1. As a warm-up exercise, consider an $SU(N)\times SU(N)$ supersymmetric gauge theory with the following chiral superfields:

$$\begin{aligned} Q_f &\in (\mathbf{N},\mathbf{1}) \quad (f=1,\ldots,N),\\ \Omega &\in (\overline{\mathbf{N}},\mathbf{N}) \quad \text{(just one)},\\ \widetilde{Q}_f &\in (\mathbf{1},\overline{\mathbf{N}}) \quad (f=1,\ldots,N). \end{aligned} \tag{1}$$

For $W_{\text{tree}}=0$, this theory has a moduli space parametrized by

$$M_{ff'}=\widetilde{Q}_f\Omega Q_{f'}\,,\quad B=\det(Q),\quad \widetilde{B}=\det(\widetilde{Q}),\quad C=\det(\Omega) \tag{2}$$

subject to a classical constraint

$$\det(M)\ =\ \widetilde{B}\times C\times B. \tag{3}$$

At generic points of this moduli space, the $SU(N)\times SU(N)$ gauge symmetry is completely Higgsed — all vector superfields are massive.

(a) Before doing anything else, write down the anomaly-free flavor symmetries of the theory.

(b) Argue that the non-perturbative quantum effects do not generate an effective superpotential for the moduli but instead modify the moduli space geometry according to

$$\det(M)\ =\ \widetilde{B}\times C\times B\ -\ \Lambda_1^{2N}\times\widetilde{B}\ -\ \Lambda_2^{2N}\times B. \tag{4}$$

(c) Despite quantum corrections (4), the theory has a vacuum state with $\langle M\rangle=\langle B\rangle=\langle\widetilde{B}\rangle=\langle C\rangle=0$ where all the flavor symmetries remain unbroken. Check 't Hooft's anomaly matching condition for this vacuum state

Hint: Quantum corrections (4) give mass to one combination of the B and $\widetilde{B}$ superfields.

2. The second problem is about the Klebanov–Strassler cascade. Let's start with a generic question.

   (a) Consider a generic SUSY gauge theory with a non-trivial infrared fixed point. Let's perturb the theory's Lagrangian by

$$\Delta\mathcal{L} \;=\; \int d^2\theta\, \lambda \hat{\mathcal{O}} \;+\; \text{H.c.} \tag{5}$$

   where λ is a small coupling and $\hat{\mathcal{O}}(x)$ is a chiral gauge-invariant operator of R-charge r. Show that in the IR limit of the theory, the perturbation (5) is: irrelevant for $r > 2$, marginal for $r = 2$, and relevant for $r < 2$.

   The Klebanov–Strassler model is an $SU(A) \times SU(B)$ SUSY gauge theory with chiral superfields $Q_1, Q_2 \in (\mathbf{A}, \overline{\mathbf{B}})$ and $\widetilde{Q}_1, \widetilde{Q}_2 \in (\overline{\mathbf{A}}, \mathbf{B})$, and a quartic superpotential

$$W_{\text{tree}} \;=\; \lambda\, \text{tr}(\widetilde{Q}_1 Q_1 \widetilde{Q}_2 Q_2 \;-\; \widetilde{Q}_1 Q_2 \widetilde{Q}_2 Q_1). \tag{6}$$

   Note the $SU(2) \times SU(2)$ flavor symmetry of this superpotential.

   But before we address the Klebanov–Strassler model in all its glory, consider a simpler version with $g_B = 0$. This model is basically SQCD with $N_c = A$ and $N_F = 2B$, perturbed by the superpotential (6).

   (b) What are the limits on flavor/color ratio of SQCD with a non-trivial infrared fixed point?

   (c) Let's go to such a fixed point and turn on a *small* $\lambda \ll (1/E)$ as a perturbation. Show that this perturbation is irrelevant for $B > A$, marginal for $B = A$, and relevant for $B < A$. Hint: use (a).

   Let's focus on the *relevant* case for $B < A$. A relevant perturbation becomes stronger at lower energies, until eventually it becomes so strong that it changes the theory. To see what happens when the relevant perturbation (6) becomes strong at low energies, consider the Seiberg dual of the un-perturbed SQCD:

   (d) Show that Seiberg duality turns the superpotential (6) into mass terms for the gauge-singlet $\Phi^{ff'}$ fields.

(e) Integrate out the massive $\Phi^{ff'}$ fields from the very-low-energy theory, and show that this generates a quartic superpotential for the *dual quarks* which looks exactly like (6) (albeit the overall coefficient may be different, $\lambda^{\text{dual}} \neq \lambda$). Also show that in the dual theory, this quartic superpotential is irrelevant because $A^{\text{dual}} < B^{\text{dual}}$.

Now let's momentarily set $\lambda = 0$ but instead turn on the $SU(B)$ gauge coupling, $g_B \neq 0$.

(f) Write down the Shifman–Vainshtein β–functions for the two gauge couplings, and show that for $A > B > \frac{3}{4}A$, the g_A has a non-trivial infrared fixed point while the g_B is infrared-free.
Note: Although $\beta_B < 0$ when both g_A and g_B are weak, β_B becomes positive when g_A flows to its fixed point.

Finally, let's turn on all three couplings, $g_A, g_B, \lambda \neq 0$. Along the renormalization flow in the IR direction, g_A grows until it reaches its fixed point, g_B grows at first but then becomes weak again, while λ — or rather dimensionless $\lambda \times E$ — first becomes weak but eventually grows strong again and disrupts the theory. At this point, we replace the $SU(A)$ with its Seiberg dual and integrate out the massless particles. Eventually, we get get an effective theory which looks exactly like the UV theory we have started from, but with $B' > A'$.

(g) Explain how this process works in terms of (c), (e), and (f).

Once we get to a theory with $B' > A'$, the process repeats itself, but with with g_A and g_B exchanging their roles: g_B grows from weak coupling to a fixed point, g_A becomes weak, while λE first becomes weak but eventually grows strong, then we go to the Seiberg dual of the $SU(B')$, integrate out massive particles — and get another Klebanov–Strassler model, this time with with $A'' > B''$. And then it's *déjà vu all over again...*

At each stage of this *duality cascade* A or B is reduced while their difference $|A - B|$ remains constant. Eventually, we end up with $A \geq 2B$ (or $B \geq 2A$) and the cascade stops.

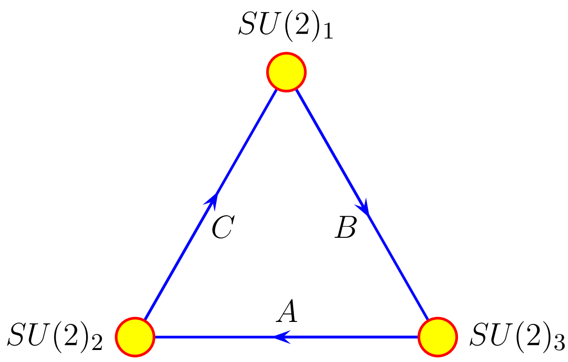
(h) During the last two stages we have $B \leq \frac{3}{4}A$ (or $A \leq \frac{3}{4}B$) and the stronger of the two groups has flavor/color ratio below the conformal window. Explain how this changes (or does not change) the renormalization+duality process. Also, show that the net outcome of this process is the same as in part (g), and the duality cascade proceeds until we get to $A \geq 2B$ (or $B \geq 2A$).

(i) Finally, let's start the cascade with $A = N \times (k+1)$ and $B = N \times k$ so that it ends with $A = 2N$ and $B = N$ or the other way around. Show that the next stage leads to the $SU(N)$ SYM theory without quarks plus a single baryonic modulus of the $SU(2N)$.

3. The third exercise is about the abelian Coulomb phase of a non-abelian SUSY gauge theory and the family of Seiberg–Witten curves for the abelian gauge coupling. Similar to the homework set#11, the UV theory is $\mathcal{N} = 1$ rather than $\mathcal{N} = 2$ supersymmetric, but this time the gauge group is $SU(2)_1 \times SU(2)_2 \times SU(2)_3$ while the chiral superfields form 3 bi-fundamental multiplets,

$$A \in (\mathbf{1}, \mathbf{2}, \mathbf{2}), \quad B \in (\mathbf{2}, \mathbf{1}, \mathbf{2}), \quad C \in (\mathbf{2}, \mathbf{2}, \mathbf{1}, \tag{7}$$

or in the quiver language



(8)

Let's start with the classical theory.

(a) Show that the theory has dim = 4 family of supersymmetric vacua parametrized by the gauge-invariant moduli

$$a = \det(A), \quad b = \det(B), \quad c = \det(C), \quad \text{and} \quad t = \tfrac{1}{2}\,\mathrm{tr}(ABC) \tag{9}$$

(in 2×2 matrix notations for the bi-fundamental fields A, B, C).

Also show that for generic values of the moduli, the $[SU(2)]^3$ gauge group is Higgsed down to a single $U(1)$, specifically $U(1) \subset SU(2)_d = \mathrm{diagonal}\big([SU(2)]^3\big)$. However, for

$$t^2 = \det(ABC) = a \times b \times c, \tag{10}$$

the whole diagonal $SU(2)_d$ becomes un-Higgsed.

Now consider the quantum theory. Similarly to what we have in problem 1 — and for the same reason — there is no non-perturbative superpotential for the moduli. Instead, the classical relation $\det(ABC) = \det(A) \times \det(B) \times \det(C)$ becomes

$$\det(ABC) = \det(A)\times\det(B)\times\det(C) - \Lambda_1^4\times\det(A) - \Lambda_2^4\times\det(B) - \Lambda_3^4\times\det(C), \tag{11}$$

or in terms of the moduli (9)

$$\det(ABC) = D(a,b,c) \stackrel{\text{def}}{=} a \times b \times c - \Lambda_1^4 \times a - \Lambda_2^4 \times b - \Lambda_3^4 \times c. \tag{12}$$

Also, the diagonal $SU(2)$ does not become un-Higgsed for $t^2 = abc$, or $t^2 = D(a,b,c)$, or for any other values of the moduli. Instead, we get massless magnetic monopoles or massless dyons when

$$D(a,b,c) - t^2 = \pm 2\Lambda_1^2\Lambda_2^2\Lambda_3^2 \tag{13}$$

Finally, the $U(1)$ gauge coupling τ becomes moduli dependent, as described by the Seiberg–Witten curve family

$$y^2 = x^3 + x^2 \times \big(D(a,b,c) - t^2\big) + x \times \Lambda_1^4\Lambda_2^4\Lambda_3^4\,. \tag{14}$$

The rest of this problem is about deriving and justifying this curve family.

(b) Let's start with the perturbative large-VEV regime. Take

$$\langle A\rangle \;=\; \langle B\rangle \;=\; \langle C\rangle \;=\; \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{15}$$

for

$$|\alpha|\,,|\beta| \;\gg\; |\alpha-\beta| \;\gg\; |\Lambda_{1,2,3}|\,. \tag{16}$$

Show that this regime corresponds to sequential Higgsing $[SU(2)]^3 \to SU(2)_d \to U(1)$ and that in this regime

$$\tau[U(1)] \;\approx\; \frac{2i}{2\pi}\log\frac{\alpha^8(\alpha-\beta)^4}{\Lambda_1^4\Lambda_2^4\Lambda_3^4} \;+\; \begin{pmatrix}\text{a numeric}\\ \text{constant}\end{pmatrix}. \tag{17}$$

(c) Now consider the Seiberg–Witten curve (14) in the same regime and show that it yields the same weak $U(1)$ coupling as in eq. (17).

Now let's momentarily make the A, B, C fields massive and heavy,

$$W_{\text{tree}} \;=\; m_a\times\det(A) \;+\; m_b\times\det(B) \;+\; m_c\times\det(C), \qquad m_{a,b,c} \;\gg\; \Lambda_{1,2,3}\,. \tag{18}$$

(d) Show that the massive theory has 8 discrete vacua and calculate the gaugino condensates $\langle S_1\rangle\,,\langle S_2\rangle\,,\langle S_3\rangle$ and VEVs of the massive scalar bilinears $\langle\det(A)\rangle\,,\langle\det(B)\rangle\,,\langle\det(C)\rangle$ in these vacua.

(e) While the above VEVs depend on the masses $m_{a,b,c}$, some VEV combinations are invariant. Show that for all non-zero masses

$$D(\langle a\rangle\,,\langle b\rangle\,,\langle c\rangle) \;=\; \pm 2\Lambda_1^2\Lambda_2^2\Lambda_3^2. \tag{19}$$

Also argue that $|\langle t\rangle| \ll |\Lambda_1\Lambda_2\Lambda_3|$.

(f) Use the results of part (d) to argue that in the massless theory, the Kähler metric for the moduli space becomes singular for $D(a,b,c) = \pm 2\Lambda_1^2\Lambda_2^2\Lambda_3^2$ and small t. Explain the physical origin of such singularities and compare them to the singularities (13) of the Seiberg–Witten curve (14).

Next, we are going to relate the SW curve (14) of the $[SU(2)]^3$ theory to the SW curves of the theories we are already familiar with. Let's take the $|\Lambda_1| \ll |\Lambda_{2,3}|$ limit and consider the intermediate-energy regime $|\Lambda_1| \ll E \ll |\Lambda_{2,3}$. In this regime, the $SU(2)_1$ gauge coupling is weak, and the theory looks exactly like the IR regime of the $SU(N)^2$ of problem 1 for $N = 2$. Specifically, the $SU(2)_2 \times SU(2)_3$ gauge groups are in the confining/Higgs regime, and the massless fields comprise the baryonic moduli a, b, and c, and the 2×2 meson matrix $M = CAB$, subject to the constraint

$$\det(M) \;=\; D(a,b,c) \;=\; c \times a \times b \;-\; \Lambda_3^4 \times c \;-\; \Lambda_2^4 \times b \;-\; [\text{negligible } \Lambda_1^4 \times a]. \quad (20)$$

From the $SU(2)_1$ point of view, the meson matrix M — or rather

$$\Phi \;=\; \frac{1}{\mu^2} M \;=\; \frac{1}{\mu^2} CAB \quad (21)$$

for some constant μ^2 of dimension mass^2 — comprises a singlet and a triplet. The triplet part Φ_3 breaks the $SU(2)_1$ group down to its $U(1)$ subgroup, which in the $|\Lambda_1| \ll |\Lambda_{2,3}|$ limit we may identify with the $U(1) \subset SU(2)_d$. Consequently, the $U(1)$ gauge coupling τ obtains from the SW curve for the $SU(2)_1$ theory with a triplet, for the appropriate identification of $U = \Phi_3^2$.

(g) Show that

$$U \;=\; \operatorname{tr}^2(\Phi) \;-\; 4\det\Phi \;=\; \frac{4}{\mu^4}\big(t^2 \;-\; D(a,b,c)\big), \quad (22)$$

hence Seiberg–Witten curve

$$y^2 \;=\; x^3 \;+\; x^2 \times (D(a,n,c) - t^2) \;+\; x \times \frac{\mu^8 \Lambda_1^4}{64}\,. \quad (23)$$

(h) Argue that the constant μ^2 in eq. (21) should be $\mu^2 = \Lambda_2\Lambda_3 \times$ a numeric constant. Specifically, in eq. (23) for the SW curve we need

$$\mu^8 \;=\; 64\Lambda_2^4\Lambda_3^4 \quad (24)$$

so the singularities of the curve would agree with part (f).

Now take an opposite limit $|\Lambda_1| \gg |\Lambda_{2,3}|$. This time, in the intermediate energy range $|\Lambda_1| \gg E \gg |\Lambda_{2,3}|$ we have weak $SU(2)_2 \times SU(2)_3$ couplings, while the $SU(2)_1$ group is in the Higgs/confinement regime. In this regime, the UV fields B and C are confined, but there are massless composite fields, namely the meson matrix $M = BC$ and the baryonic moduli b and c, constrained to have

$$\det(M) \;=\; b \times c \;-\; \Lambda_1^4. \tag{25}$$

From the $SU(2)_2 \times SU(2)_3$ point of view, the meson matrix M — or rather

$$\Psi \;=\; \frac{1}{\tilde{\mu}} M \;=\; \frac{1}{\tilde{\mu}} BC \tag{26}$$

for some factor $\tilde{\mu} = O(\Lambda_1)$ — is a bi-fundamental multiplet $(\mathbf{2}, \mathbf{2})$, just like the A multiplet. Consequently, the effective theory in the intermediate energy range is precisely the same as the UV theory in the homework set#11: The $SU(2)_2 \times SU(2)_3$ gauge group with two bi-fundamental multiplets A and Ψ of chiral superfields. The only difference are the notations, and an extra baryonic modulus in the present case.

Assuming you have done the homework, you know that $SU(2)_2 \times SU(2)_3$ theory is in the abelian Coulomb phase: the gauge group is broken to a single $U(1)$, and its gauge coupling depends on the moduli according to the Seiberg–Witten curve from eq. (HW11.12).

(i) Show that in terms of the $[SU(2)]^3$ moduli a, b, c, t, the SW curve from the homework set#11 becomes

$$y^2 \;=\; x^3 \;+\; x^2 \times \left(-\tilde{\mu}^2(\Lambda_2^4 + \Lambda_2^4) \,+\, a \times b \times c \,-\, \Lambda_1^4 \times a \,-\, t^2\right) \;+\; \tilde{\mu}^4 \Lambda_2^4 \Lambda_3^4\,. \tag{27}$$

Hint: back in homework#11 I have used $SO(4)$ notations for the dot products of bi-fundamental fields $Q_i \cdot Q_j$. In the 2×2 matrix notations of this exam, the $\mathcal{M}_{ij}$ moduli matrix becomes

$$\mathcal{M} \;=\; \begin{pmatrix} \Psi \cdot \Psi & \Psi \cdot A \\ A \cdot \Psi & A \cdot A \end{pmatrix} \;=\; \begin{pmatrix} \det(\Psi) & \frac{1}{2}\operatorname{tr}(\Psi A) \\ \frac{1}{2}\operatorname{tr}(A\Psi) & \det(A) \end{pmatrix}. \tag{28}$$

(j) Show that for $\tilde{\mu} = \Lambda_1$, the SW curve (27) agrees with the curve (14) in the $|\Lambda_1| \gg |\Lambda_{2,3}|$ limit.

The Seiberg–Witten curve (14) depends on the 4 moduli a, b, c, t of the $[SU(2)]^3$ theory via a single combination

$$\tilde{U}(a,b,c,t) \;=\; D(a,b,c) \;-\; t^2. \tag{29}$$

Consequently, we may describe the singularities of the theory — and the monodromies as one goes around the singularity — by focusing on the complex $\tilde{U}$ plane and ignoring the other 3 complex dimensions of the moduli space. Specifically, the singularities are at $\tilde{U}_1 = +2\Lambda_1^2\Lambda_2^2\Lambda_3^2$, at $\tilde{u}_2 = -2\Lambda_1^2\Lambda_2^2\Lambda_3^2$, and at $\tilde{U}_\infty = \infty$.

(k) Describe the $SL(2,\mathbf{Z})$ monodromies of the theory as $\tilde{U}$ circles each of these singularities and make sure they are consistent with each other,

$$\mathbf{M}_2 \times \mathbf{M}_1 \;=\; \mathbf{M}_\infty\,. \tag{30}$$

Also, relate these monodromies to the multiplicities and electric/magnetic charges of the particles which become massless at $\tilde{U} = \tilde{U}_1$ and at $\tilde{U} = \tilde{U}_2$.

Work by analogy with the $SU(2)$ with a triplet theory instead of re-deriving all the monodromies from scratch, as that would take way too much time. Also, please don't copy my notes on the Seiberg–Witten theory or the solutions to homework set#11, just quote the relevant result and use it.