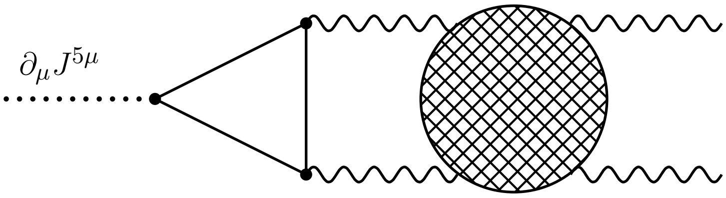
ANOMALIES, GAUGE COUPLINGS, AND THE NSVZ FORMULA

Let's start with axial or chiral anomalies in the ordinary gauge theories. The **Adler–Bardeen theorem** is often mis-stated as *the axial or chiral anomalies exist only at one loop level,* but this is not exactly true. A more accurate statement of the theorem says: *an anomaly cancels at the one loop level, then it would never show up at higher loop levels.* But if an anomaly does not cancel out at the one-loop level, then the higher loop orders may 'dress-up' the anomalous amplitudes.

For example, consider the anomalous axial symmetry of the ordinary QED with the massless electron. The anomaly originates on the one-loop triangle graph involving an axial current and two photons, but the anomalous amplitude $\mathcal{M}(\partial_\mu J^{5\mu} \to \gamma\gamma)$ receives higher-loop corrections from the re-scattering of the two photons,



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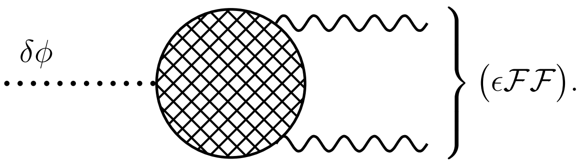
The Adler–Bardeen theorem has interesting corollary for the Θ angle of a gauge theory: *the effect of a chiral redefinition of the charged fermion fields on the* Θ *angle is exact at one loop.* For an example, consider QED with a massless electron coupled to a CP-odd modulus scalar ϕ,

$$\mathcal{L} \;=\; \frac{-1}{4e^2}\mathcal{F}_{\mu\nu}^2 \;+\; \overline{\Psi}\gamma^\mu(i\partial_\mu + e\mathcal{A}_\mu)\Psi \;+\; \frac{\Theta(\phi)}{64\pi^2}\big(\epsilon\mathcal{F}\mathcal{F}\big) \;+\; \frac{C(\phi)}{4}\,\partial_\mu\phi \times \overline{\Psi}\gamma^5\gamma^\mu\Psi. \tag{2}$$

Focus on the amplitude connecting the modulus quantum $\delta\phi$ to the CP-odd combination $(\epsilon\mathcal{F}\mathcal{F})$ of two photons,



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There is a tree-level contribution to this amplitude from the modulus dependence of the Θ

angle,

$$\mathcal{M}_1 \;=\; \frac{1}{64\pi^2}\frac{\partial\Theta}{\partial\phi}\,. \tag{4}$$

But there is also a one-loop contribution stemming from the coupling of $\partial_\mu\phi$ to the axial current $J^{5\mu}$ and the one-loop anomalous non-conservation of that current, thus

$$\frac{C(\phi)}{4}\,\partial_\mu\delta\phi \times J^{5\mu} \;\xrightarrow[\text{by parts}]{}\; -\frac{C(\phi)}{4}\,\delta\phi \times \partial_\mu J^{5\mu} \;=\; -\frac{C(\phi)}{4}\,\delta\phi \times \frac{+1}{16\pi^2}\big(\epsilon\mathcal{F}\mathcal{F}\big), \tag{5}$$

hence

$$\mathcal{M}_2 \;=\; -\frac{C(\phi)}{64\pi^2}\,. \tag{6}$$

In addition, there are going to be higher-order corrections due to photon re-scattering, but all such corrections affect the tree-level amplitude (4) and the anomalous amplitude (6) in exactly the same way, thus

$$\mathcal{M}_{\text{net}} \;=\; \frac{1}{64\pi^2}\left(\frac{\partial\Theta}{\partial\phi} \;-\; C(\phi)\right) \times \left(1 \;+\; \begin{pmatrix}\text{rescattering}\\ \text{corrections}\end{pmatrix}\right) \tag{7}$$

where the red factor comes as an *inseparable* combination of $\partial\Theta/\partial\phi$ and $C(\phi)$.

Now consider a modulus-dependent axial redefinition of the electron field,

$$\Psi'(x) \;=\; \exp\!\big(+i\alpha(\phi(x))\gamma^5\big) \times \Psi(x), \qquad \overline{\Psi}'(x) \;=\; \overline{\Psi}(x) \times \exp\!\big(+i\alpha(\phi(x))\gamma^5\big). \tag{8}$$

This redefinition does not change the vector or the axial currents of the electron field, but it does change its kinetic energy term by

$$\begin{aligned}\overline{\Psi}'\gamma^\mu(i\partial_\mu + e\mathcal{A}_\mu)\Psi' \;&=\; \overline{\Psi}\gamma^\mu(i\partial_\mu + e\mathcal{A}_\mu)\Psi \;-\; (\partial_\mu\alpha(\phi)) \times \overline{\Psi}\gamma^\mu\gamma^5\Psi \\ &=\; \overline{\Psi}\gamma^\mu(i\partial_\mu + e\mathcal{A}_\mu)\Psi \;+\; \frac{\partial\alpha}{\partial\phi}\,\partial_\mu\phi \times \big(J^{5\mu} = \overline{\Psi}\gamma^5\gamma^\mu\Psi\big).\end{aligned} \tag{9}$$

In the context of the Lagrangian (2), the extra term here can be canceled by modifying the $C(\phi)$ coefficient of the axial electron-modulus coupling,

$$C'(\phi) \;=\; C(\phi) \;-\; 4\frac{\partial\alpha}{\partial\phi}\,. \tag{10}$$

By itself, this modification may change the axionic coupling of the modulus to the photons, but since $C(\phi)$ always enters in an inseparable combination with $\partial\Theta/\partial\phi$, we may *precisely* cancel the effect of this modification by also changing the Θ angle as

$$\Theta'(\phi) \;=\; \Theta(\phi) \;-\; 4\alpha(\phi). \tag{11}$$

Note that this change of the Θ angle is completely determined at the one-loop level of the axial anomaly and is not subject to any higher-loop corrections.

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Now consider SQED with a massless electron coupled to a chiral modulus superfield M and its conjugate $\overline{M}$. The Lagrangian is

$$\begin{aligned}\mathcal{L} \;=& \int\!d^2\theta\,d^2\bar\theta\,Z(M,\overline{M})\times\Bigl(\overline{A}e^{+2V}A\,+\,\overline{B}e^{-2V}B\Bigr)\\ &+\int\!d^2\theta\frac{i\tau(M)}{16\pi}\times W^\alpha W_\alpha\;+\int\!d^2\bar\theta\,\frac{-i\tau^*(\overline{M})}{16\pi}\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha},\end{aligned} \tag{12}$$

where $\tau(M)$ is a holomorphic function of the modulus while $Z(M,\overline{M})$ is a real analytic function. Expanding this Lagrangian in components, and focusing on just the EM and electron fields — *i.e.*, disregarding the photino and the selectron fields and their interactions, — we get terms very similar to (2), namely

$$\begin{aligned}\mathcal{L}\;\supset\; &-\frac{\operatorname{Im}\tau(M)}{16\pi}\,\mathcal{F}^2_{\mu\nu}\;+\;Z(M,\overline{M})\times\overline{\Psi}\gamma^\mu\bigl(\tfrac{i}{2}\overleftrightarrow{\partial}_\mu+A_\mu\bigr)\Psi\\ &+\;\frac{\operatorname{Re}\tau(M)}{32\pi}\bigl(\epsilon\mathcal{F}\mathcal{F}\bigr)\;+\;L_\mu(M,\overline{M})\times\overline{\Psi}\gamma^5\gamma^\mu\Psi.\end{aligned} \tag{13}$$

The L_μ here is the vector component of the composite $Z(M,\overline{M})$ superfield which exists when the moduli scalars are non-constant in space,

$$Z(M,\overline{M}) \;=\; 2(\theta\sigma^\mu\bar\theta)\times L_\mu \;+\; \text{other components}, \tag{14}$$

specifically,

$$2L_\mu \;=\; -i\frac{\partial Z}{\partial M}\times\partial_\mu M \;+\; i\frac{\partial Z}{\partial\overline{M}}\times\partial_\mu\overline{M}. \tag{15}$$

Note that in the coupling $L_\mu\times\overline{\Psi}\gamma^5\gamma^\mu\Psi$, the electron fields $\Psi,\overline{\Psi}$ are non-canonically normalized because of the Z factor in their kinetic term. Consequently, the canonically normalized axial current in terms of these fields is

$$J^{5\mu} \;=\; Z \times \overline{\Psi}\gamma^5\gamma^\mu\Psi, \tag{16}$$

and the interaction of this current with the moduli scalars are given by

$$\frac{L_\mu}{Z} \times J^{5\mu} \tag{17}$$

where

$$\frac{L_\mu}{Z} \;=\; \frac{1}{2Z}\left(-i\frac{\partial Z}{\partial M}\times\partial_\mu M \,+\, i\frac{\partial Z}{\partial \overline{M}}\times\partial_\mu\overline{M}\right) \;=\; \left(-\frac{i}{2}\frac{\partial\log Z}{\partial M}\,\partial_\mu M \,+\, \frac{i}{2}\frac{\partial\log Z}{\partial\overline{M}}\,\partial\overline{M}\right). \tag{18}$$

Hence, the anomalous interactions of the moduli quanta with (the CP-odd combinations of) two photons are

$$\begin{aligned}\Gamma_{\text{anomalous}} \;&=\; \left(-\frac{i}{2}\frac{\partial\log Z}{\partial M}\times\partial_\mu\delta M \,+\, \frac{i}{2}\frac{\partial\log Z}{\partial\overline{M}}\times\partial_\mu\delta\overline{M}\right)\times J^{5\mu}\\ &\quad\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle\\ &=\; \left(\frac{i}{2}\frac{\partial\log Z}{\partial M}\times\delta M \,-\, \frac{i}{2}\frac{\partial\log Z}{\partial\overline{M}}\times\delta\overline{M}\right)\times\partial_\mu J^{5\mu}\\ &=\; \left(\frac{i}{2}\frac{\partial\log Z}{\partial M}\times\delta M \,-\, \frac{i}{2}\frac{\partial\log Z}{\partial\overline{M}}\times\delta\overline{M}\right)\times\frac{1}{16\pi^2}\big(\epsilon\mathcal{F}\mathcal{F}\big)\\ &=\; \frac{1}{32\pi^2}\big(\epsilon\mathcal{F}\mathcal{F}\big)\times\left(i\delta M\times\frac{\partial}{\partial M} \,-\, i\delta\overline{M}\times\frac{\partial}{\partial\overline{M}}\right)\log Z(M,\overline{M}).\end{aligned} \tag{19}$$

In addition, there is a tree-level interaction stemming the from moduli dependence of $2\pi\Theta = \operatorname{Re}\tau(M)$. thus

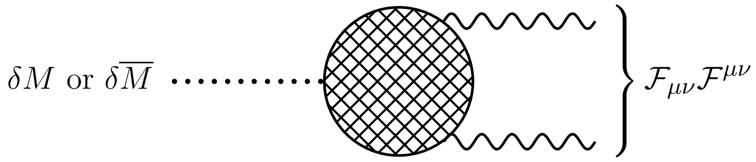
$$\begin{aligned}\Gamma_{\text{tree}} \;&=\; \frac{1}{32\pi}\big(\epsilon\mathcal{F}\mathcal{F}\big)\times\left(\delta M\times\frac{\partial}{\partial M} \,+\, \delta\overline{M}\times\frac{\partial}{\partial\overline{M}}\right)\operatorname{Re}\tau\\ &=\; \frac{1}{32\pi}\big(\epsilon\mathcal{F}\mathcal{F}\big)\times\left(i\delta M\times\frac{\partial}{\partial M} \,-\, i\delta\overline{M}\times\frac{\partial}{\partial\overline{M}}\right)\left(\operatorname{Im}\tau = \frac{1}{\alpha_w(M,\overline{M})}\right)\end{aligned} \tag{20}$$

where the second equality follows from the holomorphy of $\tau(M)$. Thus, combining the two effects, we get the net amplitudes involving one modulus quantum and a CP-odd combination of two photons:

$$\begin{aligned}
\delta M \times \big(\epsilon\mathcal{F}\mathcal{F}\big)\ \text{amplitude} &= \frac{i}{32\pi^2}\,\frac{\partial}{\partial M}\left(\frac{\pi}{\alpha_w}+\log Z\right), \\
\delta\overline{M} \times \big(\epsilon\mathcal{F}\mathcal{F}\big)\ \text{amplitude} &= \frac{-i}{32\pi^2}\,\frac{\partial}{\partial \overline{M}}\left(\frac{\pi}{\alpha_w}+\log Z\right).
\end{aligned} \tag{21}$$

Note the same combination of the $1/\alpha_w$ and $\log Z$ in both amplitudes

The reason why the CP-odd amplitudes (21) are related to the moduli dependence of CP-even parameters $\alpha(M,\overline{M})$ and $Z(M,\overline{M})$ is supersymmetry of the scattering amplitudes. Indeed, as long as the supersymmetry remains unbroken, the physical amplitudes moduli quanta and their superpartners to the photons and photinos are related to each other by SUSY. In particular, the amplitudes



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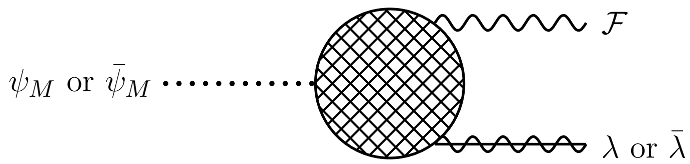

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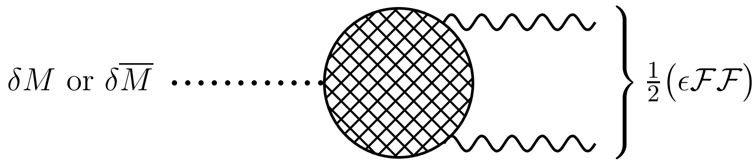


(24)

are all equal to each other modulo factors of $\pm i$. Consequently, both CP-odd amplitudes (21) are in agreement with the CP-even amplitudes involving the same modulus scalar and two photons being

$$\big(\delta M\ \text{or}\ \delta\overline{M}\big) \times \big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big)\ \text{amplitudes} = -\frac{1}{16\pi^2}\left(\frac{\partial}{\partial M}\ \text{or}\ \frac{\partial}{\partial\overline{M}}\right)\left(\frac{\pi}{\alpha_w}+\log Z\right) \tag{25}$$

where the RHS is indeed CP-even.

In the superfield formulation, we may combine the amplitudes (22) through (24) in a superfield amplitude and even derive it from the Konishi anomaly. Indeed, consider the variation of the SQED Lagrangian

$$\begin{aligned}\mathcal{L} \;=&\int\! d^2\theta\, d^2\bar\theta\, Z(M,\overline{M}) \times \left(\overline{A}e^{+2eV}A \,+\, \overline{B}e^{-2eV}B\right)\\&+\frac{i}{16\pi}\int\! d^2\theta\,\tau(M)\times W^\alpha W_\alpha \;+\; \frac{-i}{16\pi}\int\! d^2\bar\theta\,\tau^*(\overline{M})\times \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\end{aligned} \tag{26}$$

due to infinitesimal moduli variations $\delta M(y,\theta)$ and $\delta\overline{M}(\bar y,\bar\theta)$. First, there is a tree level variance due to moduli-dependent $\tau(M)$, thus

$$\Gamma_{\text{tree}} \;=\; \delta\mathcal{L}_V \;=\; \frac{i}{16\pi}\frac{\partial\tau}{\partial M}\times\int\! d^2\theta\,\delta M\times W^\alpha W_\alpha \;+\; \frac{-i}{16\pi}\frac{\partial\tau^*}{\partial\overline{M}}\times\int\! d^2\bar\theta\,\delta\overline{M}\times\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}. \tag{27}$$

For future reference, we may use holomorphy of the Wilsonian coupling τ to make both derivatives (WRT M or $\overline{M}$) act on $\operatorname{Im}\tau = -\frac{i}{2}\tau + \frac{i}{2}\tau^*$ (instead of just on τ or on τ^*), thus

$$\Gamma_{\text{tree}} \;=\; -\frac{1}{8\pi}\frac{\partial\operatorname{Im}\tau}{\partial M}\times\int\! d^2\theta\,\delta M\times W^\alpha W_\alpha \;-\; \frac{1}{8\pi}\frac{\partial\operatorname{Im}\tau}{\partial\overline{M}}\times\int\! d^2\bar\theta\,\delta\overline{M}\times\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}. \tag{28}$$

Second, these is variance of the electron Lagrangian due to moduli dependence of Z and hence anomalous one-loop coupling to the vector superfields:

$$\begin{aligned}\delta\mathcal{L}_E \;&=\int\! d^2\theta\, d^2\bar\theta\,\delta Z(M,\overline{M})\times\left(\overline{A}e^{+2eV}A \,+\, \overline{B}e^{-2eV}B \;=\; \frac{1}{Z}J_{\text{axial}}\right)\\&=\int\! d^2\theta\, d^2\bar\theta\left(\frac{\partial\log Z}{\partial M}\times\delta M \,+\, \frac{\partial\log Z}{\partial\overline{M}}\times\delta\overline{M}\right)\times J_{\text{axial}}\\&=\frac{\partial\log Z}{\partial M}\times\int\! d^2\theta\, d^2\bar\theta\,\delta M\times J_{\text{axial}} \;+\; \frac{\partial\log Z}{\partial\overline{M}}\times\int\! d^2\theta\, d^2\bar\theta\,\delta\overline{M}\times J_{\text{axial}}\end{aligned} \tag{29}$$

where

$$\begin{aligned}\int\! d^2\theta\, d^2\bar\theta\,\delta M\times J_{\text{axial}} \;&=\int\! d^2\theta\,\delta M\times\tfrac{-1}{4}\overline{D}^2 J_{\text{axial}}\\&\qquad\langle\!\langle\text{ by Konishi anomaly }\rangle\!\rangle\\&=\int\! d^2\theta\,\delta M\times\frac{-1}{8\pi^2}W^\alpha W_\alpha\end{aligned} \tag{30}$$

and likewise

$$
\begin{aligned}
\int d^2\theta\, d^2\bar\theta\, \delta\overline{M} \times J_{\rm axial} \;&=\; \int d^2\bar\theta\, \delta\overline{M} \times \tfrac{-1}{4} D^2 J_{\rm axial} \\
&\qquad \langle\!\langle \text{ by Konishi anomaly } \rangle\!\rangle \\
&=\; \int d^2\theta\, \delta\overline{M} \times \frac{-1}{8\pi^2} \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}.
\end{aligned}
\tag{31}
$$

Altogether,

$$
\Gamma^{\rm anomaly} \;=\; -\frac{1}{8\pi^2}\,\frac{\partial \log Z}{\partial M} \times \int d^2\theta\, \delta M \times W^\alpha W_\alpha \;-\; \frac{1}{8\pi^2}\,\frac{\partial \log Z}{\partial \overline{M}} \times \int d^2\theta\, \delta\overline{M} \times \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}, \tag{32}
$$

so combining it with the tree-level amplitude we get

$$
\begin{aligned}
\Gamma \;=\; & -\frac{1}{8\pi^2}\,\frac{\partial}{\partial M}\left(\frac{\pi}{\alpha_w} + \log Z\right) \times \int d^2\theta\, \delta M \times W^\alpha W_\alpha \\
& -\; \frac{1}{8\pi^2}\,\frac{\partial}{\partial \overline{M}}\left(\frac{\pi}{\alpha_w} + \log Z\right) \times \int d^2\theta\, \delta\overline{M} \times \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}.
\end{aligned}
\tag{33}
$$

The amplitude (33) includes tree-level and one-loop effects, but what about the higher-loop corrections? By holomorphy of $\tau(W)$, the tree π/α_w term is not subject to any *moduli dependent* corrections. Likewise, the Konishi anomaly is exact at the one-loop level. However, the couplings of the anomalous current $J_{\rm axial}$ to the moduli quanta follow from the electron field strength factor $Z(M,\overline{M})$, and that Z factor is subject to quantum corrections at all orders of the perturbation theory. Thus, to account for the higher-loop corrections to the amplitude (33), all we need is to replace the bare Z factor with the renormalized $Z_r(M,\overline{M},E)$, where E is the energy scale of the scattering process:

$$
\begin{aligned}
\Gamma^{\rm all\,loops} \;=\; & -\frac{1}{8\pi^2}\,\frac{\partial}{\partial M}\left(\frac{\pi}{\alpha_w(M,\overline{M})} + \log Z_r(M,\overline{M},E)\right) \times \int d^2\theta\, \delta M \times W^\alpha W_\alpha \\
& -\; \frac{1}{8\pi^2}\,\frac{\partial}{\partial \overline{M}}\left(\frac{\pi}{\alpha_w(M,\overline{M})} + \log Z_r(M,\overline{M},E)\right) \times \int d^2\theta\, \delta\overline{M} \times \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}.
\end{aligned}
\tag{34}
$$

Note that both types of amplitudes follow from the derivatives of the same *inseparable combination*

$$
T(M,\overline{M},E) \;=\; \frac{\pi}{\alpha_w(M,\overline{M})} \;+\; \log Z_r(M,\overline{M},E). \tag{35}
$$

I shall explain the physical meaning of this combination in a moment, but first let's see how to keep it invariant under the chiral redefinitions of the electron superfields.

In the ordinary QED, we have dealt with with the axial symmetries $\Psi' = \exp(i\alpha\gamma^5)\Psi$ — or in terms of the Weyl fermions, $\psi'_a = \exp(+i\alpha)\psi_a$, $\psi'_b = \exp(+i\alpha)\psi_b$ — and also moduli-dependent axial field redefinitions for non-constant $\alpha(x) = \alpha(\phi(x))$. In the SQED, we may generalize such axial field redefinitions by promoting $\alpha(x)$ to arbitrary chiral superfields $\Lambda(y, \theta)$. Thus, let

$$\begin{aligned} A'(y,\theta) &= \exp\bigl(+i\Lambda(y,\theta)\bigr) \times A(y,\theta), \qquad & B'(y,\theta) &= \exp\bigl(+i\Lambda(y,\theta)\bigr) \times B(y,\theta), \\ \overline{A}'(\bar{y},\bar{\theta}) &= \exp\bigl(-i\overline{\Lambda}(\bar{y},\bar{\theta})\bigr) \times \overline{A}(\bar{y},\bar{\theta}), \qquad & \overline{B}'(\bar{y},\bar{\theta}) &= \exp\bigl(-i\overline{\Lambda}(\bar{y},\bar{\theta})\bigr) \times \overline{B}(\bar{y},\bar{\theta}), \\ & & V'(x,\theta,\bar{\theta}) &= V(x,\theta,\bar{\theta}), \end{aligned} \tag{36}$$

where $\Lambda(y, \theta)$ must be chiral to preserve the chirality of A' and B' superfields and its conjugate $\overline{\Lambda}(\bar{y}, \bar{\theta})$ must be antichiral for similar reasons. In general, the SQED Lagrangian (26) is NOT invariant under such transforms; instead,

$$\left(\overline{A}e^{+2eV}A + \overline{B}e^{-2eV}B\right)' = \left(\overline{A}e^{+2eV}A + \overline{B}e^{-2eV}B\right) \times \exp\bigl(i\Lambda - i\overline{\Lambda}\bigr). \tag{37}$$

However, if we make Λ a function of the modulus M, and at the same time we redefine the charged fields we also change the $Z(M, \overline{M})$ factor such that

$$Z'(M,\overline{M}) = Z(M,\overline{M}) \times \exp\bigl(-i\Lambda + i\overline{\Lambda}\bigr), \tag{38}$$

then the classical Lagrangian of SQED would remain invariant. Likewise, the effective classical action of the quantum theory would remain invariant provided the *renormalized* Z factor transforms in the same way,

$$Z'_r(M,\overline{M},E) = Z_r(M,\overline{M},E) \times \exp\bigl(-i\Lambda + i\overline{\Lambda}\bigr). \tag{39}$$

But changing the moduli dependence of the Z factor changes the anomalous coupling of the moduli to the vector superfield. To compensate for this effect, we should also change the holomorphic coupling $\tau(M)$ such that the combination (35) remains invariant,

$$\pi\,\mathrm{Im}\,\tau' \;+\; \log Z'_r \;=\; \pi\,\mathrm{Im}\,\tau \;+\; \log Z_r\,. \tag{40}$$

According to eq. (39),

$$\log Z'(M,\overline{M},E) \;-\; \log Z(M,\overline{M},E) \;=\; -i\Lambda(M) \;+\; i\overline{\Lambda}(\overline{M}) \;=\; 2\,\mathrm{Im}\,\Lambda(M). \tag{41}$$

Fortunately, this is a harmonic function of the complex modulus, so it can be compensated by a holomorphic change of τ:

$$\tau'(M) \;=\; \tau(M) \;-\; \frac{2}{\pi}\Lambda(M).$$

Or in bosonic components,

$$\Theta' \;=\; \Theta \;-\; 4\,\mathrm{Re}\,\Lambda, \tag{42}$$

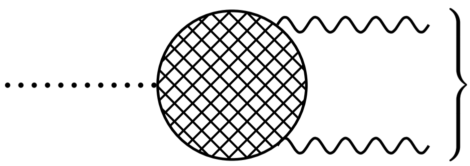
exactly as in the ordinary QED (where α plays the role of $\mathrm{Re}\,\Lambda$), and also

$$\frac{1}{\alpha'_w} \;=\; \frac{1}{\alpha_w} \;-\; \frac{2}{\pi}\,\mathrm{Im}\,\Lambda. \tag{43}$$

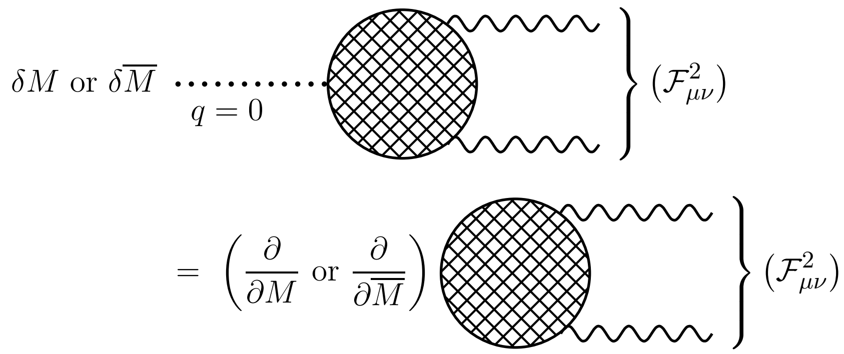
Thus, a rescaling — as opposed to a mere phase change — of SQED's charged superfields must be accompanies by an adjustment to its Wilsonian gauge coupling α_w. Consequently, the Wilsonian gauge coupling is not a physical coupling that can be directly measured by scattering amplitudes but just a parameter of the bare Lagrangian of the quantum theory. Instead, the physical gauge coupling is related to the invariant combination

$$T(M,\overline{M},E) \;=\; \frac{\pi}{\alpha(M,\overline{M})} \;+\; \log Z_r(M,\overline{M},E). \tag{35}$$

To see how this works, let's go back to the CP-even bosonic amplitudes

δM or $\delta\overline{M}$  $(\mathcal{F}^2_{\mu\nu})$. (44)

Take the momentum q of the modulus quantum δM or $\delta\overline{M}$ to zero, $q \to 0$, while the two photons have finite off-shell momenta $\pm k$, $k^2 = -E^2$. In this limit, we may treat the modulus as a global parameter, so the amplitudes (44) become simply the $\partial/\partial M$ and $\partial/\partial\overline{M}$ derivatives of the amplitude for two off-shell photons and no moduli quanta at all, thus



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But the two-photon amplitude on the bottom line here is precisely the running gauge coupling — or rather $-1/4e_r^2$ — renormalized to energy scale $E^2 = -k^2$. Thus,

$$\lim_{q\to 0}\Big(\big(\delta M \,\text{or}\, \delta\overline{M}\big) \times \mathcal{F}_{\mu\nu}^2 \text{ amplitude}\Big) \;=\; \left(\frac{\partial}{\partial M} \;\text{or}\; \frac{\partial}{\partial\overline{M}}\right)\frac{-1}{4e_r^2}\,. \tag{46}$$

At the same time, back in eq. (25) we have evaluated the same CP-even amplitude to

$$\text{(same amplitude)} \;=\; \left(\frac{\partial}{\partial M} \;\text{or}\; \frac{\partial}{\partial\overline{M}}\right)\frac{-T}{16\pi^2} \tag{47}$$

from which we conclude that

$$\frac{8\pi^2}{e_r^2(M,\overline{M},E)} \;=\; T(M,\overline{M},E) \;+\; \text{(a moduli-independent constant)}. \tag{48}$$

Note that the ‘constant’ term on the RHS should be moduli-independent but it may depend on the renormalization energy scale E. Also, the Wilsonian gauge coupling on the RHS of eq. (48) implicitly depends on the UV cutoff Λ of the theory, so the moduli-independent term may also depend on the cutoff, thus

$$\frac{\pi}{\alpha_r(M,\overline{M},E)} \;=\; \frac{\pi}{\alpha_w(M,\overline{M},\Lambda)} \;+\; \log Z_r(M,\overline{M},E) \;+\; F(E,\Lambda) \tag{49}$$

for some unknown function F of the renormalization energy scale E and the UV cutoff Λ, but not of the moduli.

To determine that function, we use two arguments: First, a theory with massless particles does not have any inherent energy scales besides the cutoff, so by dimensional analysis

$$F(E,\Lambda) \;=\; F(E/\Lambda). \tag{50}$$

Second, the running couplings α_r and Z_r depend on the renormalization energy scale E but not on the cutoff Λ, so the cutoff dependence of the other two terms in eq. (49) should cancel out,

$$\frac{\pi}{\alpha_w(M,\overline{M},\Lambda)} \;+\; F(E/\Lambda) \quad \text{should be } \Lambda\text{-independent.} \tag{51}$$

Now, by reasons of holomorphy, the Wilsonian gauge coupling's beta-function is exact at one-loop order,

$$\frac{\partial e_w(M,\overline{M},\Lambda)}{\partial \log\Lambda} \;=\; \beta_w(e) \;=\; \frac{e^3}{8\pi^2}\,, \quad \text{exactly,} \tag{52}$$

hence

$$\frac{\partial}{\partial \log\Lambda}\left(\frac{\pi}{\alpha_w} = \frac{4\pi^2}{e_w^2}\right) \;=\; -\frac{8\pi^2}{e_w^3}\times\beta_w(e_w) \;=\; -1 \tag{53}$$

and therefore

$$\frac{\pi}{\alpha_w(M,\overline{M},\Lambda)} \;=\; \frac{\pi}{\alpha_w(M,\overline{M},\Lambda_0)} \;-\; \log\frac{\Lambda}{\Lambda_0}\,, \quad \text{exactly.} \tag{54}$$

Plugging this formula into eq. (51), we immediately see that it calls for

$$F(E/\Lambda) \;=\; \log\frac{\Lambda}{E} \;+\; \text{numeric constant} \tag{55}$$

and hence

$$\frac{\pi}{\alpha_w(M,\overline{M},\Lambda)} \;+\; F(E/\Lambda) \;=\; \frac{\pi}{\alpha_w(M,\overline{M},\Lambda_0)} \;+\; \log\frac{\Lambda_0}{E} \;+\; \text{const.} \tag{56}$$

Plugging this formula into eq. (49), we arrive at the *all-loop Novikov–Shifman–Vainshtein–Zaharov formula for the running SQED gauge coupling,*

$$\frac{\pi}{\alpha_r(M,\overline{M},E)} \;=\; \frac{\pi}{\alpha_w(M,\overline{M},\Lambda_0)} \;+\; \log\frac{\Lambda_0}{E} \;+\; \text{const} + \;\log Z_r(M,\overline{M},E). \tag{57}$$

There is a related NSVZ equation for the SQED beta-function, which obtains by taking the derivatives of both sides of (57) WRT $\log E$ at fixed values of the moduli. On the LHS, we have

$$\frac{\partial}{\partial\log E}\frac{\pi}{\alpha_r(E)} \;=\; -\frac{8\pi^2}{e_r^3}\times\frac{\partial e_r}{\partial\log E} \;=\; -\frac{8\pi}{e_r^3}\times\beta(e). \tag{58}$$

On the RHS of (57), the Wilsonian gauge coupling depends on the reference cutoff Λ_0 but not on running energy scale E, while the running Z_r factor obeys

$$\frac{\partial\log Z_r}{\partial\log E} \;=\; 2\gamma(e) \tag{59}$$

where $\gamma(e)$ is the anomalous dimension of the electron superfields A and B. Thus altogether

$$-\frac{8\pi^2}{e_r^3}\times\beta(e) \;=\; -1 \;+\; 2\times\gamma(e) \tag{60}$$

and therefore

$$\beta(e) \;=\; \frac{e^3}{8\pi^2}\times\bigl(1-2\gamma(e)\bigr) \tag{61}$$

to all orders of the perturbation theory. Thus, given the electron's anomalous dimension γ to an n-loop order, this formula immediately gives us the SQED beta-function to an $(n+1)$-loop order.

Generalizations to Other Gauge Theories

All our results about the Konishi anomaly, the Wilsonian gauge couplings, and the NSVZ beta-functions in SQED may be generalized to more complicated supersymmetric gauge theories. Let's start with theories with a $U(1)$ gauge group — just like the SQED — but a general spectrum of charged chiral superfields $\Phi_1,\ldots\,\Phi_n$ of respective charges $q_1e,\ldots,q_ne$. All Wilsonian couplings of the theory may depend on some moduli, so the Lagrangian is

$$\begin{aligned}\mathcal{L} \;=&\; \int\! d^2\theta\,d^2\bar\theta\sum_{i=1}^n Z_i(M,\overline{M})\times\overline{\Phi}_i\exp(2q_iV)\Phi_i\\ &+\int\! d^2\theta\left(\frac{i\tau(M)}{16\pi}\,W^\alpha W_\alpha\;+\;W(\Phi,M)\right)\;+\;\text{H.c.}\end{aligned} \tag{62}$$

For consistency of the theory, every term in the superpotential $W(\Phi)$ must have net charge

$q=0$, and the the gauge symmetry must be anomaly-free,

$$\sum_{i=1}^{n} q_i^3 \;=\; 0 \quad \text{and} \quad \sum_{i=1}^{n} q_i \;=\; 0. \tag{63}$$

In the complete absence of the superpotential, $W(\phi)\equiv 0$, the Lagrangian (62) has a classical global $[U(1)]^n$ symmetry, $A_i \to \exp(i\alpha_i)A_i$, with the corresponding current superfield

$$J_i \;=\; Z_i \times \overline{\Phi}_i e^{2q_i V}\Phi_i\,. \tag{64}$$

Classically, each current obeys $D^2 J_i = 0$, $\overline{D}^2 J_i = 0$, but in the quantum theory there are Konishi anomalies

$$-\tfrac14 \overline{D}^2 J_i \;=\; -\frac{q_i^2}{16\pi^2} W^\alpha W_\alpha\,, \qquad -\tfrac14 D^2 J_i \;=\; -\frac{q_i^2}{16\pi^2} \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha}. \tag{65}$$

Note that a chiral redefinition of the charged fields

$$\Phi_i'(y,\theta) \;=\; \exp\bigl(i\Lambda_i(M(y,\theta))\bigr) \times \Phi_i(y,\theta), \qquad \overline{\Phi}'(\bar y,\bar\theta) \;=\; \exp\bigl(-i\overline{\Lambda}_i(\overline{M}(\bar y,\bar\theta))\bigr) \times \overline{\Phi}_i(\bar y,\bar\theta), \tag{66}$$

accompanied by

$$Z_i'^{(r)}(M,\overline{M},E) \;=\; Z_i^{(r)}(M,\overline{M},E) \times \exp\bigl(-i\Lambda_i(M) + i\overline{\Lambda}_i(\overline{M})\bigr) \tag{67}$$

preserves the effective classical Lagrangian for the charged fields, but modifies the couplings of the moduli to the Konishi anomalies (65) and hence the anomalous couplings of the moduli to the gauge fields. However, for each Φ_i rescaled by $\exp(i\Lambda_i(M))$, we may compensate for changing the $\delta M \times W^\alpha W_\alpha$ anomalous coupling by changing the tree-level holomorphich coupling $\tau(M)$ by

$$\Delta\tau(M) \;=\; -\frac{q_i^2}{\pi}\Lambda_i\,. \tag{68}$$

Altogether, we need

$$\tau'(M) \;-\; \tau(M) \;=\; -\frac{1}{\pi}\sum_{i=1}^{n} q_i^2 \times \Lambda_i(M), \tag{69}$$

which makes the combination

$$2T(M,\overline{M},R) \;=\; 2\pi\,\mathrm{Im}\,\tau(M)\;+\;\sum_{i=1}^{n} q^2\times\log Z_i^{(r)}(M,\overline{M}), \tag{70}$$

invariant under all rescalings (66) of the charged fields. Similar to SQED, it is this invariant combination which governs the modulus+two photons amplitudes.

Eventually, proceeding exactly as in SQED case, we end up with the NSVZ equation for the running gauge coupling

$$\frac{2\pi}{\alpha_r(M,\overline{M},E)} \;=\; \frac{2\pi}{\alpha_w(M,\overline{M},\Lambda_0)}\;+\;\sum_i q_i^2\times\log\frac{\Lambda_0}{E}\;+\;\sum_i q_i^2\times\log Z_i^{(r)}(M,\overline{M},E), \tag{71}$$

and ultimately the NSVZ equation for the beta-function,

$$\beta(e) \;=\; \frac{e^3}{16\pi^2}\sum_{i=1}^{n} q_i^2\times\bigl(1-2\gamma_i(e)\bigr) \tag{72}$$

where γ_i is the anomalous dimension of the charged superfield Φ_i.

But what if $W(\Phi)\neq 0$ and there are some masses and/or Yukawa couplings? In this case, many — if not all — of the J_i currents are no longer classically conserved. But the Konishi anomalies remain exactly the same, they simply add up to the classical mis-conservations of the currents,

$$\begin{aligned} \tfrac14\overline{D}^2 J_i \;&=\; \frac{q_i^2}{16\pi^2}\,W^\alpha W_\alpha\;+\;\Phi_i\times\frac{\partial W}{\partial\Phi_i}\,,\\ \tfrac14 D^2 J_i \;&=\; \frac{q_i^2}{16\pi^2}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\;+\;\overline{\Phi}_i\times\frac{\partial W^*}{\partial\overline{\Phi}_i}\,. \end{aligned} \tag{73}$$

But under a chiral field redefinition (66), the $Z_i^{(r)}$ factors transform exactly as in eq. (67), while the Wilsonian holomorphic gauge coupling $\tau(M)$ transforms exactly as in eq. (69), regardless of the superpotential $W(\Phi)$. (Although the superpotential couplings should also transform to keep $W'(\Phi')=W(\Phi)$.) Consequently, the Novikov–Shifman–Vainshtein–Zaharov equations for the gauge coupling and its beta-function remain exactly as in (71) and (72).

Note: in presence of the superpotential Yukawa couplings $W \supset \frac{1}{6} g_{ijk} \Phi_i \Phi_j \Phi_k$, the anomalous dimensions γ_i depend both on the gauge coupling e and on the Yukawa couplings g_{ijk}. But once we calculate all the anomalous dimensions to some n-loop order, we get the beta-functions for both kinds of couplings:

$$\beta[g_{ijk}] \;=\; g_{ijk} \times \bigl(\gamma_i + \gamma_j + \gamma_k\bigr) \tag{74}$$

to the n-loop order, and

$$\beta[e] \;=\; \frac{e^3}{16\pi^2} \sum_{i=1}^{n} q_i^2 \times \bigl(1 - 2\gamma_i\bigr) \tag{75}$$

to the $(n+1)$ loop order.

⋆ ⋆ ⋆

Next, consider the SQCD as an example of a non-abelian SUSY gauge theory. In matrix notations, the vector superfield V is a traceless hermitian $N_c \times N_c$ matrix, the quark superfields A and $\overline{B}$ are $N_c \times N_f$ matrices, while the antiquark superfields $\overline{A}$ and B are $N_f \times N_c$ matrices, and the Lagrangian is

$$\begin{aligned} \mathcal{L} \;=& \int d^2\theta\, d^2\bar\theta\, Z(M,\overline{M}) \times \mathrm{tr}\Bigl(\overline{A} \exp(+2V) A \;+\; B \exp(-2V) \overline{B}\Bigr) \\ &+ \int d^2\theta \left(\mathrm{tr}\bigl(m(M) BA\bigr) \;+\; \frac{i\tau(M)}{8\pi}\, \mathrm{tr}\bigl(W^\alpha W_\alpha\bigr) \right) \;+\; \mathrm{H.\,c.} \end{aligned} \tag{76}$$

where m is the $N_f \times N_f$ quark mass matrix. When $m = 0$, — which we assume through the rest of this section — the Lagrangian (76) has a $U(N_f) \times U(N_f)$ chiral symmetry; there is also a $U(1)$ R-symmetry, but let's put that aside for a moment. Similar to the ordinary QCD, the $SU(N_f) \times SU(N_f) \times U(1)_V$ flavor symmetries remain good symmetries of the quantum theory, but the axial $U(1)_A$ is anomalous:

$$\partial_\mu j^{5\mu}[U(1)_A] \;=\; \frac{N_f}{16\pi^2}\, \mathrm{tr}\bigl(\epsilon \mathcal{F}\mathcal{F}\bigr). \tag{77}$$

The $\mathcal{F}_{\mu\nu}$ on the RHS here are complete non-abelian tension fields, so $\mathrm{tr}\bigl(\epsilon\mathcal{F}\mathcal{F}\bigr)$ includes both 2-photon and 3-photon terms. Supersymmetrizing the anomaly equation (77) we get the non-abelian Konishi anomaly,

$$J_{\rm ax} = Z\times {\rm tr}\Big(\overline{A}\exp(+2V)A + B\exp(-2V)\overline{B}\Big), \tag{78}$$

$$\tfrac14\overline{D}^2 J_{\rm ax} = \frac{N_f}{8\pi^2}\,{\rm tr}\big(W^\alpha W_\alpha\big), \tag{79}$$

$$\tfrac14 D^2 J_{\rm ax} = \frac{N_f}{8\pi^2}\,{\rm tr}\big(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\big). \tag{80}$$

In the superfield formulation, the right hand sides here comprise infinite series of n-vector amplitudes for $n=2,3,4,\ldots$, so a direct superfield calculation of the anomalies (79) and (80) involves an infinite series of non-trivial one-loop diagrams.⋆ Fortunately, in the background field formalism, the non-abelian anomaly calculation reduces to the same few triangle graphs as in the abelian case. Alas, the background field method — and hence this calculation — are to technical for this introductory SUSY class. The interested students are referred to the Superspace book, §6.5–7.

For other flavor symmetries generated by

$$\delta A = iA\times Q_a\,, \qquad \delta B = iQ_b\times B \tag{81}$$

for some $N_f\times N_f$ matrices Q_a and Q_b, we have currents

$$J[Q_a,Q_b] = Z\,tr\big(\overline{A}e^{+2V}A\times Q_a\big) + Z\,{\rm tr}\big(Q_b\times Be^{-2V}\overline{B}\big) \tag{82}$$

whose color anomalies are proportional to

$$C[Q_a,Q_b] = {\rm tr}(Q_a) + {\rm tr}(Q_b): \tag{83}$$

$$\tfrac14\overline{D}^2 J[Q_a,Q_b] = \frac{C[Q_a,Q_b]}{16\pi^2}\,{\rm tr}\big(W^\alpha W_\alpha\big), \tag{84}$$

$$\tfrac14 D^2 J[Q_a,Q_b] = \frac{C[Q_a,Q_b]}{16\pi^2}\,{\rm tr}\big(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\big). \tag{85}$$

Indeed, all the traceless $SU(N_f)\times SU(N_f)$ flavor symmetry have $C=0$, and the vector $U(1)_V$ also has $C=0$, so all these symmetries are anomaly free. On the other hand, the axial $U(1)$ symmetry has $C=2N_f$, hence eqs. (79) and (80).

⋆ But only one-loop, since all the multi-loop diagrams may be regulated by covariant higher-derivative terms for the vector superfields, and such CHD terms do not break the axial symmetry. But the one-loop diagrams take another regulator such as Pauli–Villars, which does break the axial symmetry, and that's what causes the anomaly.

⋆ ⋆ ⋆

Next, consider the SQCD beta-function. As we have learned last semester, at the one-loop level the beta-function for a gauge coupling depends only on the spectrum of the field multiplets WRT the gauge symmetry in question, but it does not depend on any other interactions of those fields. Specifically,

$$\beta^{1\,\text{loop}}(g) \;=\; B\times\frac{g^3}{16\pi^2} \tag{86}$$

for

$$B \;=\; \sum_{\substack{\text{all physical}\\ \text{multiplets}}} R(\text{multiplet})\times\begin{cases} -\frac{11}{3} & \text{for the gauge fields,}\\ +\frac{4}{3} & \text{for Dirac fermions,}\\ +\frac{2}{3} & \text{for Majorana fermions,}\\ +\frac{2}{3} & \text{for chiral Weyl fermions,}\\ +\frac{1}{3} & \text{for complex scalar fields,}\\ +\frac{1}{6} & \text{for real scalar fields,}\end{cases} \tag{87}$$

see my notes on QCD beta function for details. The $R(\text{multiplet})$ in eq. (87) is the multiplet's quadratic index: for any two gauge group generators T^a and T^b,

$$\text{tr}_{\text{multiplet}}\bigl(T^aT^b\bigr) \;=\; \delta^{ab}\times R(\text{multiplet}). \tag{88}$$

For a supersymmetric theory, eq. (87) works verbatim in terms of the component vector, fermionic, or scalar fields, and the multiplets they form WRT the gauge group. In particular, counting only the physical component fields of SQCD, we have the following multiplets:

- Adjoint multiplet of vector fields.
- Adjoint multiplet of Weyl gauginos.
- N_f fundamental multiplets of Dirac quarks.
- $2N_f$ fundamental multiplets of squarks (which are complex scalars).

Altogether, their contributions to the one-loop beta-functions add up to

$$\begin{aligned} B \;&=\; -\tfrac{11}{3}\times R(\text{adjoint})\;+\;\tfrac{2}{3}\times R(\text{adjoint}) \\ &\qquad +\;\tfrac{4}{3}\times N_f\times R(\text{fundamental})\;+\;\tfrac{1}{3}\times 2N_f\times R(\text{fundamental}) \\ &=\; -3\times R(\text{adjoint})\;+\;2\times N_f\times R(\text{fundamental}). \end{aligned} \tag{89}$$

For the $SU(N_c)$ gauge group, the adjoint multiplet has index N_c while the fundamental multiplet has index $\frac{1}{2}$, thus

$$B \;=\; -3N_c \;+\; N_f \tag{90}$$

and therefore

$$\beta^{1\,\text{loop}}(g) \;=\; \frac{g^3}{16\pi^2}\times(-3N_c+N_f). \tag{91}$$

Consequently, SQCD is asymptotically (UV limit) free only for $N_f < 3N_c$. On the other hand, SQCD with $N_f > 3N_c$ *massless* flavors becomes weakly coupled and almost free in the deep IR limit of the RG flow. This regime of SQCD-like theories is called *the non-abelian Coulomb phase.*

The higher-loop terms in SQCD beta-function obtain from the NSVZ equation. To derive this equation, let's start with the coupling of the moduli quanta δM and $\delta\overline{M}$ to the CP-off combination $\text{tr}(\epsilon\mathcal{F}\mathcal{F})$ of two or three gluons. Just like in SQED, there are several contribution to these amplitudes:

1. Tree-level coupling of $\Theta(M,\overline{M}) = 2\pi\,\text{Re}\,\tau(M)$ to the gluons,

$$\begin{aligned} \big(\delta M\ \text{or}\ \delta\overline{M}\big)\times\text{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big)\ \text{amplitude}_1 \;&=\; \frac{1}{32\pi}\left(\frac{\partial\tau}{\partial M}\ \text{or}\ \frac{\partial\tau^*}{\partial\overline{M}}\right) \\ &=\; \frac{1}{16\pi}\left(i\frac{\partial}{\partial M}\ \text{or}\ -i\frac{\partial}{\partial\overline{M}}\right)\frac{1}{\alpha_w(M,\overline{M})}\,. \end{aligned} \tag{92}$$

2. Coupling of the axial $U(1)_A$ current of the quarks

$$J^{5\mu} \;=\; Z\times\overline{\Psi}\gamma^5\gamma^\mu\Psi \qquad \langle\!\langle\,\text{summed over all colors and flavors}\,\rangle\!\rangle \tag{93}$$

to the gradients of the moduli field as

$$\mathcal{L} \;\supset\; L_\mu\times\frac{1}{Z}J^{5\mu} \tag{94}$$

for

$$L_\mu \;=\; -\frac{i}{2}\frac{\partial Z}{\partial M}\times\partial_\mu M \;+\; \frac{i}{2}\frac{\partial Z}{\partial\overline{M}}\times\partial_\mu\overline{M}. \tag{95}$$

Consequently,

$$\begin{aligned}
\delta\mathcal{L} \;&=\; -\frac{i}{2Z}\frac{\partial Z}{\partial M}\,\partial\delta M\times J^{5\mu} \;+\; \text{H.c.}\\
&\cong\; +\frac{i}{2}\frac{\partial\log Z}{\partial M}\,\delta M\times\partial_\mu J^{5\mu} \;+\; \text{H.c.}\\
&\qquad\langle\!\langle\,\text{using axial anomaly of the current}\,\rangle\!\rangle\\
&=\; \frac{i}{2}\frac{\partial\log Z}{\partial M}\,\delta M\times\frac{N_f}{16\pi^2}\,\mathrm{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big) \;+\; \text{H.c.},
\end{aligned} \tag{96}$$

hence anomalous couplings of moduli quanta to the gluons with amplitude

$$\big(\delta M\text{ or }\delta\overline{M}\big)\times\mathrm{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big)\text{ amplitude}_2 \;=\; \frac{N_f}{32\pi^2}\left(i\frac{\partial}{\partial M}\text{ or }-i\frac{\partial}{\partial\overline{M}}\right)\log Z(M,\overline{M}). \tag{97}$$

3. The chiral chirrent of the gluinos also couples to the moduli quanta via moduli dependence of the Θ angle:

$$\begin{aligned}
\mathcal{L} \;&\supset\; \frac{\Theta(M,\overline{M})}{16\pi^2}\times\partial_\mu\,\mathrm{tr}\big(\bar\lambda\bar\sigma^\mu\lambda\big)\\
\delta\mathcal{L} \;&=\; \frac{1}{16\pi^2}\left(\frac{\partial\Theta}{\partial M}\,\delta M \;+\; \frac{\partial\Theta}{\partial\overline{M}}\,\delta\overline{M}\right)\times\partial_\mu\,\mathrm{tr}\big(\bar\lambda\bar\sigma^\mu\lambda\big).
\end{aligned} \tag{98}$$

In SQED there is a similar coupling of moduli quanta to the divergence of the photino current, but that current is not anomalous since the photinos are neutral. But in SQCD the gluinos are in the adjoint multiplet, so its chiral current is anomalous. Indeed, for any non-trivial multiplet (m) of LH Weyl fermions χ_i^α without the corresponding RH counterparts, the LH chiral current

$$j_L^\mu \;=\; \sum_i \bar\chi_i\bar\sigma^\mu\chi_i \tag{99}$$

has gauge anomaly

$$\partial_\mu j_L^\mu \;=\; \frac{R(m)}{8\pi^2}\,\mathrm{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big). \tag{100}$$

In particular, for the gluino current

$$\partial_\mu j_L^\mu[\text{gluinos}] \;=\; \frac{N_c}{8\pi^2}\,\operatorname{tr}\bigl(\epsilon\mathcal{F}\mathcal{F}\bigr). \tag{101}$$

However, the gluinos $\lambda^{a,\alpha}$ in eq. (98) are non-canonically normalized,

$$\mathcal{L} \;\supset\; \frac{1}{g^2}\,\operatorname{tr}\Bigl(i\bar\lambda_{\dot\alpha}\bar\sigma^{\mu,\dot\alpha\beta}\overleftrightarrow{\mathcal{D}}_\mu\lambda_\beta\Bigr) \;=\; \frac{1}{g^2}\sum_a \tfrac{i}{2}\bar\lambda^a_{\dot\alpha}\bar\sigma^{\mu,\dot\alpha\beta}\overleftrightarrow{\mathcal{D}}_\mu\lambda^a_\beta\,, \tag{102}$$

so their properly normalized chiral current — which obeys eq. (101) — is

$$j_L^\mu \;=\; \frac{1}{g^2}\times\sum_a \bar\lambda^a\bar\sigma^\mu\lambda^a \;=\; \frac{2}{g^2}\times\operatorname{tr}\bigl(\bar\lambda\bar\sigma^\mu\lambda\bigr). \tag{103}$$

Consequently,

$$\partial_\mu \operatorname{tr}\bigl(\bar\lambda\bar\sigma^\mu\lambda\bigr) \;=\; \frac{N_c g^2}{16\pi^2}\,\operatorname{tr}\bigl(\epsilon\mathcal{F}\mathcal{F}\bigr), \tag{104}$$

so eq. (98) becomes

$$\delta\mathcal{L} \;=\; \frac{N_c g^2(M,\overline{M})}{256\pi^4}\left(\frac{\partial\Theta}{\partial M}\,\delta M \;+\; \frac{\partial\Theta}{\partial\overline{M}}\,\delta\overline{M}\right)\times\operatorname{tr}\bigl(\epsilon\mathcal{F}\mathcal{F}\bigr). \tag{105}$$

Furthermore, at the one-loop level of accuracy we may use the tree-level couplings

$$g^2 \;=\; \frac{4\pi}{\operatorname{Im}\tau(M)}\,, \quad \Theta \;=\; 2\pi\operatorname{Re}\tau(M) \tag{106}$$

for a holomorphic function $\tau(M)$, hence

$$\begin{gathered}\frac{\partial\operatorname{Re}\tau}{\partial M} \;=\; i\frac{\partial\operatorname{Im}\tau}{\partial M}\\ \Downarrow\\ g^2\times\frac{\partial\Theta}{\partial M} \;=\; \frac{8\pi^2}{\operatorname{Im}\tau}\,\frac{\partial\operatorname{Re}\tau}{\partial M} \;=\; \frac{8\pi^2 i}{\operatorname{Im}\tau}\,\frac{\partial\operatorname{Im}\tau}{\partial M} \;=\; 8\pi^2 i\,\frac{\partial\log(\operatorname{Im}\tau)}{\partial M}\end{gathered} \tag{107}$$

and likewise

$$g^2\times\frac{\partial\Theta}{\partial\overline{M}} \;=\; -8\pi^2 i\,\frac{\partial\log(\operatorname{Im}\tau)}{\partial\overline{M}}\,. \tag{108}$$

Plugging these formulae into eq. (105), we get the amplitudes

$$\bigl(\delta M \text{ or } \delta\overline{M}\bigr)\times\operatorname{tr}\bigl(\epsilon\mathcal{F}\mathcal{F}\bigr)\text{ amplitude}_3 \;=\; \frac{N_c}{32\pi^2}\left(i\frac{\partial}{\partial M}\text{ or } -i\frac{\partial}{\partial\overline{M}}\right)\log\left(\operatorname{Im}\tau \;=\; \frac{1}{\alpha}\right). \tag{109}$$

4. Combining the tree amplitude (92) and the one-loop amplitudes (97) and (109), we get the net amplitude connecting the moduli quanta to the CP-odd combinations of 2 or 3 gluons:

$$\begin{aligned}&\big(\delta M \text{ or } \delta\overline{M}\big) \times \mathrm{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big) \text{ amplitude } = \\ &= \frac{1}{32\pi^2}\left(i\frac{\partial}{\partial M} \text{ or } -i\frac{\partial}{\partial\overline{M}}\right)\left[\frac{2\pi}{\alpha_w} + N_f \times \log Z + N_c \times \log\frac{1}{\alpha_w}\right].\end{aligned} \tag{110}$$

Similar to SQED, there are also higher-loop contributions which modify the couplings of the anomalous quark and gluino currents to the moduli quanta. To account for such effects, we should replace the bare $Z(M,\overline{M})$ factor for the quarks with the renormalized factor $Z_r(M,\overline{M},E)$ where E is the energy scale of the amplitude. Likewise, in the $\log(1/\alpha)$ term — and only in that term — we should replace the Wilsonian gauge coupling $\alpha_w(M,\overline{M})$ with the running gauge coupling $\alpha_r(M,\overline{M},E)$. With these corrections, we get the *all-loop* amplitudes

$$\begin{aligned}&\text{net } \big(\delta M \text{ or } \delta\overline{M}\big) \times \mathrm{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big) \text{ amplitude } = \\ &= \frac{1}{32\pi^2}\left(i\frac{\partial}{\partial M} \text{ or } -i\frac{\partial}{\partial\overline{M}}\right)\left[\frac{2\pi}{\alpha_w} + N_f \times \log Z_r + N_c \times \log\frac{1}{\alpha_r}\right].\end{aligned} \tag{111}$$

Thus far, we have used the axial and chiral anomalies to calculate the amplitudes connecting moduli quanta to the CP-odd combinations of the gluons. The direct calculations involving the CP-even combinations of the gluons would be much more difficult. Fortunately, by to supersymmetry the CP-even amplitudes are simply equal to the CP-odd amplitudes, up to overall factors $\pm i$. (And also factor of 2 since $(\epsilon\mathcal{F}\mathcal{F}) = 2\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}$.) Thus, without any further calculations, we immediately get the CP-even amplitudes as

$$\begin{aligned}&\text{net } \big(\delta M \text{ or } \delta\overline{M}\big) \times \mathrm{tr}\big(\mathcal{F}_{\mu\nu}^2\big) \text{ amplitude } = \\ &= -\frac{1}{16\pi^2}\left(\frac{\partial}{\partial M} \text{ or } \frac{\partial}{\partial\overline{M}}\right)\left[\frac{2\pi}{\alpha_w} + N_f \times \log Z_r + N_c \times \log\frac{1}{\alpha_r}\right].\end{aligned} \tag{112}$$

Next, we take the limit where the momentum q of the modulus quantum goes to zero while the gluon momenta remain off-shell at some energy scale E. In this limit,

$$\begin{aligned}
&\text{net } \big(\delta M \text{ or } \delta\overline{M}\big) \times \mathrm{tr}\big(\mathcal{F}_{\mu\nu}^2\big) \text{ amplitude} \\
&\qquad \mathop{\longrightarrow}_{q\to 0} \left(\frac{\partial}{\partial M} \text{ or } \frac{\partial}{\partial\overline{M}}\right)\Big(\mathrm{tr}\big(\mathcal{F}_{\mu\nu}^2\big) \text{ amplitude}\Big) \qquad (113)\\
&\text{where } \mathrm{tr}\big(\mathcal{F}_{\mu\nu}^2\big) \text{ amplitude} = -\frac{1}{2g_r^2(M,\overline{M},E)}\,, \qquad (114)
\end{aligned}$$

the (inverse) renormalized gauge coupling at scale E. Comparing the last 3 equations, we immediately see that

$$\left(\frac{\partial}{\partial M} \text{ or } \frac{\partial}{\partial\overline{M}}\right)\frac{8\pi^2}{g_r^2} = \left(\frac{\partial}{\partial M} \text{ or } \frac{\partial}{\partial\overline{M}}\right)\left[\frac{2\pi}{\alpha_w} + N_f \times \log Z_r + N_c \times \log\frac{1}{\alpha_r}\right] \qquad (115)$$

and therefore

$$\begin{aligned}
\frac{8\pi^2}{g_r^2(M,\overline{M},E)} = {}& \frac{8\pi^2}{g_w^2(M,\overline{M})} + N_f \times \log Z_r(M,\overline{M},E) + N_c \log\frac{4\pi}{g_r^2(M,\overline{M},E)} \qquad (116)\\
&+ \text{ a moduli-independent `constant'}
\end{aligned}$$

On the bottom line here, the 'constant' is in quotes because it is actually a function of the renormalization scale E, which for a massless theory means a function of the ratio of energy scale E to the UV cutoff Λ. The Wilsonian gauge coupling g_w is also cutoff-dependent, thus

$$\begin{aligned}
\frac{8\pi^2}{g_r^2(M,\overline{M},E)} = {}& \frac{8\pi^2}{g_w^2(M,\overline{M},\Lambda)} + F(E/\Lambda) \\
&+ N_f \times \log Z_r(M,\overline{M},E) + N_c \log\frac{4\pi}{g_r^2(M,\overline{M},E)}\,.
\end{aligned} \qquad (117)$$

Finally, the LHS here is cutoff-independent, so the net RHS should also independent, and since only the first two terms on the RHS depend on Λ, we need the

$$\frac{8\pi^2}{g_w^2(M,\overline{M},\Lambda)} + F(E/\Lambda) \qquad (118)$$

combination.to be cutoff-independent. By holomorphy of $\tau(M)$, the Wilsonian beta-function for the gauge coupling is exact at one loop order, thus

$$\beta_g^w = (N_f - 3N_c) \times \frac{g^3}{16\pi^2}, \quad \text{exactly}, \tag{119}$$

hence

$$\frac{\partial}{\partial \log \Lambda} \frac{8\pi^2}{g_w^2} = (3N_c - N_f), \quad \text{exactly}, \tag{120}$$

and therefore

$$\frac{8\pi^2}{g_w^2(\Lambda)} = \frac{8\pi^2}{g_w^2(\Lambda_0)} + (3N_c - N_f)\log\frac{\Lambda}{\Lambda_0}. \tag{121}$$

Hence, to keep the combination (118) Λ-independent we need

$$F(E/\Lambda) = (N_f - 3N_c)\log\frac{\Lambda}{E} + \text{numeric constant}, \tag{122}$$

and plugging this formula into eq. (117) we finally arrive at the NSVZ equation for SQCD:

$$\begin{aligned}\frac{8\pi^2}{g_r^2(M,\overline{M},E)} = {} & \frac{8\pi^2}{g_w^2(M,\overline{M},\Lambda_0)} + (N_f - 3N_c)\log\frac{\Lambda_0}{E} + \text{const} \\ & + N_f \times \log Z_r(M,\overline{M},E) + N_c \log\frac{4\pi}{g_r^2(M,\overline{M},E)}.\end{aligned} \tag{123}$$

Or rather, this is the integral form of the NSVZ equation for the running gauge coupling governing both its moduli dependence and energy scale dependence. To get the NSVZ equation for the SQCD beta function, we focus on the energy dependence and take the derivatives of both sides of eq. (123) WRT $\log E$. Thus

$$-\frac{16\pi^2}{g_r^3} \times \frac{dg_r}{d\log E} = (3N_c - N_F) + N_f \times \frac{d\log Z_r}{d\log E} - \frac{2N_c}{g_r} \times \frac{dg_r}{d\log E} \tag{124}$$

and hence

$$-\frac{16\pi^2}{g^3} \times \beta(g) = (3N_c - N_f) + N_f \times 2\gamma(g) - \frac{2N_c}{g} \times \beta(g) \tag{125}$$

where γ is the anomalous dimension of the quarks and squarks. Finally, solving this equation for the beta-function, we arrive at the NSVZ equation for the SQCD beta-function:

$$\beta(g) = \frac{g^3}{16\pi^2 - 2N_c g^2} \times \Big[-3N_c + N_f\big(1 - 2\gamma(e)\big)\Big], \quad \text{exactly}. \tag{126}$$

$\star \quad \star \quad \star$

Finally, let me write down the NSVZ equations for completely general SUSY gauge theories. Let's allow for a completely general gauge group G, simple or a product

$$G = G_1 \otimes G_2 \otimes \cdots \tag{127}$$

with the corresponding gauge couplings $g_1, g_2, \ldots$. Likewise, let the chiral superfields Φ_i form any multiplets (m_1), (m_2), $\ldots$ of G, as long as all such multiplets are complete and all the gauge^3 anomalies cancel out,

$$\sum_{(m)} \mathrm{tr}_{(m)}\bigl(t^a\{t^b,t^c\}\bigr) = 0 \quad \text{for any 3 generators } t^a, t^b, t^c \text{ of } G. \tag{128}$$

For the abelian factors of G — if any — the trace anomalies should also vanish.

Besides the gauge couplings, the theory may have a superpotential $W(\Phi)$ with any Yukawa couplings we like, as long as $W(\Phi)$ is gauge invariant.

With these assumptions, the one-loop beta function for the coupling g_a of the gauge group gactor G_a is simply

$$\beta_a^{1\,\mathrm{loop}} = B_a \times \frac{g_a^3}{16\pi^2} \tag{129}$$

regardless of any other gauge or Yukawa coupling, where

$$B_a = -3R[G_a](\mathrm{adj}) + \sum_{(m)} R[G_a](m), \tag{130}$$

with $R[G_a]$(multiplet) denoting the Index of that multiplet WRT G_a group.

For the Wilsonian gauge couplins, the one-loop beta-functions (129) are exact, but for the physical running gauge couplings there are higher-loop corrections related to the anomalous dimensions of the chiral superfields. Specifically,

$$\beta_a = \frac{g^3}{16\pi^2 - 2R[G_a](\mathrm{adj})g^2} \times \left[-3R[G_a](\mathrm{adj}) + \sum_{(m)} R[G_a](m) \times \bigl(1 - 2\gamma[(m)]\bigr)\right] \tag{131}$$

where $\gamma[(m)]$ is the anomalous dimensions of the fields in multiplet (m). Note: same γ for all scalar and fermionic components of every chiral superfield in the multiplet. Also note that such anomalous dimensions generally depend on all gauge and Yukawa couplings of the theory, so at the two-loop and higher levels, the beta-function for a gauge coupling g_a depends not only on the g_a but also on all the other couplings.

Finally, the moduli dependence of any Wilsonian guage coupling is a harmonic function

$$\frac{4\pi}{(g_a^w)^2(M,\overline{M})} \;=\; \operatorname{Im}\tau_a(M), \tag{132}$$

but the moduli dependence of the physical gauge couplings is generally non-harmonic. Instead,

$$\begin{aligned}\frac{8\pi^2}{(g_a^r)(M,\overline{M},E)} \;=\;& 2\pi\operatorname{Im}\tau(M,\Lambda) \;+\; B\times\log\frac{\Lambda}{E} \\ &+\; R[G_a](\mathrm{adj})\times\log\frac{1}{(g_r^a)(M,\overline{M},E)} \\ &+\; \sum_{(m)} R[G_a](m)\times\log Z^r_{(m)}(M,\overline{M},E).\end{aligned} \tag{133}$$