Please do not waste time and paper by copying the homework solutions or any notes I posted on the web, simply reference the appropriate question or equation and go ahead. Ditto for anything explicitly derived in class or explained in any textbook. But if you are quoting a book or an article, make sure to spell out all the intermediate steps.

1. As a warm-up exercise, consider a semi-classical SUSY gauge theory with $G = SO(N_c)$ gauge group (assume $N_c > 4$) and $N_f \times N_c$ chiral superfields Q_{fc} forming N_f copies of the N_c–vector multiplet $\mathbf{N_c}$ of the gauge group. There is no classical superpotential, $W_{\rm tree}(Q) = 0$.

   FYI: For $SO(N)$ groups, Index(vector) = 1 and Index(adjoint) = $N - 2$.

   (a) Let's start with $N_f = 1$. Write down the classical scalar potential of this theory and describe its flat directions. Specifically, show that modulo $SO(N_c)$ there is just one flat direction and write down the gauge-invariant chiral modulus $\mathcal{M}$ for this direction.

   (b) Describe the Higgs mechanism for $\langle\mathcal{M}\rangle \neq 0$. Work in the component field formalism and show how all the components of a massive vector multiplet get their masses.

   (c) Now consider several flavors, $N_f > 1$ (but $N_f \leq N_c - 2$). Turn on the 'squark' VEVs $\langle Q\rangle$, first $f = 1$, then $f = 2$, *etc.*, *etc.*, ending with $f = N_f$, while keeping $V_{\rm scalar} = 0$ (and hence unbroken SUSY) at each stage. Show that generically, this sequence Higgses the $SO(N_c)$ gauge symmetry down to $SO(N_c - N_f)$.

   (d) Argue that for $N_f > 1$ but $N_f \leq N_c - 2$, the independent moduli form a symmetric $N_f \times N_f$ matrix $\mathcal{M}_{ff'} = \mathcal{M}_{f'f} = \sum_c Q_{cf}Q_{cf'}$.

2. Now consider the quantum $\mathbf{SO}(N_c)$ SUSY gauge theory. For simplicity, let's assume just one flavor, $N_f = 1$.

   (a) Take $N_f = 1$ and assume a large Higgs VEV $\langle Q\rangle \gg \Lambda$. Describe the low-energy EFT for $E \ll \langle Q\rangle$ and calculate its Wilsonian gauge coupling τ_w^{low} — or rather its dimensional transmutant Λ^{low} — in terms of the Λ^{high} and the modulus $\mathcal{M}$.

   (b) Let us fix the modulus field for a moment and focus on the infra-red behavior of the gauge theory. How many vacua does it have, and what is the order parameter distinguishing those vacua?

   (c) Now un-fix the modulus field $\mathcal{M}$ and calculate the non-perturbative effective superpotential $W_{\text{n.p.}}(\mathcal{M})$ due to gaugino condensation.

   (d) For $\langle Q\rangle \gg \Lambda$ the gauge coupling is weak at the Higgs scale, so we may approximate the Kähler function for the modulus field by the tree-level

$$K_{\text{tree}}(\mathcal{M}^*, \mathcal{M}) \;=\; Q^\dagger Q \;=\; \sqrt{\mathcal{M}^*\mathcal{M}}. \tag{1}$$

   Use this Kähler function and the effective superpotential you have obtained in part (c) to calculate the scalar potential $V(\mathcal{M}^*, \mathcal{M})$ for the modulus and show that this potential leads to runaway $\mathcal{M} \to \infty$.

   Now let $W_{\text{tree}} = \frac{m}{2}\sum_c Q_c^2$.

   (e) For small mass $m \ll \Lambda$, the effective superpotential for the modulus comprises $W_{\text{tree}}(\mathcal{M}) + W_{\text{n.p.}}(\mathcal{M})$. Show that this superpotential has several SUSY vacua, and calculate the expectation values $\langle\mathcal{M}\rangle$ and $\langle S\rangle$ for each vacuum.

   (f) For large $m \gg \Lambda$, we may integrate out the Q fields perturbatively, and then study the IR behavior in terms of the low-energy EFT, which now is pure $SO(N)$ SYM theory. Calculate the Λ_{low} of that effective theory — and hence the gaugino condensate $\langle S\rangle$ in the $m \gg \Lambda$ regime — as a function of Λ and m parameters of the original (UV) theory.

   (g) Compare the formulae for the gaugino condensate $\langle S\rangle = \text{function}(m, \Lambda)$ in the two regimes, $m \ll \Lambda$ and $m \gg \Lambda$. What does this comparison tell you about the phase structure of the theory?

3. Next, an exercise in superfield Feynman rules. Consider SQED with an extra neutral chiral superfield C with a Yukawa coupling to the charged chiral superfields A (charge $=$ $+1$) and B (charge $= -1$). For simplicity, let all the fields be massless, thus

$$\begin{aligned}\mathcal{L}_{\text{phys}} &= \int d^4\theta \left(\overline{A}e^{+2V}A + \overline{B}e^{-2V}B + \overline{C}C\right) \\ &\quad + \int d^2\theta \left(\frac{1}{4e^2}W^\alpha W_\alpha - yABC\right) + \text{H. c.}\end{aligned} \tag{2}$$

Let's focus on the wave-function renormalization of the neutral chiral field C. At the one-loop level, this renormalization depends only on the Yukawa coupling y, but at the two-loop and higher levels it depends on both the Yukawa and the gauge couplings.

(a) Draw all the *two-loop* 1PI Feynman diagrams contributing to the $C\overline{C}$ amplitude. If a diagram has a nested divergence (*i.e.*, includes a UV-divergent one-loop sub-diagram), draw the one-loop diagram containing a counter-term vertex that would cancel this sub-divergence *in situ*.

(b) Evaluate the $D, \overline{D}$ algebra for each diagram and write the resulting amplitude as

$$\int d^4\theta\, C(k,\theta)\overline{C}(-k,\theta) \times \text{a prefactor} \times \text{a momentum integral} \tag{3}$$

where the prefactor comprises powers of e and y and purely numeric factors, if any. Hint: $\int d^4\theta\, D^2C \times \overline{D}^2\overline{C} = 16k^2 \times \int d^4\theta\, C\overline{C}$.

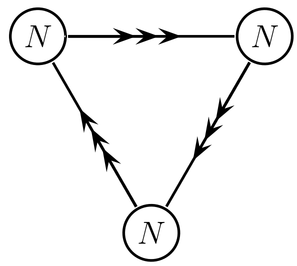
(c) Without actually evaluating the momentum integrals, show that for one of the two-loop diagrams the momentum integral is finite, while every other two-loop diagram involves exactly the same logarithmically divergent integral.

(d) Use part (c) and the prefactors you obtained in part (b) to show that the two-loop contribution to the anomalous dimension of the C field has form

$$\gamma^{2\,\text{loop}}(C) = |y|^2 \times \left(|y|^2 - 4e^2\right) \times \text{an overall numeric constant.} \tag{4}$$

The numeric constant here follows from the regularized divergence of the momentum integral you should have identified in part (c). But evaluating this integral would take way too much time for this exam, so the problem ends here.

4. Finally, an exercise on NSVZ beta-functions and IR fixed points. Consider the $[SU(N)]^3$ SUSY gauge theory with a quiver diagram



(5)

That is, there are $3(N^2-1)$ vector superfields of the $SU(N)\times SU(N)\times SU(N)$ gauge symmetry and $9N^2$ chiral superfields comprising the following multiplets:

$$\begin{aligned} A_1, A_2, A_3 &\in (\mathbf{1}, \mathbf{N}, \overline{\mathbf{N}}), \\ B_1, B_2, B_3 &\in (\overline{\mathbf{N}}, \mathbf{1}, \mathbf{N}), \\ C_1, C_2, C_3 &\in (\mathbf{N}, \overline{\mathbf{N}}, \mathbf{1}). \end{aligned} \tag{6}$$

In $N\times N$ matrix notations, there are 9 matrices of chiral superfields, and the tree-level superpotential is

$$W \;=\; \lambda \sum_{i,j,k=1,2,3} \epsilon^{ijk}\,\mathrm{tr}(A_i B_j C_k). \tag{7}$$

Note the $SU(3)$ global symmetry of the theory: $A_i \mapsto U_i^{\ j} A_j$, $B_i \mapsto U_i^{\ j} B_j$, $C_i \mapsto U_i^{\ j} C_j$, all for the same $SU(3)$ matrix U.

For your information, this gauge theory obtains in type IIB superstring when a stack of N coincident D3–branes sits at the orbifold singularity $\mathbf{C}^3/\mathbf{Z}_3$. When $N\times g_{\text{string}} \ll 1$, the open strings connecting the D3–branes give rise to the gauge and chiral superfields of the quiver (5). In the opposite limit $N\times g_{\text{string}} \gg 1$, the branes curve the 10D spacetime and give rise to the gravity dual of the gauge theory, namely SUGRA on the smooth manifold $AdS_5\times(S^5/\mathbf{Z}_3)$.

But you do not need to know any string theory or gauge-gravity duality to do this exam. This problem focuses only on the field-theoretical aspects of the supersymmetric gauge theory in question.

(a) The quiver theory (5) has 4 couplings, namely 3 gauge couplings g_1, g_2, g_3, and the Yukawa coupling λ. Write down the exact β–functions for all 4 couplings in terms of the anomalous dimensions γ_A, γ_B, and γ_C.

(b) Show that the fixed points of the renormalization group satisfy

$$\gamma_A = \gamma_B = \gamma_C = 0. \tag{8}$$

Also, show that the solutions to eqs. (8) form a *fixed line* in the coupling space parametrized by

$$g_1 = g_2 = g_3 = g^*(|\lambda|) \tag{9}$$

for some function $g^*(|\lambda|)$.

(c) Now calculate the anomalous dimensions to one-loop order. To save time, open the solutions to the homework set#6, read the preamble to problem 1(c), and use eq. (S.17).

Show that the fixed line (9) goes through the week-coupling region, and calculate the $g^*(|\lambda|)$ in that region.

(d) Finally, show that the fixed line is IR stable, or at least IR stable in the weak coupling regime. That is, if we start with weak UV couplings which do not exactly satisfy eqs. (9) but are not too far from it in the coupling space, then in the IR direction of the renormalization group, the couplings flow towards the fixed line rather than away from it.