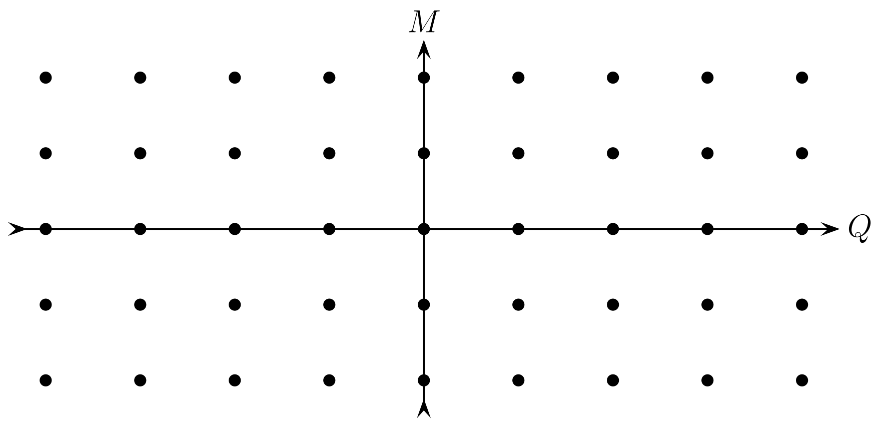
$$Q \;=\; e \times \left(n + \frac{\Theta}{2\pi} \times m\right). \tag{24}$$

Thus, for generic values of the Θ angle, the apparent electric charges of the magnetic monopoles or dyons are irrational in units of e! Only for $\Theta \equiv 0 \pmod{2\pi}$ do the electric charges become integer (in units of e) for all the magnetically charged particles.

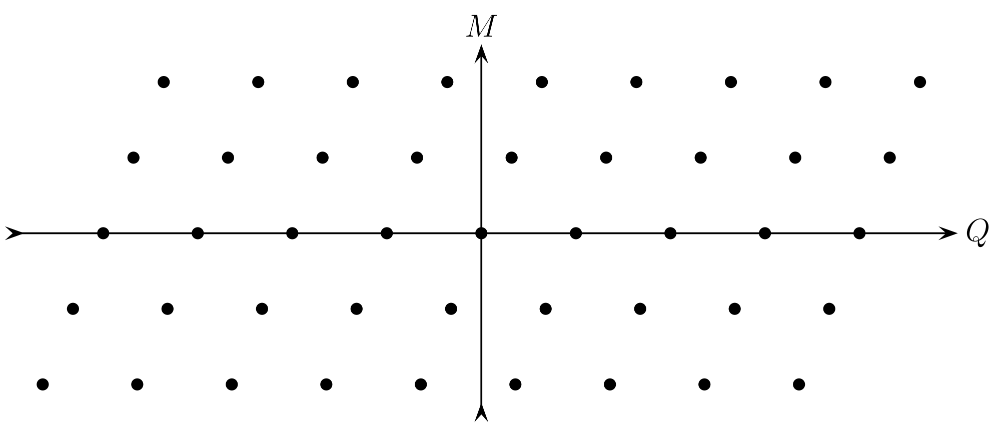
But despite the non-integer electric charges (24) of the magnetic monopoles or dyons, they are consistent with the Dirac's charge quantization condition: for any two particles with integer m_1 and m_2 and integer or half-integer n_1 and n_2, their electric and magnetic charges obey

$$\begin{aligned} Q_1 \times M_2 \ - \ Q_2 \times M_1 \ &= \ e\left(n_1 \,+\, \frac{\Theta}{2\pi}\,m_1\right) \times \frac{4\pi m_2}{e} \ - \ e\left(n_2 \,+\, \frac{\Theta}{2\pi}\,m_2\right) \times \frac{4\pi m_1}{e} \\ &= \ 4\pi\bigl(n_1 \times m_2 \ - \ n_2 \times m_1\bigr) \quad \langle\!\langle \,\text{regardless of } \Theta \,\rangle\!\rangle \\ &= \ 2\pi \times \text{an integer}. \end{aligned} \tag{25}$$

Now consider the 2D space of electric and magnetic charges of all the particles — massless or massive, elementary or composite, stable or unstable, whatever. The charge quantization makes this space a 2D periodic lattice: a rectangular lattice for $\Theta \equiv 0 \pmod{2\pi}$:



but a *tilted lattice* for $\Theta \not\equiv 0 \pmod{2\pi}$:



The 2D plane spanned by the (Q, M) can be thought as a complex plane spanned by the complex charges

$$\begin{aligned} Z &\stackrel{\text{def}}{=} Q + iM \\ &= e \times \left(n + \frac{\Theta}{2\pi} m\right) + \frac{4\pi i}{e} \times m = n \times e + m \times e \times \left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\right) \\ &= e \times \left(n + m \times \tau\right) \end{aligned} \tag{26}$$

for

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{e^2}\,. \tag{27}$$

Note: in a supersymmetric theory, this τ is precisely the holomorphic gauge coupling of the $U(1)$ gauge theory!

The complex charges $Z = e(n + m\tau)$ are convenient for the complex components of the $F_{\mu\nu}$ tensor with definite (j_L, j_R) Lorentz quantum numbers:

$$\begin{aligned} j_L = 1, j_R = 0: \quad & F_{\alpha\beta} \propto \mathbf{E} + i\mathbf{B} = \frac{Z\mathbf{n}}{4\pi r^2}\,, \\ J_L = 0, J_R = 1: \quad & F_{\dot\alpha\dot\beta} \propto \mathbf{E} - i\mathbf{B} = \frac{Z^*\mathbf{n}}{4\pi r^2}\,. \end{aligned} \tag{28}$$

Canonical Quantization and the Group of EM Dualities

Let's start with the free EM fields — without electric or magnetic currents — but with the Θ angle,

$$\mathcal{L}_{\text{free}} = \frac{1}{2e^2}\big(\vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2\big) - \frac{\Theta}{8\pi^2}\,\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}. \tag{29}$$

Classically, treating $\vec{\mathcal{A}}(\mathbf{x})$ as independent dynamical variables, we get the canonically conjugate variables

$$\frac{\partial\mathcal{L}}{\partial(\partial_0\vec{\mathcal{A}})} = -\frac{\partial\mathcal{L}}{\partial\vec{\mathcal{E}}} = \frac{-1}{4\pi}\left(\vec{\mathcal{C}} = \frac{1}{\alpha}\vec{\mathcal{E}} - \frac{\Theta}{2\pi}\vec{\mathcal{B}}\right). \tag{30}$$

This gives us the classical Poisson brackets of the $\vec{\mathcal{C}}$ and $\vec{\mathcal{B}}$ fields,

$$\big[\mathcal{C}^i(\mathbf{x}), \mathcal{B}^j(\mathbf{y})\big]_P = 4\pi i\epsilon^{ijk}\frac{\partial}{\partial x^k}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{31}$$

and hence the equal-time commutators of the quantum fields:

$$
\begin{aligned}
\left[\hat{\mathcal{C}}^i(\mathbf{x}),\hat{\mathcal{C}}^j(\mathbf{y})\right] &= 0,\\
\left[\hat{\mathcal{B}}^i(\mathbf{x}),\hat{\mathcal{B}}^j(\mathbf{y})\right] &= 0,\\
\left[\hat{\mathcal{C}}^i(\mathbf{x}),\hat{\mathcal{B}}^j(\mathbf{y})\right] &= 4\pi i\epsilon^{ijk}\frac{\partial}{\partial x^k}\delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned}
\tag{32}
$$

Note that these commutation relations are invariant under symplectic linear transforms of the $\hat{\vec{\mathcal{C}}}$ and $\hat{\vec{\mathcal{B}}}$ fields into each other,

$$
\begin{pmatrix} \hat{\vec{\mathcal{C}}}'(x) \\ \hat{\vec{\mathcal{B}}}'(x) \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \begin{pmatrix} \hat{\vec{\mathcal{C}}}(x) \\ \hat{\vec{\mathcal{B}}}(x) \end{pmatrix}
\tag{33}
$$

for

$$
\begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \in \mathrm{Sp}(2,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R}),
\tag{34}
$$

$$
\text{meaning :}\quad \text{real } \alpha,\beta,\gamma,\delta,\quad \alpha\delta-\beta\gamma=1.
\tag{35}
$$

Now consider the free EM Hamiltonian. Classically,

$$
\begin{aligned}
\mathcal{H}_{\text{free}} &= \frac{1}{4\pi}\vec{\mathcal{C}}\cdot\vec{\mathcal{E}} - \mathcal{L}_{\text{free}}\\
&= \frac{1}{4\pi\alpha}\vec{\mathcal{E}}^2 - \frac{\Theta}{8\pi^2}\vec{\mathcal{E}}\cdot\vec{\mathcal{B}} - \frac{1}{2e^2}\left(\vec{\mathcal{E}}^2-\vec{\mathcal{B}}^2\right) + \frac{\Theta}{8\pi^2}\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\\
&= \frac{1}{2e^2}\left(\vec{\mathcal{E}}^2+\vec{\mathcal{B}}^2\right)^2 = \frac{1}{2e^2}\left|\vec{\mathcal{E}}+i\vec{\mathcal{B}}\right|^2
\end{aligned}
\tag{36}
$$

regardless of the Θ angle, but in terms of the canonical $\vec{\mathcal{B}}$ and $\vec{\mathcal{C}}$ field,

$$
\mathcal{H}_{\text{free}} = \frac{1}{8\pi\,\mathrm{Im}\,\tau}\left|\vec{\mathcal{C}}+\tau\vec{\mathcal{B}}\right|^2.
\tag{37}
$$

Indeed,

$$
\vec{\mathcal{C}}+\tau\vec{\mathcal{B}} = \frac{1}{\alpha}\vec{\mathcal{E}} - \cancel{\frac{\Theta}{2\pi}}\vec{\mathcal{B}} + \left(\cancel{\frac{\Theta}{2\pi}}+\frac{i}{\alpha}\right)\vec{\mathcal{B}} = \frac{1}{\alpha}\left(\vec{\mathcal{E}}+i\vec{\mathcal{B}}\right),
\tag{38}
$$

while

$$\frac{\alpha^2}{2e^2} = \frac{\alpha}{8\pi} = \frac{1}{8\pi\,\mathrm{Im}\,\tau}, \tag{39}$$

hence

$$\frac{1}{2e^2}\left|\vec{\mathcal{E}}+i\vec{\mathcal{B}}\right|^2 = \frac{\alpha^2}{2e^2}\left|\vec{\mathcal{C}}+\tau\vec{\mathcal{B}}\right|^2 = \frac{1}{8\pi\,\mathrm{Im}\,\tau}\left|\vec{\mathcal{C}}+\tau\vec{\mathcal{B}}\right|^2. \tag{40}$$

To keep the classical Hamiltonian density (37) — or its quantum counterpart

$$\widehat{\mathcal{H}}_{\text{free}} = \frac{1}{8\pi\,\mathrm{Im}\,\tau}\big(\hat{\vec{\mathcal{C}}} + \tau\hat{\vec{\mathcal{B}}}\big)^\dagger\cdot\big(\hat{\vec{\mathcal{C}}} + \tau\hat{\vec{\mathcal{B}}}\big) \tag{41}$$

— invariant under the symplectic transforms (33), we should accompany such transforms by changing the τ parameter to

$$\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}. \tag{42}$$

Note that this transform preserves the positivity of $\mathrm{Im}\,\tau = (4\pi/e^2)$. Indeed,

$$\mathrm{Im}\,\tau' = \frac{\mathrm{Im}\big[(\alpha\tau+\beta)(\gamma\tau^*+\delta)\big]}{|\gamma\tau+\delta|^2} = \frac{\mathrm{Im}\,\tau\times(\alpha\delta-\beta\gamma)}{|\gamma\tau+\delta|^2} = \frac{\mathrm{Im}\,\tau}{|\gamma\tau+\delta|^2}, \tag{43}$$

so if $\mathrm{Im}\,\tau>0$ then also $\mathrm{Im}\,\tau'>0$.

In the context of the Hamiltonian (37), the combination $\vec{\mathcal{C}}+\tau\vec{\mathcal{B}}$ transforms to

$$\begin{aligned}
\vec{\mathcal{C}}' + \tau'\times\vec{\mathcal{B}}' &= (\alpha\vec{\mathcal{C}} - \beta\vec{\mathcal{B}}) + \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\times(-\gamma\vec{\mathcal{C}}+\delta\vec{\mathcal{B}})\\
&= \frac{1}{\gamma\tau+\delta}\Big[(\gamma\tau+\delta)(\alpha\vec{\mathcal{C}}-\beta\vec{\mathcal{B}}) + (\alpha\tau+\beta)(-\gamma\vec{\mathcal{C}}+\delta\vec{\mathcal{B}})\Big]\\
&= \frac{1}{\gamma\tau+\delta}\Big[\vec{\mathcal{C}}\times\big(\cancel{\gamma\tau\alpha}+\delta\alpha-\cancel{\alpha\tau\gamma}-\beta\gamma\big) + \vec{\mathcal{B}}\times\big(-\gamma\tau\beta-\cancel{\delta\beta}+\alpha\tau\delta+\cancel{\beta\delta}\big)\Big]\\
&= \frac{1}{\gamma\tau+\delta}\Big[\big(\vec{\mathcal{C}}+\tau\vec{\mathcal{B}}\big)\times\big(\alpha\delta-\beta\gamma = 1\big)\Big]\\
&= \frac{\vec{\mathcal{C}}+\tau\vec{\mathcal{B}}}{\gamma\tau+\delta},
\end{aligned} \tag{44}$$

hence

$$\frac{\left|\vec{\mathcal{C}}' + \tau\vec{\mathcal{B}}'\right|^2}{\operatorname{Im}\tau'} = \left|\frac{\vec{\mathcal{C}} + \tau\vec{\mathcal{B}}}{\gamma\tau+\delta}\right|^2 \times \frac{|\gamma\tau+\delta|^2}{\operatorname{Im}\tau} = \frac{\left|\vec{\mathcal{C}} + \tau\vec{\mathcal{B}}\right|^2}{\operatorname{Im}\tau} \tag{45}$$

and therefore $\mathcal{H}' = \mathcal{H}$: The free Hamiltonian (37) is indeed invariant under the combined transforms (33) and (42) of the fields $\vec{\mathcal{C}}$ and $\vec{\mathcal{B}}$ and the τ parameter.

Next, let's couple the EM fields to the electrically and magnetically charged particles. Let me skip the Hamiltonian terms for the interaction of the EM fields with the currents and focus on the Gauss Laws, which in the quantum theory are implemented as time-independent operatorial constraints,

$$\begin{aligned} \nabla\cdot\hat{\vec{\mathcal{C}}}(\mathbf{x}) &= 4\pi n \times \delta^{(3)}(\mathbf{x}), \\ \nabla\cdot\hat{\vec{\mathcal{B}}}(\mathbf{x}) &= 4\pi m \times \delta^{(3)}(\mathbf{x}), \end{aligned} \tag{46}$$

where n and m are the integer coefficients of the complex charge $Z = Q + iM = e(n + \tau m)$. To maintain these Gauss Law equations under symplectic transforms (33) of the EM fields, the (n, m) coefficients should transform exactly like the $(\vec{\mathcal{C}}, \vec{\mathcal{B}})$ fields, namely

$$\begin{pmatrix} n' \\ m' \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}. \tag{47}$$

On the other hand, the (n, m) coefficients — both before and after a symplectic transform (47) — must have integer values. Therefore, the matrix elements $\alpha, \beta, \gamma, \delta$ of the symplectic transform must be integers! The matrices

$$\mathcal{M} = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix} \quad \text{with } \textbf{integer } \alpha, \beta, \gamma, \delta, \quad \alpha\delta - \beta\gamma = 1 \tag{48}$$

form an infinite discrete non-abelian group called $\mathrm{SL}(2, \mathbf{Z})$. This group is generated by

$$S = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{49}$$

meaning it comprises all possible products of these matrices in any order — for example

$STTTSTTS$, — subject to relations

$$S^2 \;=\; (ST)^3 \;=\; -1. \tag{50}$$

From the Physics point of view, this means that once we understand how the generators S and T act on the fields, charges, and coupling(s) of a theory in question — and what are the physical meanings of those transforms, — we can figure out the actions of the other $\mathrm{ZL}(2,\mathbf{Z})$ matrices by writing them as products of S's and T's. Often, the action of the S generator is particularly important, so the whole $\mathrm{ZL}(2,\mathbf{Z})$ group of dualities is called the S-duality group.

In particular, in QED S acts as electric-magnetic duality

$$n' \;=\; m, \quad m' \;=\; -n, \quad \vec{\mathcal{C}}' \;=\; \vec{\mathcal{B}}, \quad \vec{\mathcal{B}}' \;=\; -\vec{\mathcal{C}}, \tag{51}$$

while

$$\tau' \;=\; \frac{-1}{\tau}\,. \tag{52}$$

Note that the Θ angle changes the naive $\alpha' = 1/\alpha$ coupling duality to

$$\alpha' \;=\; \frac{1}{\alpha} \;+\; \frac{\alpha\Theta^2}{4\pi^2}\,, \quad \Theta' \;=\; \frac{-\alpha^2\Theta}{1+(\alpha\Theta/2\pi)^2}\,. \tag{53}$$

The other generator T of the S-duality group acts as

$$\vec{\mathcal{B}}' \;=\; \vec{\mathcal{B}}, \quad \vec{\mathcal{C}}' \;=\; \vec{\mathcal{C}} \;-\; \vec{\mathcal{B}}, \quad m' \;=\; m, \quad n' \;=\; n \;-\; m\,: \tag{54}$$

the magnetic charges remain invariant, but the electric charges of the magnetic monopoles and dyons change by (minus) their magnetic charges. At the same time,

$$T:\tau \;\to\; \tau' \;=\; \tau \;+\; 1 \quad\Longrightarrow\quad \alpha' \;=\; \alpha \quad \text{but} \quad \Theta' \;=\; \Theta \;+\; 2\pi. \tag{55}$$

Consequently, two theories with the same α but $\Theta_2 - \Theta_1 = 2\pi k$ (for an integer k) are not quite identical but are related by the T^k transform: They should have similar non-magnetic particles, but the magnetically charged particles should have different electric charges, $n_2 - n_1 = -k(m_1 = m_2)$.