GAUGINO CONDENSATION IN SYM AND IN SQCD

Super Yang–Mills Theories

As a warm-up exercise, consider the ordinary QCD with a few light flavors. This theory has uniformly negative $\beta(g)$, hence asymptotic freedom in the UV limit. On the other hand, at lower energies the running coupling gets stronger until at $E \sim \Lambda_{\rm QCD}$ $g(E)$ becomes singular. Physically, this leads to a complete re-arrangement of the low-energy degrees of freedom: The quarks and the gluons becomes confined, while the free particle spectrum is comprised of the color-singlet composite particles — the mesons, the baryons, and the glueballs. Also, the strong quark-antiquark attraction leads to the Bose–Einstein condensation of $q\bar{q}$ bound states and hence spontaneous breaking of the chiral symmetry

$$SU(N_f)_L \times SU(N_f)_R \ \to \ SU(N_f)_V \,. \tag{1}$$

In LH Weyl fermion notations — where $\psi_f^{\alpha,i}$ denotes the LH quarks and $\tilde{\psi}_{\alpha,i,f}$ the LH antiquarks — condensate VEV is

$$\mathcal{M}_{f,f'} \ \stackrel{\rm def}{=} \ \left\langle \psi_f^{\alpha,i} \tilde{\psi}_{\alpha,i,f'} \right\rangle \ = \ \delta_{f,f'} \times \mathcal{M} \ \neq \ 0, \tag{2}$$

hence χSB as in eq. (1).

Now consider an $\mathcal{N}=1$ SYM theory for some simple gauge group G such as $SU(N)$. In components, this is a gauge theory with Majorana fermions in the adjoint multiplet but no other fermions and no scalar fields at all. So its behavior strongly resembles that of the ordinary QCD with $N_f = N_c$: no couplings other than the gauge coupling with $\beta(g) < 0$, hence asymptotic freedom in the UV limit but stronger coupling at lower energies until at $E \sim \Lambda_{SYM}$ (the analogue of $\Lambda_{\rm QCD}$) the coupling blows up. Consequently, the elementary quanta of the theory — the gluons and the gluinos* — become confined, while the free particle spectrum is comprised of the color-singlet glueballs and oddballs. Also, the gluinos — being in a real representation of the gauge group — strongly attract each other, so their

$\star$ For simplicity, I use SQCD terminology: gluons for the gauge field quanta and gluinos for their superpartners. More generally, such superpartners are called *gauginos*, but what the heck. . .

bound pairs form a Bose–Einstein condensate

$$\mathcal{S} \;=\; \langle \lambda^{\alpha,a}\lambda^a_\alpha \rangle \;\neq\; 0 \tag{3}$$

similar to the quark-antiquark condensate (2) in the ordinary QCD. Strictly speaking, the condensate (3) should be called the 'gluino-bilinear condensate' or 'gaugino-bilinear condensate', but everybody calls it simply the *'gaugino condensate'*.

Now consider the chiral symmetry and its spontaneous breakdown in the SYM theory. Classically, the SYM theory has a $U(1)$ chiral symmetry: In Weyl fermion notations, it acts as

$$A^a_\mu(x) \;\to\; \text{itself}, \qquad \lambda^a_\alpha(x) \;\to\; e^{+i\rho} \times \lambda^a_\alpha(x), \qquad \bar\lambda^a_{\dot\alpha}(x) \;\to e^{-i\rho} \times \bar\lambda^a_{\dot\alpha}(x) \tag{4}$$

for an arbitrary global phase ρ. From the supersymmetry point of view this chiral symmetry is an R-symmetry — it does not commute with the supercharges but instead rotates their phases as

$$Q_\alpha \;\to\; e^{-i\rho} \times Q_\alpha\,, \qquad \overline{Q}_{\dot\alpha} \;\to\; e^{+i\rho} \times \overline{Q}_{\dot\alpha}\,, \tag{5}$$

but from the component-field theory's point of view it's just a global $U(1)$.

In the quantum SYM theory, the classical $U(1)$ chiral symmetry is anomalous, although we may cancel the anomaly by adjusting the theory's instanton angle Θ as

$$\Theta \;\to\; \Theta + 2N \times \rho. \tag{6}$$

(The N here is the index of the adjoint multiplet of the gauge group G; for $G = SU(N)$ this index is N, hence the notation.) However, the instanton angle Θ appears in the functional integral of the theory only as the phase $e^{i\Theta}$, so a $\Theta' = \Theta + 2\pi \times \text{integer}$ is indistinguishable from Θ. In other words, the instanton angle Θ is only defined modulo 2π. Consequently, for

$$\rho \;=\; \frac{2\pi}{2N} \times \text{integer} \tag{7}$$

the chiral symmetry works without changing the Θ angle, and that makes it non-anomalous. Thus, while the continuous $U(1)$ chiral symmetry is anomalous, its discrete subgroup $\mathbf{Z}_{2N}$ is anomaly-free.

Now consider the gaugino condensate (3). Under the chiral symmetry, it transforms as

$$\mathcal{S} \;\to\; e^{2i\rho} \times \mathcal{S}, \tag{8}$$

which is invariant for $\rho = 0, \pi$ but no other values of ρ. Consequently, the gaugino condensate spontaneously breaks the discrete $\mathbf{Z}_{2N}$ chiral symmetry down to its $\mathbf{Z}_2$ subgroup. As usual for spontaneously broken symmetries, the factor group $\mathbf{Z}_{2N}/\mathbf{Z}_2 = \mathbf{Z}_N$ relates distinct vacuum states of the theory to each other. Specifically, there N vacuum states — in perfect agreement with the Witten index of the SYM being $I = +N$ — distinguished by the phases of the gaugino condensate:

$$\text{phase}(\mathcal{S}\text{ in vacuum}\#j) \;=\; \text{common} \;+\; \frac{2\pi j}{N}\,. \tag{9}$$

Moreover, the common term here is related to the instanton angle Θ. Indeed, using a chiral field redefinition

$$(\lambda^a_\alpha)' \;=\; e^{i\rho} \times \lambda^a_\alpha \quad\Longrightarrow\quad S' \;=\; e^{2i\rho} \times S \quad\text{while}\quad \Theta' \;=\; \Theta + 2N\rho \tag{10}$$

we may always set $\Theta' = 0$, which makes the redefined SYM theory CP-invariant. Consequently, it should have a vacuum state with a real value of the condensate $\mathcal{S}'$, hence the other vacua have

$$\text{phase}(\mathcal{S}'[\text{vac}_j]) \;=\; 0 \;+\; \frac{2\pi j}{N}\,. \tag{11}$$

In terms of the condensate phases before the field redefinition, this means

$$\text{phase}(\mathcal{S}[\text{vac}_j]) \;=\; \frac{\Theta}{N} \;+\; \frac{2\pi j}{N}\,. \tag{12}$$

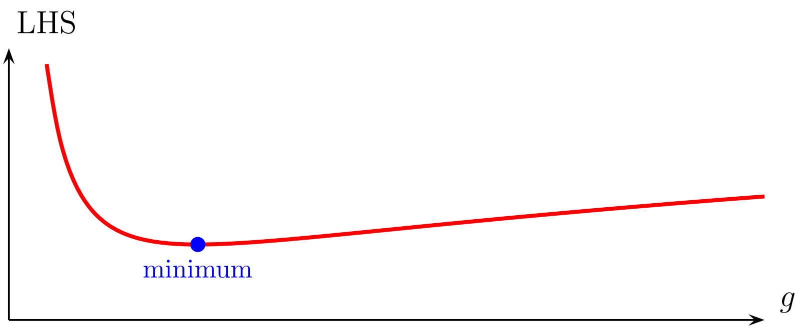
Now consider the magnitude of the gaugino condensate. By dimensional analysis, we should have

$$|\langle\lambda\lambda\rangle| \;=\; \Lambda^3_{\text{SYM}} \times \begin{pmatrix}\text{a numeric}\\ \text{constant}\end{pmatrix} \tag{13}$$

where Λ_{SYM} is the energy scale E where the running gauge coupling $g(E)$ becomes singular, or rather where the RG flow to the IR hits a singularity. Let's relate this scale to the Wilsonian gauge coupling τ for some UV cutoff scale $\Lambda_{\rm UV}$. The Novikov–Shifman–Vainshtein–Zaharov equation for the physical gauge coupling of the SYM theory can be written as

$$\frac{8\pi^2}{g^2(E)}\ -\ N\times\log\frac{1}{g^2(E)}\ =\ 2\pi\,{\rm Im}\,\tau(\Lambda_{\rm UV})\ -\ 3N\times\log\frac{\Lambda_{\rm UV}}{E}+\begin{pmatrix}\text{a numeric}\\ \text{constant}\end{pmatrix}. \tag{14}$$

As a function of energy, the RHS here is $+3N\log E+{\rm const}$, so it monotonically decreases with the RG flow from UV to IR. On the other hand, the LHS of eq. (14) is a non-monotonic function of g: it decreases with g for a weak coupling, but then hits a minimum and starts increasing:



(15)

The minimum happens for $Ng^2=8\pi^2$, which is the point where the NSVZ beta-function

$$\beta(g)\ =\ \frac{-3N_cg^3}{16\pi^2-2Ng^2} \tag{16}$$

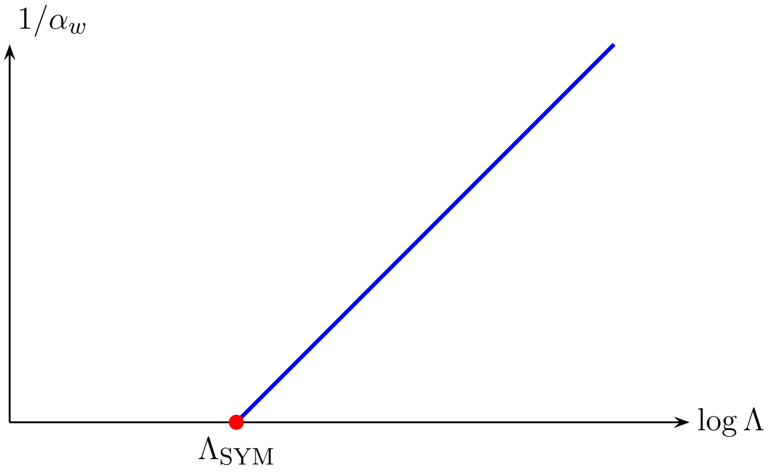
blows up. In terms of eq. (14), this minimum on the LHS translates on the RHS to

$$2\pi\,{\rm Im}\,\tau(\Lambda_{\rm UV})\ -\ 3N\times\log\frac{\Lambda_{\rm UV}}{E}\ \geq\ \begin{pmatrix}\text{a numeric}\\ \text{constant}\end{pmatrix} \tag{17}$$

and hence

$$E\ \geq\ \Lambda_{\rm SYM}\quad\text{for}\quad\Lambda_{\rm SYM}\ =\ \Lambda_{\rm UV}\times\exp\left(-\frac{2\pi}{3N}\,{\rm Im}\,\tau(\Lambda_{\rm UV})\right)\times\begin{pmatrix}\text{a numeric}\\ \text{factor}\end{pmatrix}. \tag{18}$$

So (up to an $O(1)$ numerical factor I don't care about), the strong-interaction scale of SYM obtains from simple exponentiation of the Wilsonian gauge coupling ${\rm Im}\,\tau=1/\alpha_w$. In other words, we simply use the Wilsonian beta-function — which is exact at one loop — to find the scale at which the Wilsonian $(1/\alpha_w)(\Lambda_{SYM})$ crosses zero:



(19)

In terms of the gaugino condensate's magnitude, eq. (18) means

$$|\langle\lambda\lambda\rangle| \;=\; \Lambda_{\rm UV}^3\times\exp\left(-\frac{2\pi}{N}\,{\rm Im}\,\tau(\Lambda)\right)\times\begin{pmatrix}\text{a numeric}\\ \text{factor}\end{pmatrix}. \tag{20}$$

At the same time, we have eq. (12) for the gaugino condensate phase; in terms of the complex gauge coupling τ, it becomes

$$\text{phase}(\langle\lambda\lambda\rangle) \;=\; \frac{2\pi}{N}\times{\rm Re}\,\tau \;+\; \frac{2\pi}{N}\times\text{integer}. \tag{21}$$

Clearly, we may combine the last two formulae into a holomorphic formula

$$\langle\lambda\lambda\rangle \;=\; \Lambda_{\rm UV}^3\times\exp\left(\frac{2\pi i}{N}\times\tau(\Lambda_{\rm UV})\right)\times\begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix}\times\sqrt[N]{1}, \tag{22}$$

where $\sqrt[N]{1}$ stands for *any* $N^{\rm th}$ root of unity, different root for a different SUSY vacuum of the SYM theory. Eq. (22) indicates *holomorphic dimensional transmutation,* in which we trade a dimensionless holomorphic parameter τ specified for a particular cutoff scale $\Lambda_{\rm UV}$ for a dimensionful holomorphic parameter

$$\Lambda_{\rm SYM}^{3N} \;\stackrel{\rm def}{=}\; \Lambda_{\rm UV}^{3N}\times\exp\bigl(2\pi i\tau(\Lambda_{\rm UV})\bigr) \tag{23}$$

that is independent of the UV cutoff. By holomorphic parameters, I mean that in a SYM theory coupled to a chiral modulus superfield Φ, $\tau(\Phi)$ should be a holomorphic function, and consequently $\Lambda_{\rm SYM}$ is also a holomorphic function of the modulus Φ because it's a holomorphic function of τ. Anyway, in terms of the holomorphic $\Lambda_{\rm SYM}$, the gaugino condensate is simply

$$\langle\lambda\lambda\rangle \;=\; \Lambda_{\rm SYM}^3 \times \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \sqrt[N]{1}. \tag{24}$$

Note: in the holomorphic formulae (22) or (24) for the gaugino condensate $\langle\lambda\lambda\rangle$, the gaugino fields are non-canonically normalized. Instead, they are normalized as superpartners of the gauge fields $\mathcal{A}_\mu^a$ in covariant derivatives $\nabla_\mu = \partial_\mu + i\mathcal{A}_\mu^a t^a$. Or in superfield terms, we normalize

$$\lambda_\alpha \;=\; \lambda_\alpha^a t^a \;=\; \text{lowest component of } \mathcal{W}_\alpha \;=\; -\tfrac{1}{8}\overline{D}^2\bigl(e^{-2V} D_\alpha e^{+2V}\bigr). \tag{25}$$

Consequently, the gaugino bilinear $\operatorname{tr}(\lambda^\alpha\lambda_\alpha)$ is the lowest component of the gauge-invariant chiral superfield

$$S \;=\; \operatorname{tr}\bigl(\mathcal{W}^\alpha\mathcal{W}_\alpha\bigr). \tag{26}$$

A general rule for all supersymmetric vacua says that *a non-zero VEV of a component field must belong to the lowest component of a gauge-invariant superfield that is not a total D, $\overline{D}$, or ∂ derivatives of another gauge-invariant superfield.* The S superfield from eq. (26) is a total $\overline{D}$ derivative — because the $\mathcal{W}_\alpha$ tension is itself a total $\overline{D}$ derivative, — but it is not a total derivative of anything gauge-invariant. Consequently, the lowest component of S may have a non-zero VEV in a supersymmetric vacuum and that's how we get the gaugino condensate $\langle\operatorname{tr}(\lambda\lambda)\rangle = \langle S\rangle \neq 0$.

Another general rule says that the a VEV of the lowest components of a *chiral* superfield such as S must be holomorphic functions of the gauge coupling τ or its dimensional transmutant $\Lambda_{\rm SYM}$, and indeed eqs. (22) and (24) are manifestly holomorphic. But again, these holomorphic formulae apply to the non-canonically normalized gaugino condensate. The canonically normalized condensate is

$$\bigl\langle\operatorname{tr}\bigl(\lambda_{\rm can}\lambda_{\rm can}\bigr)\bigr\rangle \;=\; g^2 \times \langle S\rangle \;=\; g^2 \times \Lambda_{\rm SYM}^3 \times \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \sqrt[N]{1} \tag{27}$$

where g^2 is the physical gauge coupling[2] at the $\Lambda_{\rm SYM}$ scale, and that g^2 factor may be a non-holomorphic function of the parameters.

Veneziano–Yankielowicz Superpotential

Consider small fluctuations

$$\delta S(y,\theta) \;=\; S(y,\theta) \;-\; \langle S\rangle \tag{28}$$

of the gaugino bilinear around a SUSY vacuum of the SYM theory. Since the spontaneously broken continuous $U(1)$ R-symmetry is anomalous, δS is not a Goldstone mode of the SYM, so we do not expect the quanta of δS to be massless. However, by analogy with the η-mesons in real-life QCD we expect those quanta to be light relative to the other massive glueballs or oddball particles.

Indeed, the pions and the eta-mesons are pseudo-Goldstone bosons of the spontaneous breaking of the approximate $U(2)_L \times U(2)_R$ symmetry down to $U(2)_V$. But the explicit breaking of the $SU(2)_L \times SU(2)_R$ symmetry by the u, d quark masses is weaker that the explicit breaking of the $U(1)_A$ symmetry by the axial anomaly, and that's why the pions are much lighter than the eta meson. However, the eta-meson itself is significantly lighter that the rho meson or other non-pseudo-Goldstone bosons, so it makes sense to include it into an effective low-energy theory for the light particles.

Similarly, the δS superfield in SYM is not a true Goldstone mode, but it's a pseudo-Goldstone mode of an approximate symmetry (where the approximation is neglecting the anomaly), so we expect its quanta to be significantly lighter than all the other massive particles of the theory. Consequently, it makes sense to write down the effective low-energy theory for just the S superfield, and back in 1982 Gabriele Veneziano and Shimon Yankielowicz wrote an analytic formula for the exact non-perturbative superpotential $W(S)$ for this effective theory. But could only guess the general shape of the effective theory's K̈hler function $K(S,\overline{S})$; alas, the non-holomorphic non-perturbative effects cannot be calculated *exactly.*

In a moment I shall derive the Veneziano–Yankielowicz superpotential, albeit in a different way from how Veneziano and Yankielowicz did it. First, I shall couple the SYM to a chiral modulus superfield Φ via modulus-dependent gauge coupling $\tau(\Phi)$. Second, I shall completely integrate out the SYM and look at the effective ultra-low-energy theory for just the modulus Φ. Specifically, I shall derive the effective superpotential $W_{\text{eff}}(\Phi)$ for the modulus generated by the gaugino condensation in the SYM sector. Third, I shall look at non-so-low energies and 'integrate in' the S superfield into an effective theory for S and Φ. And only then I shall drop the modulus and focus on the effective superpotential for just the gaugino condensate S.

So let's start with the SYM Lagrangian with a modulus-dependent gauge coupling,

$$\mathcal{L} \;=\; \frac{i}{8\pi}\int\! d^2\theta\,\tau(\Phi)\times\mathrm{tr}\big(\mathcal{W}^\alpha\mathcal{W}_\alpha\big)\;+\;\mathrm{H.\,c.} \tag{29}$$

Among other things, this Lagrangian includes the coupling of the modulus's auxiliary component F_Φ to the gaugino bilinear,

$$\mathcal{L}\;\supset\;\frac{i}{8\pi}\,\frac{\partial\tau}{\partial\Phi}\,F_\Phi\times\mathrm{tr}(\lambda^\alpha\lambda_\alpha)+\;\mathrm{H.\,c.}, \tag{30}$$

so gaugino condensation gives rise to the F-term for the modulus,

$$\mathcal{L}\;\supset\;\frac{i}{8\pi}\,\frac{\partial\tau}{\partial\Phi}\,F_\Phi\times\langle S\rangle\;+\;\mathrm{H.\,c.} \tag{31}$$

By itself, such F-term may break SUSY in the modulus sector, but we assume the modulus also couples to something else (besides the SYM) which may cancel this F-term.

Now, let's completely integrate out the SYM theory and consider an effective ultra-low-energy theory just for the modulus superfield Φ. In terms of that effective theory, the F-term for Φ obtains from an effective superpotential $W_{\text{eff}}(\Phi)$,

$$\mathcal{L}\;\supset\;F_\Phi\times\frac{\partial W_{\text{eff}}}{\partial\Phi}\;+\;\mathrm{H.\,c.}, \tag{32}$$

so to match the F-term (31) we should have

$$\frac{\partial W_{\text{eff}}}{\partial\Phi}\;=\;\frac{i}{8\pi}\,\frac{\partial\tau}{\partial\Phi}\times\langle S\rangle\,. \tag{33}$$

Moreover, the gaugino condensate $\langle S\rangle$ here depends on the modulus itself via the gauge coupling $\tau(\Phi)$,

$$\langle S\rangle \;=\; \Lambda_{\rm UV}^3\times\exp\left(\frac{2\pi i}{N}\tau(\Phi,\Lambda_{\rm UV})\right). \tag{34}$$

(Up to an overall numeric factor I do not care about.) Consequently,

$$\frac{\partial W_{\rm eff}}{\partial\Phi} \;=\; \frac{i\Lambda_{\rm UV}^3}{8\pi}\times\exp\left(\frac{2\pi i}{N}\tau(\Phi,\Lambda_{\rm UV})\right)\times\frac{\partial\tau}{\partial\Phi}\,, \tag{35}$$

and therefore

$$W_{\rm eff}(\Phi) \;=\; \frac{N\Lambda_{\rm UV}^3}{16\pi^2}\times\exp\left(\frac{2\pi i}{N}\tau(\Phi,\Lambda_{\rm UV})\right). \tag{36}$$

Or in terms of the holomorphic $\Lambda_{\rm SYM}(\Phi)$,

$$W_{\rm eff}(\Phi) \;=\; \frac{N}{16\pi^2}\times\Lambda_{\rm SYM}^3(\Phi). \tag{37}$$

Now consider the effective theory at low but not too-low energies, so it includes both the modulus Φ and the lightest composite superfield S of the SYM theory. The superpotential $W(S,\Phi)$ of this EFT obtains by 'integrating in' the S superfield into the superpotential (37) just for the Φ. The integration-in works by reversal of the integration-out procedure: We assume a $W(S,\Phi)$, integrate out the S, and then try to match the result to eq. (37). As to the integration-out, we first treat $W(S,\Phi)$ as the superpotential for just the S but with modulus-dependent couplings and look for a SUSY vacuum with $F_S=0$. That is, we

$$\text{solve the}\quad \frac{\partial W(S,\Phi)}{\partial S} \;=\; 0\quad\text{equation for } S\quad\Longrightarrow\quad\text{solution }\langle S\rangle(\Phi). \tag{38}$$

Then we plug the solution $\langle S\rangle$ back into $W(S,\Phi)$, and this gives the effective superpotential for the modulus,

$$W_{\rm eff}(\Phi) \;=\; W(\langle S\rangle(\Phi),\Phi). \tag{39}$$

Thus, in light of eq. (37), the $W(S,\Phi)$ should obey

$$\text{for } S(\Phi) \text{ such that}\quad \frac{\partial W(S,\Phi)}{\partial S} \;=\; 0:\quad W(S(\Phi),\Phi) \;=\; \frac{N}{16\pi^2}\,\Lambda_{\rm SYM}^3(\Phi). \tag{40}$$

Furthermore, the VEV $\langle S\rangle$ is the gaugino condensate in the full SYM theory, and in that theory we already know $\langle S\rangle=\Lambda_{\rm SYM}^3(\Phi)$. Combining this knowledge with eqs. (38) and (40), we arrive at

$$\text{for } S \;=\; \Lambda^3_{\rm SYM}(\Phi):\quad W(S,\Phi)\;=\;\frac{N}{16\pi^2}\,\Lambda^3_{\rm SYM}(\Phi)\quad\text{and}\quad \frac{\partial W(S,\Phi)}{\partial S}\;=\;0. \tag{41}$$

By themselves, these conditions do not completely determine the superpotential $S(S,\Phi)$, so let's bring in additional arguments.

Gabriele Veneziano and Shimon Yankielowicz themselves use the following heuristic argument: From the S field point of view, the SYM Lagrangian can be thought of as a tree-level superpotential

$$W_{\rm tree}(S,\Phi)\;=\;\frac{i\tau(\Phi)}{8\pi}\times S. \tag{42}$$

There are no loop corrections to this effective superpotential, but there should be non-perturbative contributions. Since such non-perturbative effects should come from the low-energy effects in the SYM, they should depend on the condensate itself rather than on the UV gauge coupling $\tau(\Phi)$, so we expect $W_{\rm n.p.}(S\text{ only})$ and hence

$$\begin{aligned} W(S,\Phi)\;=\;W_{\rm tree}\;+\;W_{\rm n.p.}\;&=\;\frac{i\tau(\Phi)}{8\pi}\times S\;+\;W_{\rm n.p.}(S\text{ only})\\ &=\;\frac{N}{16\pi^2}\log\Lambda^3_{\rm SYM}\times S\;+\;W_{\rm n.p.}(S\text{ only}). \end{aligned} \tag{43}$$

This general form for the superpotential $W(S,\Phi)$ is somewhat of a guesswork, but let's plug it into eqs. (41) anyway:

$$\text{for } S\;=\;\Lambda^3_{\rm SYM}:\quad \begin{cases} \dfrac{N}{16\pi^2}\log\Lambda^3_{\rm SYM}\times S\;+\;W_{\rm n.p.}(S)\;=\;\dfrac{N}{16\pi^2}\times\Lambda^3_{\rm SYM}\,,\\ \dfrac{N}{16\pi^2}\log\Lambda^3_{\rm SYM}\;+\;\dfrac{dW_{\rm n.p.}}{dS}\;=\;0. \end{cases} \tag{44}$$

These equations have a unique solution for the non-perturbative term

$$W_{\rm n.p.}(S)\;=\;\frac{N}{16\pi^2}\,S\times\bigl(1-\log S\bigr), \tag{45}$$

hence

$$W(S,\Phi)\;=\;\frac{N}{16\pi^2}\times S\times\left(1\;-\;\log\frac{S}{\Lambda^3_{\rm SYM}(\Phi)}\right). \tag{46}$$

A more rigorous way to find this superpotential involves combining eqs. (41) with holomorphy, symmetry, and asymptotic constraints. Let's start with the anomalous R-symmetry, under which we should have

$$S' = e^{2i\rho} \times S, \quad \Lambda'^3_{\rm SYM} \to e^{2i\rho} \times \Lambda^3_{\rm SYM}\,, \quad W(S', \Lambda'^3_{\rm SYM}) = e^{2i\rho} \times W(S, \Lambda^3_{\rm SYM}). \tag{47}$$

This symmetry requires

$$W(S, \Lambda^3_{\rm SYM}) = S \times f(S/\Lambda^3_{\rm SYM}) \tag{48}$$

where $f(x)$ is some holomorphic function of a single argument $x = S/\Lambda^3_{\rm SYM}$. In terms of this function, eqs. (41) become

$$f(1) = -f'(1) = \frac{N}{16\pi^2}\,. \tag{49}$$

Finally, the effective theory should not have spurious SUSY vacua, so the derivative $\partial W/\partial S$ should not vanish for $|S| \neq |\Lambda^3_{\rm SYM}|$. In terms of $f(x)$ this means $f(x) + f'(x) \neq 0$ for $|x| \neq 1$, and in particular for $|x| > 1$. For a complex analytic function $f(x)$, it means at large x it should not grow faster than $O(\log x)$, and with this constraint there is only one solution to eqs. (49):

$$f(x) = \frac{N}{16\pi^2} \times \bigl(1 - \log x\bigr), \tag{50}$$

hence the Veneziano–Yankielowicz superpotential

$$W(S, \Phi) = \frac{N}{16\pi^2} \times S \times \left(1 - \log \frac{S}{\Lambda^3_{\rm SYM}(\Phi)}\right). \tag{46}$$

Note: holomorphic dimension transmutation of the Wilsonian τ coupling produces a single-valued $\Lambda^{3N}_{\rm SYM}$ parameter, but the $\Lambda^3_{\rm SYM}$ in the Veneziano–Yankielowicz superpotential (46) has N different values for N distinct SUSY vacua of the SYM. Thus, there is no single Veneziano–Yankielowicz superpotential for all N vacua of SYM but N different superpotentials, one for each vacuum. Sometimes it's convenient to write a common formula

$$W(S) = -\frac{1}{16\pi^2}\, S \times \left(\log \frac{S^N}{\Lambda^{3N}_{\rm SYM}} - N\right) \tag{51}$$

which seems to cover all N vacua, but the devil is in the detail: Each vacuum state corresponds to a different branch of the log function, so the superpotential (51) isn't really common to all the SUSY vacua of the SYM theory. Physically, this means that if we try to interpolate between the different vacua — say, but putting a domain wall with different vacua on its two sides, — then somewhere in the middle of that wall we would get extra degrees of freedom that cannot be accounted just by the S superfield. But alas, such domain walls are beyond the scope of this class.

SQCD with 1 Massive Flavor

Let's move on from a pure SYM theory to SQCD with N colors and 1 massive flavor. This theory has two holomorphic parameters — the quark mass m and the gauge coupling τ, or equivalently, its dimensional transmutant

$$\Lambda_{SQCD}^{3N-1} \;=\; \Lambda_{\rm UV}^{3N-1} \times \exp\bigl(2\pi i\tau\bigr). \tag{52}$$

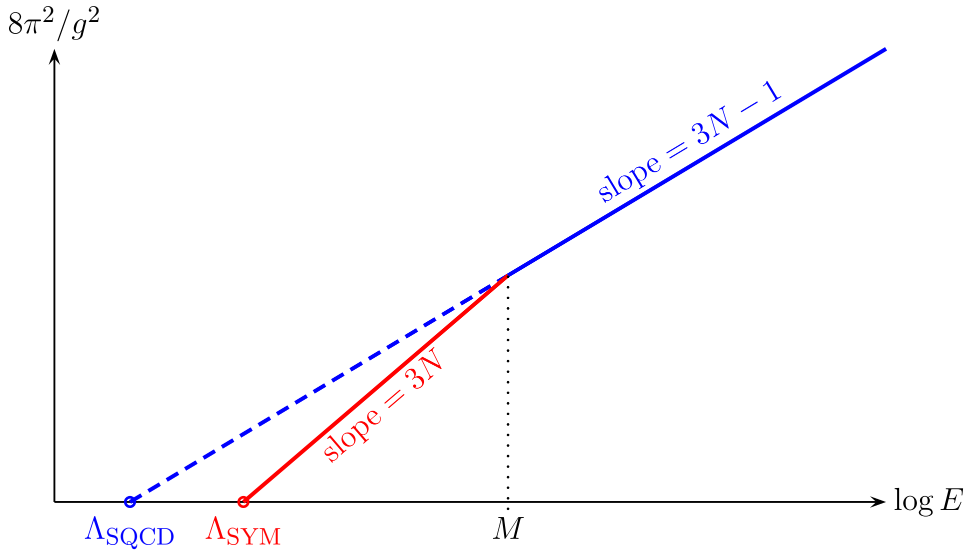
Note the $(3N-1)$ power of Λ on both sides of this formula because the one-loop beta-function coefficient of the 1-flavor SQCD is $-B_1 = 3N_c - N_f = 3N-1$.

The theory also has two independent VEVs of composite gauge-invariant chiral operators, namely the gaugino condensate $\langle S\rangle = \langle \mathrm{tr}(\lambda\lambda)\rangle$ and the squark-antisquark bilinear $\langle\mathcal{M}\rangle = \bigl\langle A^i B_i\bigr\rangle$. By analogy with the ordinary QCD, the chiral superfield $\mathcal{M}$ is called the *meson*, but the analogy is misleading because its made of two scalars rather than two fermions, so in the Higgs regime of SQCD $\langle\mathcal{M}\rangle$ is dominated by the semiclassical squark VEV rather than quantum squark-antisquark pairing.

As long as we define S and $\mathcal{M}$ as non-canonically normalized but chiral operators, their expectation values should be holomorphic functions of m and $\Lambda_{\rm SQCD}$, and we shall derive such holomorphic functions in this section. Moreover, we shall see that despite rather different regimes of the theory for $m \gg \Lambda$ and $m \ll \Lambda$, we end up with exactly the same holomorphic functions $S(m,\Lambda)$ and $\mathcal{M}(m,\Lambda)$ for both regimes. This indicates that SQCD with 1 massive has only one phase and there are no phase transitions at finite m.

Heavy Quark Limit

Let's start with the high quark mass limit $m \gg \Lambda_{\rm SQCD}$. In this regime, the heavy quark decouples from the physics at energies $E \ll m$, so the effective theory of those energies is the SYM without any quarks. The $\Lambda_{\rm SYM}$ parameter of that effective theory obtains from the RG flow which has a threshold at the quark mass M:



(53)

The straight lines on this picture correspond to the one-loop approximation. To work at all-loop precision, let's use the NSVZ equations: At high energies above the threshold

$$\frac{8\pi^2}{g^2(E)} \;-\; N\log\frac{1}{g^2(E)} \;=\; (3N-1)\times\log\frac{E}{|\Lambda_{\rm SQCD}|} \;-\; \log Z(E) \tag{54}$$

where $Z(E)$ is the running quark field normalization factor, while at low energies below the threshold

$$\frac{8\pi^2}{g^2(E)} \;-\; N\log\frac{1}{g^2(E)} \;=\; 3N\times\log\frac{E}{|\Lambda_{\rm SYM}|}\,. \tag{55}$$

Both equations should yield the same running gauge coupling $g^2(E)$ at the threshold $E=M$, which is the physical mass of the quark. In terms of the holomorphic mass m,

$$M \;=\; \frac{|m|}{Z(E=M)}\,. \tag{56}$$

Thus, comparing eq. (54) and (55) at $E=M$, we immediately get

$$(3N-1)\times\log\frac{M}{|\Lambda_{\rm SQCD}|} \;-\; \log Z(E) \;=\; 3N\times\log\frac{M}{|\Lambda_{\rm SYM}|} \tag{57}$$

and hence

$$\left|\Lambda_{\rm SYM}\right|^{3N} \;=\; \left|\Lambda_{\rm SQCD}\right|^{3N-1}\times\Bigl(MZ\,=\,|m|\Bigr). \tag{58}$$

Naturally, this formula for the magnitudes suggests the holomorphic relation

$$\Lambda_{\rm SYM}^{3N} \;=\; \Lambda_{\rm SQCD}^{3N-1}\times m, \tag{59}$$

so let's check the phases of the two sides of this formula. On the LHS

$$\text{phase}\bigl(\Lambda_{\rm SYM}^{3N}\bigr) \;=\; \text{phase}\bigl(\exp(2\pi i\tau_{\rm SYM})\bigr) \;=\; \Theta_{\rm SYM}\,, \tag{60}$$

and likewise on the RHS of eq. (59)

$$\text{phase}\bigl(\Lambda_{\rm SQCD}^{3N-1}\bigr) \;=\; \Theta_{\rm SQCD}\,. \tag{61}$$

However, in SQCD the proper CP-violating parameter is not the instanton angle Θ by itself but rather

$$\overline{\Theta}_{\rm SQCD} \;=\; \Theta_{\rm SQCD} \;+\; \text{phase}(m), \tag{62}$$

— *cf.* my notes on instantons from the QFT 2 class, pages 22–30, — and it is this combination that should match the $\Theta_{\rm SYM}$ of the low-energy theory,

$$\Theta_{\rm SYM} \;=\; \Theta_{\rm SQCD} \;+\; \text{phase}(m). \tag{63}$$

In terms of the holomorphic Λ parameters, this means

$$\text{phase}\bigl(\Lambda_{\rm SYM}^{3N}\bigr) \;=\; \text{phase}\bigl(\Lambda_{\rm SQCD}^{3N-1}\bigr) \;+\; \text{phase}(m), \tag{64}$$

in perfect agreement with the holomorphic relation (59).

Finally consider the supersymmetric vacuum states of SQCD. Since the non-perturbative effects in this theory are low-energy, they are governed by the effective low-energy theory, namely SYM. Thus, there are N vacua distinguished by the phases of the gaugino condensate:

$$\begin{aligned}\langle S\rangle &= \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \left[\Lambda_{\text{SYM}}^{3N}\right]^{1/N} \times \sqrt[N]{1}\\ &= \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \left[\lambda_{SQCD}^{3N-1} \times m\right]^{1/N} \times \sqrt[N]{1},\end{aligned} \tag{65}$$

where the second equality follows from eq. (22). Note that the magnitude of the gaugino condensate grows for large $m \to \infty$ and shrinks for small $m \to 0$.

As to the 'meson' VEVs $\langle AB\rangle$, we cannot derive it from the effective theory from which the quark fields are integrated out. But it can be calculated from the SQCD itself, and I shall calculate it in a later section.

Light quark Limit

Thus far we have dealt with the heavy quark limit $m \gg \Lambda_{\text{SQCD}}$, but now consider the opposite limit of the light quark $m \ll \Lambda_{\text{SQCD}}$. In the extreme limit of $m = 0$, the classical potential for the quark fields has a flat direction parametrized by the 'meson' $\mathcal{M} = A^i B_i$ acting as a modulus. Later in this section I shall show that the non-perturbative scalar potential is non-flat but decreases for $\mathcal{M} \to \infty$. Without the quark mass, this causes a runaway — thus no stable vacuum states — but for a small but non-zero m, the potential reaches a supersymmetric minimum — or rather N supersymmetric minima — at large but finite values of $\mathcal{M}$. For the moment, let's not worry how this works but simply assume large meson VEV $\langle\mathcal{M}\rangle \gg \Lambda_{\text{SQCD}}^2$.

Note that an SQCD 'meson' is made from 2 scalar particles — a squark and an antisquark — rather than two fermions, so the meson field can develop a large VEV without any strong interactions. Instead, it simply reflects the large VEVs of the quark field themselves — up to a gauge symmetry,

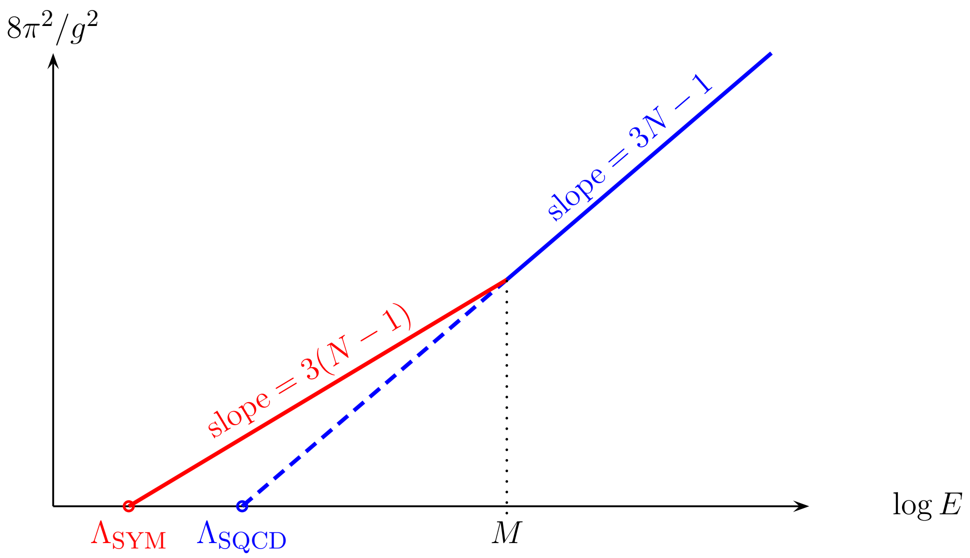
$$\langle A\rangle = \begin{pmatrix}0\\ \vdots\\ 0\\ \Phi\end{pmatrix}, \quad \langle B\rangle = \begin{pmatrix}0 & \cdots & 0 & \Phi\end{pmatrix}, \quad \Phi = \sqrt{\mathcal{M}} \gg \Lambda_{\text{SQCD}}. \tag{66}$$

The gauge coupling at the $E = \Phi \gg \Lambda_{\rm SQCD}$ scale is not strong, so the squark VEVs (66) lead to the semiclassical Higgs mechanism in which the $SU(N)$ gauge group is broken down to $SU(N-1)$. Or down to nothing for $N = 2$, but let's assume $N \geq 3$, so the effective theory below the Higgs scale is not just the free modulus $\mathcal{M}$ but also the $SU(N-1)$ SYM.

The gauge coupling of that SYM — or rather its dimensional transmutant $\Lambda_{\rm SYM}$ — obtains from the RG flow across the vector mass threshold

$$M = g \times |\langle\Phi_{\rm can}\rangle| = g \times \frac{|\langle\Phi\rangle|}{\sqrt{Z}} = g \times \frac{\left|\sqrt{\mathcal{M}}\right|}{\sqrt{Z}}. \tag{67}$$

Graphically,



(68)

where the lines are straight in the one-loop approximation. To all loop order, we use the NSVZ equations: At high energies above the Higgs threshold

$$\frac{8\pi^2}{g^2(E)} - N \times \log\frac{1}{g^2(E)} + \log Z(E) = (3N-1) \times \log\frac{E}{|\Lambda_{\rm SQCD}|} \tag{69}$$

while at low energies below the threshold

$$\frac{8\pi^2}{g^2(E)} - (N-1)\log\frac{1}{g^2(E)} = (3N-3) \times \log\frac{E}{|\Lambda_{\rm SYM}|}. \tag{70}$$

At the threshold $E = M$ both equations should agree for the same gauge coupling $g^2(E)$, hence

$$(3N-3)\times\log\frac{M}{|\Lambda_{\rm SYM}|}\ -\ (3N-1)\times\log\frac{M}{|\Lambda_{\rm SQCD}|}\ =\ \log\frac{1}{g^2(M)}\ -\ \log Z(M) \tag{71}$$

and therefore

$$|\Lambda_{\rm SYM}|^{(3N-3)}\ =\ \bigl|\Lambda_{\rm SQCD}\bigr|^{(3N-1)}\times\left(\frac{g^2Z(M)}{M^2}\ =\ \frac{1}{|\mathcal{M}|}\right). \tag{72}$$

Naturally, this formula for the magnitudes suggests the holomorphic relation

$$\Lambda_{\rm SYM}^{(3N-3)}\ =\ \frac{\Lambda_{\rm SQCD}^{(3N-1)}}{\mathcal{M}}\,, \tag{73}$$

so let's check the phases of the two sides of this formula. In terms of the instanton angles of the high-energy and low-energy theories, the phase part of eq. (73) translates to

$$\Theta_{\rm SYM}\ =\ \Theta_{\rm SQCD}\ -\ {\rm phase}(\mathcal{M}), \tag{74}$$

so let me explain the second term on the RHS. When SQCD is Higgsed down by the squark VEVs of a single flavor, the massive vector superfields include two fundamental $(\mathbf{N}-\mathbf{1})$ multiplets of Dirac fermions: One is made from ψ^i_A and $\overline{\lambda^N_i}$ (for $i=1,\ldots,(N-1)$), and the other from $\lambda^i{}_N$ and $\overline{\psi_{B,i}}$. Both types of fermions get their masses from the Yukawa couplings to the Φ^* rather than to Φ, hence

$${\rm phase(fermion\ mass)}\ =\ -\,{\rm phase}(\Phi)\ =\ -\tfrac{1}{2}\,{\rm phase}(\mathcal{M}). \tag{75}$$

Consequently, the strong CP-violating parameter of the theory is

$$\overline{\Theta}_{\rm SQCD}\ =\ \Theta_{\rm SQCD}\ +\ 2\times{\rm phase(fermion\ mass)}\ =\ \Theta_{\rm SQCD}\ -\ {\rm phase}(\mathcal{M}). \tag{76}$$

In the low-energy effective SYM theory, the corresponding CPV parameter is simply $\Theta_{\rm SYM}$, and that's why we have

$$\Theta_{\rm SYM}\ =\ \Theta_{\rm SQCD}\ -\ {\rm phase}(\mathcal{M}), \tag{74}$$

which is the phase part of the holomorphic relation (73).

Now let's use the relation

$$\Lambda_{\rm SYM}^{(3N-3)} \;=\; \frac{\Lambda_{\rm SQCD}^{(3N-1)}}{\mathcal{M}} \tag{73}$$

to find the vacuum states of the theory and the VEVs of composite chiral superfields $\mathcal{M}$ and S. For a given meson VEV $\langle\mathcal{M}\rangle$, the gaugino condensate obtains as

$$\langle S\rangle \;=\; \Lambda^3_{\rm SYM} \times \sqrt[(N-1)]{1} \;=\; \left[\frac{\Lambda_{\rm SQCD}^{(3N-1)}}{\mathcal{M}}\right]^{1/(N-1)} \times \sqrt[(N-1)]{1}, \tag{77}$$

so there seems to be only $N-1$ SUSY vacuum states of the theory. However, an $SU(N)$ SQCD with a massive flavor has the same Witten index N regardless of quark mass being large or small, so there should be at least N vacua rather than $N-1$.

To resolve this paradox, we notice that the meson VEV $\langle\mathcal{M}\rangle$ itself is determined dynamically, so let's see how this works. The low-energy EFT comprises the SYM and the meson $\mathcal{M}$; they are connected to each other via modulus-dependent $\Lambda_{\rm SYM}$, *cf.* eq. (73). Through this connection, gaugino condensation in the SYM provides a non-perturbative effective superpotential for the meson,

$$W_{\rm n.p.}(\mathcal{M}) \;=\; \frac{(N-1)}{16\pi^2} \times \Lambda^3_{\rm SYM} \;=\; \frac{(N-1)}{16\pi^2} \times \left[\frac{\Lambda_{\rm SQCD}^{(3N-1)}}{\mathcal{M}}\right]^{1/(N-1)} . \tag{78}$$

In addition, the quark mass term acts as a tree-level superpotential for $\mathcal{M}$,

$$W_{\rm tree} \;=\; mA^iB_i \;=\; m\times\mathcal{M}\,. \tag{79}$$

Altogether

$$W_{\rm net}(\mathcal{M}) \;=\; W_{\rm tree}(\mathcal{M}) \;+\; W_{\rm n.p.}(\mathcal{M}), \tag{80}$$

and the supersymmetric vacua obtain for $\mathcal{M}$ such that

$$\frac{\partial W_{\rm net}}{\partial\mathcal{M}} \;=\; 0. \tag{81}$$

Evaluating the derivative on the LHS here, we get

$$\frac{\partial W_{\rm net}}{\partial\mathcal{M}} \;=\; m \;-\; \frac{1}{16\pi^2}\times\Bigl[\Lambda_{\rm SQCD}\Bigr]^{(3N-1)/(N-1)}\times\Bigl[\mathcal{M}\Bigr]^{-N/(N-1)}, \tag{82}$$

so demanding it vanishes leads to

$$\mathcal{M}^N \;=\; \Lambda_{\rm SQCD}^{3N-1}\times(16\pi^2 m)^{1-N}. \tag{83}$$

Thus, there are indeed N SUSY vacua distinguished by phases of the meson VEV

$$\langle\mathcal{M}\rangle \;=\; \binom{\rm numeric}{\rm factor}\times\left[\frac{\Lambda_{\rm SQCD}^{3N-1}}{m^{N-1}}\right]^{1/N}\times\sqrt[N]{1}. \tag{84}$$

As to the gaugino condensate $\langle S\rangle$, we may rewrite eq. (82) as

$$0 \;=\; \frac{\partial W_{\rm net}}{\partial\mathcal{M}} \;=\; m \;-\; \frac{1}{16\pi^2}\times\frac{S=\Lambda_{\rm SYM}^3}{\mathcal{M}}\,, \tag{85}$$

hence in any SUSY vacuum

$$m\times\langle\mathcal{M}\rangle \;=\; \frac{\langle S\rangle}{16\pi^2}\,. \tag{86}$$

Thus, plugging eq. (84) into this formula, we get

$$\langle S\rangle \;=\; \binom{\rm numeric}{\rm factor}\times\Bigl[m\times\Lambda_{\rm SQCD}^{3N-1}\Bigr]^{1/N}\times\sqrt[N]{1} \tag{87}$$

for the same $N^{\rm th}$ root of unity as in eq. (84) for $\langle\mathcal{M}\rangle$.

Note that according to eqs. (87) and (84), the gaugino condensates increases with growing quark mass while the meson VEV decreases. And in the opposite direction of $m\to 0$, we have $\langle S\rangle\to 0$ while $\langle\mathcal{M}\rangle\to\infty$. In particular, as I promised in the beginning of this section, in the low-mass limit $m\ll\Lambda_{\rm SQCD}$ we have a large meson VEV $\langle\mathcal{M}\rangle\gg\Lambda_{\rm SQCD}^2$ and hence semiclassical Higgs mechanism.

The growing meson VEV $\langle\mathcal{M}\rangle \to \infty$ in the $m \to 0$ limit suggests that in a truly massless SQCD, the squark VEVs run away to infinity instead of having a stable vacuum value. To see how this works, consider the effective theory for the $\Phi = \sqrt{\mathcal{M}}$ field whose lowest component is the squark VEV. For large $\Phi \gg \Lambda_{\rm SQCD}$, the gauge coupling is fairly weak at the Higgs threshold, so we should have

$$\begin{aligned} K(\Phi,\overline{\Phi}) &= K_{\rm tree} + \textit{small}\ \text{quantum corrections} \\ &\approx K_{\rm tree} = 2\Phi\overline{\Phi}. \end{aligned} \tag{88}$$

Consequently, the scalar potential for Φ becomes approximately

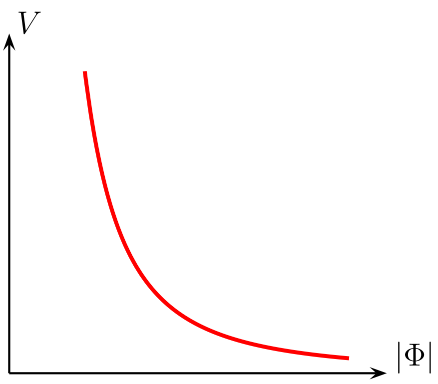
$$V \approx \frac{1}{2}\left|\frac{\partial W}{\partial\Phi}\right|^2, \tag{89}$$

where in the absence of quark mass

$$\begin{aligned} W(\Phi) = W_{\rm n.p.}(\Phi) &= \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \left[\frac{\Lambda_{\rm SQCD}^{3N-1}}{\mathcal{M}=\Phi^2}\right]^{1/(N-1)} \\ &= \text{const} \times \Phi^{\rm negative}. \end{aligned} \tag{90}$$

Therefore, at large Φ the scalar potential becomes

$$V(\Phi) = \text{const} \times |\Phi|^{\rm negative}, \tag{91}$$



(92)

which indeed has no stable minima but pushes $\langle\Phi\rangle$ to infinity.

Higgs–Confinement Complementarity

Thus far we have seen two different regimes of SQCD:

1. The confinement regime for $m \gg \Lambda_{\rm SQCD}$: The heavy quark decouples, and then the remaining $SU(N)$ SYM theory confines all N colors and generates the gaugino condensate.

2. The Higgs regime for $m \ll \Lambda_{\rm SQCD}$: The squarks get large semiclassical VEVs and Higgs the $SU(N)$ gauge theory down to $SU(N-1)$. And then the remaining $SU(N-1)$ SYM confines $N-1$ colors out of N, and also generates the gaugino condensate.

The two regimes look quite different, nevertheless they both have the same number N of SUSY vacua, even the same analytic formula for the gaugino condensates in these vacua:

$$\langle S\rangle \;=\; \binom{\text{numeric}}{\text{factor}} \times \left[m\times\Lambda_{\rm SQCD}^{3N-1}\right]^{1/N} \times\ \sqrt[N]{1} \tag{93}$$

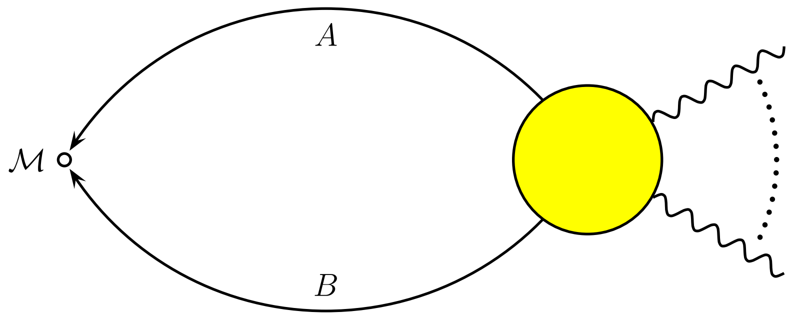
Moreover, a more careful analysis of the numerical factor here would show it has the same value in both confinement and Higgs regimes.

For the Higgs regime we have also calculated the 'meson' VEV $\langle\mathcal{M}\rangle = \left\langle A^i B_i\right\rangle$ which acts as a gauge-invariant measure of the $\langle\text{squark}\rangle^2$. Specifically,

$$\begin{aligned}\langle M\rangle \;&=\; \frac{\langle S\rangle}{16\pi^2 m} \quad \langle\!\langle\,\text{exactly}\,\rangle\!\rangle\\ &=\; \binom{\text{numeric}}{\text{factor}} \times \left[\frac{\Lambda_{\rm SQCD}^{3N-1}}{m^{N-1}}\right]^{1/N} \times\ \sqrt[N]{1}.\end{aligned} \tag{94}$$

But what about the meson VEV in the confinement regime of $m \gg \Lambda_{\rm SQCD}$? In this regime, there are no semiclassical squark VEVs, but the strong attraction between the massive squarks and antisquarks leads to a *small* bilinear condensate $\langle\mathcal{M}\rangle \neq 0$. To see how it works, consider the composite meson superfield $\mathcal{M}$ in the background of gauge superfields.

Perturbatively, we have



(95)

where each diagram loos exactly like the similar diagram for the Konishi anomaly calculated in the Pauli–Villars regularization scheme as

$$-\tfrac{1}{4}\overline{D}^2 J_{rmaxial} \;=\; 2M_{\rm PV}\times\langle XY\rangle \;=\; \frac{1}{8\pi^2}\times\mathrm{tr}\big(\mathcal{W}^\alpha\mathcal{W}_\alpha\big) \tag{96}$$

(where X and Y are the Pauli–Villars compensating fields and $M_{\rm PV}$ is their mass). For the massive squark fields — and negligible momenta of all external particles — we get exactly the same formula

$$m\times\langle XY\rangle \;=\; \frac{1}{16\pi^2}\times\mathrm{tr}\big(\mathcal{W}^\alpha\mathcal{W}_\alpha\big), \tag{97}$$

which in presence of the gaugino condensate

$$\langle S\rangle \;=\; \big\langle\mathrm{tr}\big(\mathcal{W}^\alpha\mathcal{W}_\alpha\big)\big\rangle \tag{98}$$

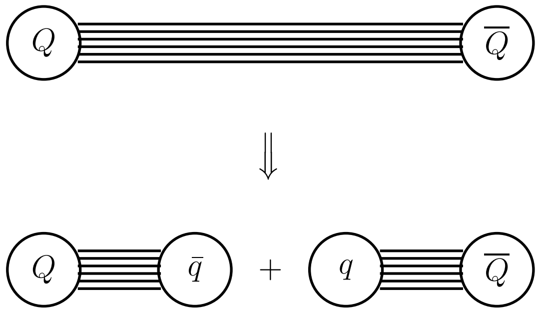
immediately implies

$$\langle\mathcal{M}\rangle \;=\; \frac{\langle S\rangle}{16\pi^2 m}. \tag{99}$$

Comparing this formula for the confinement regime with eq. (94) for the Higgs regime, we see exactly the same analytic formula for both regimes!

Physically, have exactly the same analytic formulae for the gaugino condensate $\langle S\rangle$ and meson VEV $\langle\mathcal{M}\rangle$ in both regimes indicates that there is no sharp phase transition between the two regimes. Instead, there is a smooth cross-over between the confinement and the Higgs regimes, so the difference between them is a difference of degree rather than a difference of kind. Technically, this means that any order parameter of SQCD would vary continuously with the $m/\Lambda_{\rm SQCD}$ ratio rather than suddenly jump at some transition point.

Naively, this contradicts the qualitative difference in the Wilson loop asymptotics between confining theories (area law) and non-confining theories (perimeter law). However, the perfect area law — and hence perfect confinement of quark-like external probes — is possible only in theories without any particles at all in the fundamental multiplets of $SU(N)$, thus no quarks or squarks at all, however massive. But once we introduce dynamical quarks and/or squarks in the fundamental **N** multiplets of the gauge symmetry, a flux tube between external probes can break in two by pair-creating a $q\bar{q}$ pair in the middle of the tube:



(100)

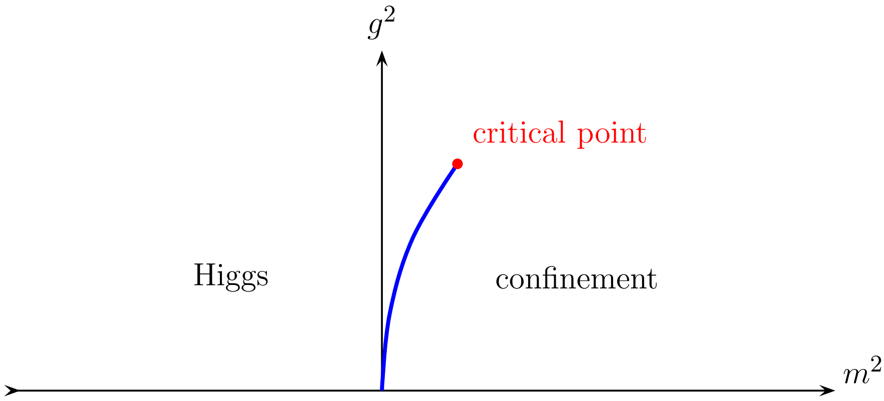
For a heavyish dynamical quark q, this process has exponentially low amplitude $O(\pi m_q^2/(\text{flux tube tension}))$, but it's possible, so for an exponentially large Wilson loop the area low turns into the perimeter law. For smaller quark masses, the area law turn to perimeter law for smaller Wilson loops, until eventually for small m there is no area law at all. But formally, the order parameter for the confinement is the asymptotic behavior of Wilson loops of size $\to \infty$, so we get the perimeter law for any finite quark mass, however large.

And that's how we get a smooth crossover between the confining and the Higgs regimes of SQCD!

While supersymmetry helps establish the smooth crossover between the confinement at the Higgs regimes, such smooth crossovers also happen in non-supersymmetric theories. This so-called *confinement–Higgs complementarity* was discovered back in 1979 in lattice models by Fradkin and Shenker and elaborated by many other people.

In general, whenever two regimes of the same system differ in degree rather than in kind, the transition between could be continuous or discontinuous, or even depend on some other parameter. For example, the gas and the liquid phases of some material have similar symmetries but different densities, so this is a difference in degree rather than in kind. And the transition between the two phases as a function of temperature could be continuous at supercritical pressures but first-order at low pressures.

For a QFT example, consider an $SU(2)$ gauge theory coupled to a single doublet of scalars of some mass^2 and quartic self-coupling λ. At weak gauge coupling, the theory is in the Higgs regime for $m^2 < 0$, and in the confining regime for $m^2 > 0$, or at least for $m^2 \gg \Lambda^2$. Moreover, there is a clear phase transition between the two regimes. But the lattice calculations show that at strong gauge couplings the phase transition disappears and becomes a continuous crossover, just like the gas-liquid transition at a supercritical pressure. And as usual in such situations, there is a critical point where the phase transition disappears and becomes a smooth crossover. Here is the diagram:



(101)

But unlike the above theory, the SQCD does not have a critical point. Instead, the transition between the Higgs and the confinement regimes is always continuous, that's why we have exactly the same analytic formulae for the $\langle S \rangle$ and $\langle \mathcal{M} \rangle$ as functions of m and Λ_{SQCD} for both regimes.

Veneziano Yankielowicz Superpotential

Now consider the effective superpotential for both the meson $\mathcal{M}$ and the gaugino condensate S, or rather for their fluctuations $\delta\mathcal{M}$ and δS around their VEVs is some supersymmetric vacuum. Physically, the EFT for just the $\delta\mathcal{M}$ and δS fields makes sense when these are the lightest particles of the theory, so we are going to derive the VY superpotential $W(S, \mathcal{M})$ for the Higgs regime of $m \ll \Lambda_{\text{SQCD}}$. However, thanks to the Higgs-confinement complementarity, exactly the same holomorphic superpotential $W(S, \mathcal{M})$ should work for all the regimes of SQCD and not just the Higgs regime.

The tree-level superpotential for the meson $\mathcal{M}$ is simply the quark mass term

$$W_{\text{tree}}(\mathcal{M}) \;=\; mB_iA^i \;=\; m\times\mathcal{M}, \tag{102}$$

and there are no perturbative corrections. There are of course the non-perturbative corrections due to gaugino condensation, but they should not directly depend on the quark mass m but only indirectly via the meson VEV $\langle\mathcal{M}\rangle$ itself.[⋆] Thus altogether we should have

$$W(\mathcal{M},S;m,\Lambda_{\text{SQCD}}) \;=\; m\times M \;+\; W_{\text{n.p.}}(\mathcal{M},S;\cancel{m},\Lambda_{\text{SQCD}}). \tag{103}$$

Next, we know that for any mass $\langle\mathcal{M}\rangle = \langle S\rangle/16\pi^2 m$. In terms of the effective superpotential (102), this means that

$$\frac{\partial W}{\partial\mathcal{M}} \;=\; 0 \quad \text{when} \quad \mathcal{M} \;=\; \frac{S}{16\pi^2 m} \tag{104}$$

and hence

$$\frac{\partial W_{\text{n.p.}}}{\partial\mathcal{M}} \;=\; -m \;=\; -\frac{S}{16\pi^2\mathcal{M}}\,. \tag{105}$$

Solving this differential equation, we get

$$W_{\text{n.p.}}(\mathcal{M},S,\Lambda_{\text{SQCD}}) \;=\; -\frac{S}{16\pi^2}\times\log(\mathcal{M}) \;+\; f(S;\Lambda_{\text{SQCD}}\ \text{only}) \tag{106}$$

where the last term is the integration 'constant' — it does not depend on $\mathcal{M}$ but may depend of S or Λ_{SQCD}. To determine this integration 'constant', let's integrate out the meson $\mathcal{M}$ to get the effective theory for just the gaugino condensate S. The result should match the Veneziano–Yankielowicz superpotential of the effective SYM theory we have derived in the previous section,

$$W_{\text{eff}}(S) \;=\; -\frac{S}{16\pi^2}\left(\log\frac{S^N}{\Lambda_{\text{SYM}}^{3N}} \;-\; C\right) \tag{107}$$

for some numerical constant C. On the other hand, integrating out the meson superfield from the superpotential (103) amounts to solving the equation $\partial W/\partial\mathcal{M} = 0$ for $\mathcal{M}$ and

⋆ This heuristic argument is due to Veneziano and Yankielowicz. It's hardly rigorous, but the result works!

plugging the solution back into the superpotential, thus

$$\begin{aligned} W(\mathcal{M}, S; m, \Lambda_{\text{SQCD}})) &= m \times \mathcal{M} - \frac{S}{16\pi^2} \times \log(\mathcal{M}) + f(S; \Lambda_{\text{SQCD}}) \\ \text{for } \mathcal{M} &= \frac{S}{16\pi^2 m} \\ \text{becomes} &= \frac{S}{16\pi^2}\left(1 - \log\frac{S}{16\pi^2 m}\right) + f(S; \Lambda_{\text{SQCD}}). \end{aligned} \tag{108}$$

Comparing the two effective superpotentials (108) and (107), we see that

$$\begin{aligned} f(S; \Lambda_{\text{SQCD}}) &= -\frac{S}{16\pi^2}\left(\log\frac{S^N}{\Lambda^{3N}_{\text{SYM}} = m\Lambda^{3N-1}_{\text{SQCD}}} - C\right) - \frac{S}{16\pi^2}\left(1 - \log\frac{S}{16\pi^2 m}\right) \\ &= -\frac{S}{16\pi^2}\left(\log\frac{16\pi^2 S^{N-1}}{\Lambda^{3N-1}_{\text{SQCD}}} + 1 - C\right) \end{aligned} \tag{109}$$

and hence

$$W(\mathcal{M}, S; m, \Lambda_{\text{SQCD}})) = m \times \mathcal{M} - \frac{S}{16\pi^2}\left(\log\frac{S^{N-1} \times \mathcal{M}}{\Lambda^{3N-1}_{\text{SQCD}}} - C'\right) \tag{110}$$

for a numeric constant $C' = C - 1 - \log(16\pi^2)$. This is the Veneziano–Yankielowicz superpotential for the SQCD with 1 massive flavor. By the Higgs-confinement complementarity, this superpotential works for all regimes of SQCD, high-mass, low-mass, or anything it between. On the other hand, it does not work for all the SUSY vacua of SQCD but only for the vicinity of one vacuum state at a time but not for any interpolation between two different vacua. Specifically, choosing a particular Riemann sheet of the log function in the superpotential (110) chooses a particular vacuum state,

SQCD with Several Massive Flavors

Now let's turn our attention to SQCD with N_c colors and $N_f > 1$ flavors, all massive. In matrix notations,⋆ the theory has bare Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; &\int d^2\theta\, d^2\bar{\theta}\Big(\operatorname{tr}\big(Z_A\overline{A}\exp(+2V)A\big) \;+\; \operatorname{tr}\big(Z_B B\exp(-2V)\overline{B}\big)\Big)\\ &+\int d^2\theta\left(\operatorname{tr}(mBA) \;+\; \frac{i\tau}{8\pi}\operatorname{tr}\big(W^\alpha W_\alpha\big)\right) \;+\; \text{H.c.}\end{aligned} \tag{111}$$

Suppose all quark masses — *i.e.*, all the eigenvalues of the m matrix — are heavy, much larger than the Λ_{SQCD} scale. This allows us to integrate out all the quarks from the low-energy effective theory — which then is just the SYM.

To find the coupling of that SYM, let's start with the $\Theta = 2\pi\operatorname{Re}\tau$. In the ordinary QCD, the CP violating parameter is not the instanton angle Θ per se but the combination

$$\overline{\Theta} \;=\; \Theta \;+\; \text{phase}(\det(m)) \tag{112}$$

that is invariant under anomalous chiral transforms of the quarks. In SQCD, the chiral redefinitions of the quark fields become chiral linear redefinitions of the whole quark superfields, but that does not affect the anomaly, so the CP violating parameter is exactly as in eq. (112). And when we integrate out the quark superfields from the effective low-energy SYM theory, the result should not depend on the redefinitions of the integrated-out fields, hence

$$\Theta_{\text{SYM}} \;=\; \overline{\Theta}_{\text{SQCD}} \;=\; \Theta_{\text{SQCD}} \;+\; \text{phase}(\det(m)). \tag{113}$$

In terms of the complex Λ_{SYM} and Λ_{SQCD} parameters of the low-energy and the high-energy theories, this relation becomes

$$\text{phase}\big(\Lambda_{\text{SYM}}^{3N_c}\big) \;=\; \text{phase}\big(\Lambda_{\text{SQCD}}^{3N_c-N_f}\big) \;+\; \text{phase}(\det(m)), \tag{114}$$

⋆ The quark superfields A and $\overline{B}$ are $N_c\times N_f$ matrices, the antisquark superfields $\overline{A}$ and B are $N_f\times N_c$ matrices, the vector superfield V is a traceless hermitian $N_c\times N_c$ matrix, the quark masses form a complex $N_f\times N_f$ matrix m, and the the normalization factors for the A and B superfields form hermitian $N_f\times N_f$ matrices Z_A and Z_B.

and by holomorphy, it should extend to the relation of both phases and magnitudes:

$$\Lambda_{\text{SYM}}^{3N_c} = \Lambda_{\text{SQCD}}^{3N_c-N_f} \times \det(m) \times \begin{pmatrix}\text{numeric}\\ \text{constant}\end{pmatrix}. \tag{115}$$

For the hierarchical quark masses $m_1 \gg m_2 \gg \cdots \gg m_{N_f} \gg \Lambda_{\text{SQCD}}$ we may derive the magnitude part of eq. (115) from the RG flow over several thresholds, but let me leave this exercise for your next homework. Instead, I am going to derive eq. (115) in all its holomorphic glory from the Konishi anomaly of the quark superfield redefinitions.

Indeed, consider a general linear redefinition of the quark superfields; in matrix notations

$$A' = A \times U_A, \quad B' = U_B \times B, \quad \overline{A}' = U_A^\dagger \times \overline{A}, \quad \overline{B}' = \overline{B} \times U_B^\dagger \tag{116}$$

for some arbitrary invertible $N_f \times N_f$ matrices U_A and U_B. Note: invertible, but generally not unitary! This field redefinition should be accompanied by changing the m, Z_A, and Z_B matrices of the theory according to

$$m' = U_A^{-1} \times m \times U_B^{-1}, \quad Z_A = U_A^{-1} \times Z_A \times \big(U_A^\dagger\big)^{-1}, \quad Z_B = \big(U_B^\dagger\big)^{-1} \times Z_B \times U_B^{-1} \tag{117}$$

to preserve the bare Lagrangian (111), and we should also shift the bare gauge coupling τ to cancel the Konishi anomaly of the transform (116):

$$\tau' = \tau - \frac{i}{2\pi}\Big(\log\det(U_A) + \log\det(U_B)\Big). \tag{118}$$

At the same time,

$$\begin{aligned}\log\det(m') &= \log\det\big(U_A^{-1}\big) + \log\det(m) + \log\det\big(U_B^{-1}\big)\\ &= \log\det(m) - \log\det(U_A) - \log\det(U_B),\end{aligned} \tag{119}$$

hence the holomorphic combination

$$T = 2\pi i\tau + \log\det(m) \tag{120}$$

remains invariant under the transform,

$$T' = T. \tag{121}$$

Moreover, this is the only invariant combination of the holomorphic parameters τ and $m_{ff'}$ of the theory.

When we integrate out the squark superfields, their redefinitions should not matter to the low-energy effective SYM, so the SYM gauge coupling should depend on the invariant T rather then $\tau_{\rm SQCD}$ itself, and the only holomorphic dependence $\tau_{\rm SYM}(T)$ that makes sense in the context of gauge couplings is

$$2\pi i\tau_{\rm SYM}\ =\ T\ +\ \text{const}\ =\ 2\pi i\tau_{\rm SQCD}\ +\ \log\det(m). \tag{122}$$

Or in terms of dimensional transmutants of the two theories,

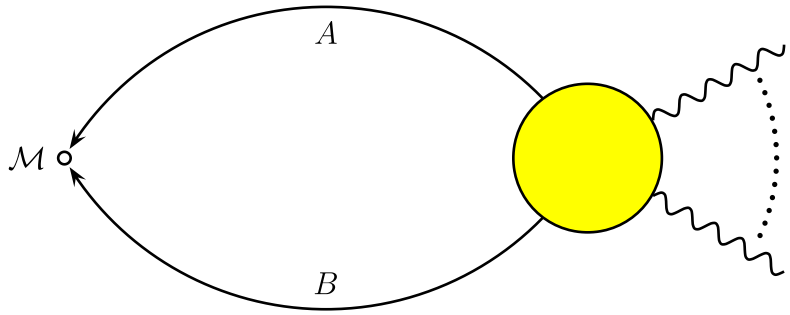
$$\Lambda_{\rm SYM}^{3N_c}\ =\ \Lambda_{\rm SQCD}^{3N_c-N_f}\times\det(m)\times\binom{\text{numeric}}{\text{constant}}. \tag{115}$$

Anyway, the N_c supersymmetric vacua of SQCD — and the gaugino condensates $\langle S\rangle$ which distinguish between them — can be obtained from the effective low-energy SYM theory. Given its $\Lambda_{\rm SYM}$ parameter from eq. (115), we immediately have

$$\begin{aligned}\langle S\rangle\ &=\ \binom{\text{numeric}}{\text{factor}}\times\Lambda_{\rm SYM}^3\times\sqrt[N_c]{1}\\ &=\ \binom{\text{numeric}}{\text{factor}}\times\left[\Lambda_{\rm SQCD}^{3N_c-N_f}\times\det(m)\right]^{1/N_c}\times\sqrt[N_c]{1}\end{aligned} \tag{123}$$

where different $\sqrt[N_c]{1}$ roots of unity correspond to the different SUSY vacuum states of the theory.

Next, let's find the VEVs of the meson matrix $\langle\mathcal{M}_{ff'}\rangle$ for all the vacua of SQCD in the high-mass regime. For a single flavor, we saw that the anomaly-like diagrams



(124)

lead to

$$\langle\mathcal{M}\rangle\ =\ \langle BA\rangle\ =\ \frac{\langle S\rangle}{16\pi^2 m} \tag{125}$$

At zero external momenta, only the one-loop diagrams (124) contribute to the relation (125) Consequently, in the multi-flavor SQCD with a diagonal mass matrix $m_{ff'}=m_f\delta_{ff'}$, we have completely independent diagrams (124) for each flavor, hence

$$\left\langle B_f A_f \right\rangle \;=\; \frac{\langle S\rangle}{16\pi^2 m_f}\,. \tag{126}$$

At the same time, thanks to the $[U(1)_V]^{N_f}$ vector symmetry of the theory — one $U(1)V$ symmetry for each flavor — the meson VEVS do not mix flavors,

$$\left\langle B_f A_{f'} \right\rangle \;=\; 0 \text{ for } f' \neq f. \tag{127}$$

In terms of the diagonal mass matrix m this means that the meson VEV matrix is proportional to the m^{-1} matrix,

$$\langle\mathcal{M}\rangle_{ff'} \;=\; (m^{-1})_{ff'} \times \frac{S}{16\pi^2}\,. \tag{128}$$

Moreover, in a different flavor basis where the quark mass matrix m is not diagonal, we get exactly the same matrix relation by changing the basis to the eigenbasis of m and then going back to the original basis, thus

$$\langle\text{matrix }\mathcal{M}\rangle \;=\; \bigl(\text{matrix } m\bigr)^{-1} \times \frac{S}{16\pi^2}\,. \tag{129}$$

Or in terms of Λ_{SQCD} and the mass matrix,

$$\langle\text{matrix }\mathcal{M}\rangle \;=\; \bigl(\text{matrix } m\bigr)^{-1} \times \Bigl[\Lambda_{\text{SQCD}}^{3N_c-N_f} \times \det(m)\Bigr]^{1/N_c} \times \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \sqrt[N_c]{1}. \tag{130}$$

The Higgs-confinement complementarity should work in the multi-flavor SQCD just as well as in the single-flavor theory, so we expect the holomorphic relations (123) and (130) to work in both heavy-lass and light-mass regimes. So let's see what happens to the meson VEVs in the low-quark-mass limit. To be specific, let's start with some general mass matrix m and then uniformly scale it by some factor $t < 1$, $m' = tm$, then

$$(m')^{-1} \;=\; \frac{1}{t} \times m^{-1} \quad\text{while}\quad \det(m') \;=\; t^{N_f} \times \det(m). \tag{131}$$

Plugging this scaling into eq. (130), we find that

$$\left\langle\text{matrix }\mathcal{M}'\right\rangle \;=\; \langle\text{matrix }\mathcal{M}\rangle \times \frac{t^{N_f/N_c}}{t} \;\propto\; t^{(N_f-N_C)/N_c}\,. \tag{132}$$

Thus, in the zero mass limit in the $t \to 0$ direction, the meson VEVs grow for $N_f < N_c$, stay finite for $N_f = N_c$, and shrink for $N_f > N_c$. Physically, this means that the large semiclassical squark VEVs obtain in the low-mass limit **only** for $N_f < N_c$. Likewise, for a completely massless theory, the runaway squark VEVS $\langle A\rangle, \langle B\rangle \to \infty$ happen **only** for $N_F < N_c$. Instead, SQCD theories with $N_f > N_c$ *massless* flavors have continuous families of supersymmetric vacua with $V = 0$, and we shall spend quite a few lectures discussing them in class. But through the remainder of these notes, I am limiting the discussion to SQCD theories with $N_f < N_c$ and assume that all the flavors are massive.

For $N_f < N_c$, the low-quark-mass behavior of the theory is the Higgs regime in which large semi-classical squark VEVs spontaneously break the $SU(N_c)$ gauge symmetry down to $SU(N_c - N_f)$. Or down to nothing for $N_f = N_c - 1$, but in that case, the non-perturbative effects have a rather different origin. So let's focus on the $N_f \leq N_c - 2$ cases where the low-energy effective theory is the $SU(N_c - N_f)$ SYM coupled to N_f^2 mesonic moduli $\mathcal{M}_{ff'}$, and the non-perturbative superpotential is generated by the SYM gaugino condensate

$$\langle S\rangle \;=\; \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix} \times \Lambda^3_{\text{SYM}}(\mathcal{M}). \tag{133}$$

To find the moduli dependence of the Λ_{SYM} — or equivalently, of the SYM Wilsonian gauge coupling τ_{SYM}, — we start with the high-energy SQCD theory and its linear redefinitions

$$A' \;=\; A \times U_A, \quad B' \;=\; U_B \times B, \quad \overline{A}' \;=\; U_A^\dagger \times \overline{A}, \quad \overline{B}' \;=\; \overline{B} \times U_B^\dagger \tag{116}$$

of the quark superfields. Under such a redefinition, the meson matrix $\mathcal{M}$ becomes

$$\mathcal{M}' \;=\; U_B \times \mathcal{M} \times U_A \tag{134}$$

while the Wilsonian gauge coupling τ becomes

$$\tau' \;=\; \tau \;-\; \frac{i}{2\pi}\Bigl(\log\det(U_A) \;+\; \log\det(U_B)\Bigr) \tag{118}$$

to compensate for the Konishi anomaly. The only holomorphic invariant combination of the

τ and the moduli matrix $\mathcal{M}$ is

$$P \;=\; 2\pi i\tau \;-\; \log\det(\mathcal{M}). \tag{135}$$

The gauge coupling of the low-energy SYM should be invariant under redefinition of the quark and meson fields, hence

$$2\pi i\tau_{\rm SYM} \;=\; P_{\rm SQCD} \;+\; {\rm const} = \; 2\pi i\tau_{\rm SQCD} \;-\; \log\det(\mathcal{M}) \;+\; {\rm const}. \tag{136}$$

Or in terms of the respective complex $\Lambda_{\rm SYM}$ and $\Lambda_{\rm SQCD}$,

$$\Lambda_{\rm SYM}^{3(N_c-N_F)} \;=\frac{\Lambda_{\rm SQCD}^{3N_c-N_f}}{\det(\mathcal{M})}\times\binom{\rm numeric}{\rm factor}. \tag{137}$$

Alternatively, we can get the same formula by running the RG flow through several vector-mass thresholds, but let's leave that exercise for the next homework set.

Instead, I am going to use eq. (137) to find the gaugino condensate as a function of the meson VEVs, then derive the Veneziano–Yankielowicz superpotential for both S and the meson matrix, and eventually the vevs of both the mesons and the gaugino condensate. Indeed, plugging eq. (137) into eq. (133) we get

$$\begin{aligned}\langle S\rangle \;&=\; \binom{\rm numeric}{\rm factor}\times\Lambda_{\rm SYM}^3\\ &=\; \binom{\rm numeric}{\rm factor}\times\left[\frac{\Lambda_{\rm SQCD}^{3N_c-N_f}}{\det(\mathcal{M})}\right]^{1/(N_c-N_f)}.\end{aligned} \tag{138}$$

Moreover, treating the meson VEVs as moduli, we can get the Veneziano–Yankielowicz superpotential for both S and $\mathcal{M}$ by simply plugging eq. (137) for the $\Lambda_{\rm SYM}(\mathcal{M})$ into the Veneziano–Yankielowicz superpotential for just the S superfield:

$$W \;=\; -\frac{S}{16\pi^2}\times\left(\log\frac{S^{N_c-N_f}}{\Lambda^{3(N_c-N_f)}} \;-\; C\right) \;=\; -\frac{S}{16\pi^2}\times\left(\log\frac{S^{N_c-N_f}\times\det(\mathcal{M})}{\Lambda_{\rm SQCD}^{3N_c-N_f}} \;-\; C'\right) \tag{139}$$

where C and C' are some numeric constants. Or rather, this is the non-perturbative part of the Veneziano–Yankielowicz superpotential, but there is also a tree-level term ${\rm tr}(m\mathcal{M})$ from the quark masses. Altogether, the Veneziano–Yankielowicz superpotential for SQCD is

$$W_{\rm VY}(\mathcal{M},S)\ =\ {\rm tr}(m\mathcal{M})\ -\ \frac{S}{16\pi^2}\times\left(\log\frac{S^{N_c-N_f}\times\det(\mathcal{M})}{\Lambda_{\rm SQCD}^{3N_c-N_f}}\ -\ C'\right). \tag{140}$$

Finally, the VEVs of the S and meson fields obtain by solving the algebraic equations

$$\frac{\partial W_{\rm VY}}{\partial\mathcal{M}_{ff'}}\ =\ 0\quad\text{and}\quad\frac{\partial W_{\rm VY}}{\partial S}\ =\ 0. \tag{141}$$

In particular,

$$\frac{\partial W_{\rm VY}}{\partial\mathcal{M}_{ff'}}\ =\ m_{f'f}\ -\ \frac{S}{16\pi^2}\times\left(\mathcal{M}^{-1}\right)_{f'f}, \tag{142}$$

which vanishes when

$$(\text{matrix }\mathcal{M})\times(\text{matrix }m)\ =\ \frac{S}{16\pi^2}\times(\text{unit }N_f\times N_f\text{ matrix}), \tag{143}$$

exactly as in eq. (129) for the confinement-regime.

At the same time,

$$\frac{\partial W_{\rm VY}}{\partial S}\ =\ -\frac{1}{16\pi^2}\left(\log\frac{S^{N_c-N_f}\times\det(\mathcal{M})}{\Lambda_{\rm SQCD}^{3N_c-N_f}}\ +\ (N_c-N_F)\ -\ C'\right), \tag{144}$$

which vanishes for

$$S^{N_c-N_f}\ =\ \frac{\Lambda_{\rm SQCD}^{3N_c-N_f}}{\det(\mathcal{M})}\times\binom{\text{numeric}}{\text{factor}}, \tag{145}$$

exactly as eq. (138). Finally, combining this formula with eq. (143), we get

$$\langle S\rangle^{N_c-N_f}\ =\ \Lambda_{\rm SQCD}^{3N_c-N_f}\times\frac{\det(m)}{\langle S\rangle^{N_f}}\times\binom{\text{numeric}}{\text{factor}}, \tag{146}$$

hence

$$\langle S\rangle\ =\ \left[\Lambda_{\rm SQCD}^{3N_c-N_f}\times\det(m)\right]^{1/N_c}\times\binom{\text{numeric}}{\text{factor}}\times\sqrt[N_c]{1}\,, \tag{147}$$

and therefore

$$\langle\text{matrix }\mathcal{M}\rangle\ =\ (\text{matrix }m)^{-1}\times\left[\Lambda_{\rm SQCD}^{3N_c-N_f}\times\det(m)\right]^{1/N_c}\times\binom{\text{numeric}}{\text{factor}}\times\sqrt[N_c]{1}\,, \tag{148}$$

exactly as in eqs. (123) and (130) for the confinement regime.

For completeness sake, we should also check that the same formula for the VEVs apply to the mixed regimes in which some quark masses are heavy (compared to the Λ_{SQCD}) while other quark masses are light. In this case, we get large semiclassical squark VEVs for the light quark flavors only, the $SU(N_c)$ gauge symmetry is Higgsed down to $SU(N_c - N_f^{\text{light}})$, and once we integrate out both the heavy flavors and the massive vectors, we end up with an effective SYM whose gaugino condensation generates the non-perturbative superpotential. And after some algebra — which I leave as an optional exercise for the interested students — we end up with exactly the same holomorphic formulae for the $\langle S\rangle$ and $\langle\mathcal{M}\rangle$ VEVs as in the all-flavors-heavy and all-flavors-light regimes.

What happens for $N_f = N_c - 1$?

Finally, consider SQCD with $N_f = N_1 - 1$. In the regimes where all quark masses are heavy, or some flavor's masses are heavy and some are light, this theory behaves exactly as SQCD with $N_f \leq N_c - 2$, and has exactly the same type of the Veneziano–Yankielowicz superpotential

$$W_{\text{VY}}(\mathcal{M}, S) \;=\; \operatorname{tr}(m\mathcal{M}) \;-\; \frac{S}{16\pi^2} \times \left(\log \frac{S^{(N_c - N_f = 1)} \times \det(\mathcal{M})}{\Lambda_{\text{SQCD}}^{3N_c - N_f}} \;-\; C'\right) \tag{149}$$

and hence similar VEVs of the gaugino condensate and of the meson matrix. But in the regime where all quark flavors are light, hence all $N_f = N_c - 1$ squark flavors have large semiclassical VEVs, the $SU(N)$ gauge theory is Higgsed down to nothing, so the effective low-energy theory has mesons but no gauge fields. Consequently without the low-energy SYM theory to generate the gaugino condensate, where does the non-perturbative term in the superpotential (149) come from?

To answer this question — or at least to get some clues, — let's integrate out the S field from the superpotential (149) and get the effective superpotential just for the meson matrix. Thus, solving $\partial W_{\text{VY}}/\partial S$ for S we find

$$\langle S\rangle \;=\; \binom{\text{numeric}}{\text{factor}} \times \frac{\Lambda_{\text{SQCD}}^{3N_c - N_f}}{\det(\mathcal{M})} \tag{150}$$

and hence

$$W_{\text{eff}}(\mathcal{M}) \;=\; W_{\text{VY}}(\langle S\rangle(\mathcal{M}),\mathcal{M}) \;=\; \operatorname{tr}(m\mathcal{M}) \;+\; \begin{pmatrix}\text{numeric}\\ \text{factor}\end{pmatrix}\times\frac{\Lambda_{\text{SQCD}}^{3N_c-N_f}}{\det(\mathcal{M})}\,. \tag{151}$$

Now, a close look at the non-perturbative term here gives us a couple of useful clues:

Clue#1:
The non-perturbative term here is a single-valued function of the meson matrix $\mathcal{M}$. This means it's origin does not involve any spontaneous symmetry breaking by any hidden degrees of freedom (such as the a gaugino condensate).

Clue#2:
The overall coefficient of the non-perturbative term is

$$\Lambda_{\text{SQCD}}^{3N_c-N_f} \;\propto\; \exp(2\pi i\tau) \;=\; \exp(i\Theta)\times\exp\left(-\frac{8\pi^2}{g_w^2}\right). \tag{152}$$

Both factors on the RHS here suggest a one-instanton effect. Indeed, a single YM instanton has action $8\pi^2/g^2$, so a one-instanton effect's amplitude is suppressed by the factor $\exp(-\text{action}) = \exp(-8\pi^2/g^2)$. Also, for $\Theta\neq 0$, so the the amplitudes originating in the one-instanton sector carry the $\exp(i\Theta)$ phase.

And indeed, in 1983 Ian Affleck, Michael Dine, and Nathan Seiberg found that the instantons in a completely Higgsed down SQCD indeed generate a non-perturbative superpotential of the form (151). How this works is beyond the scope of these notes, but I shall explain it in class one I remind you how the instantons work in the ordinary gauge theories.