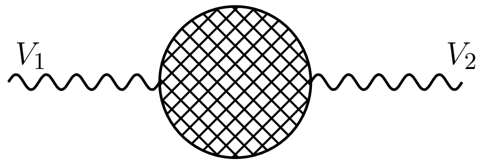
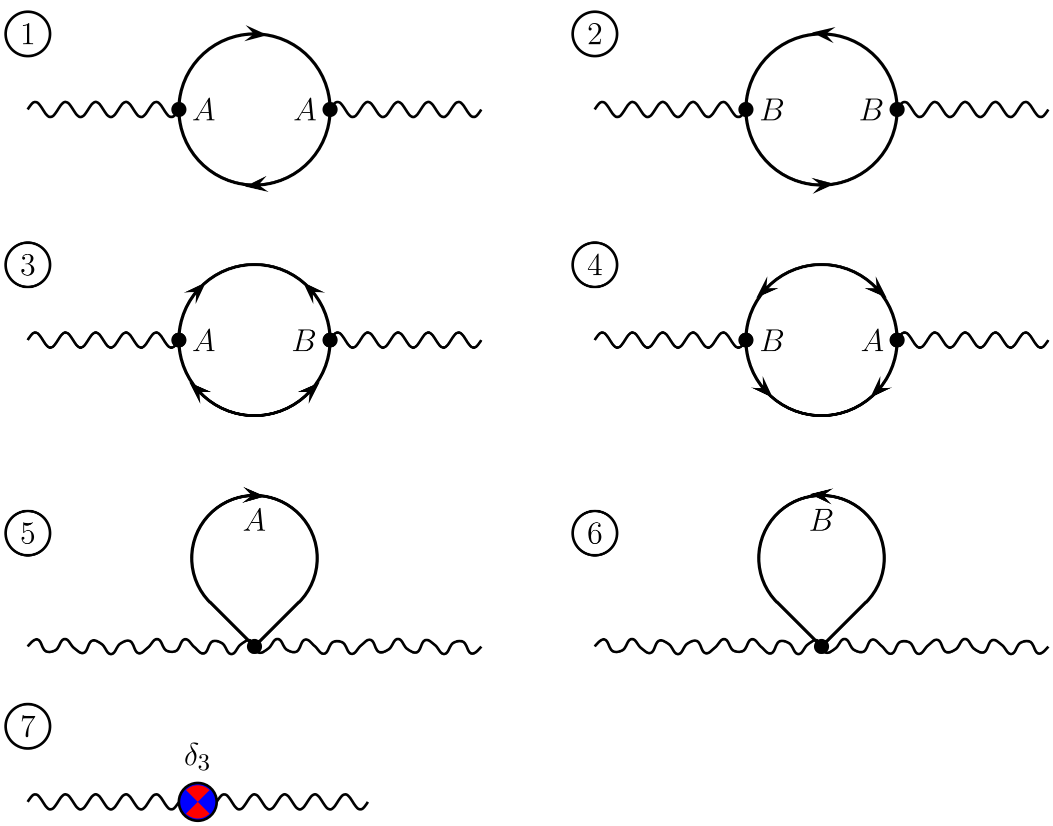
SQED Two–Photon Amplitude at the One–Loop Level

Consider the two-vector-superfields amplitude in SQED,

$i\mathcal{M}$ = 

(1)

At the one-loop level, this amplitude comes from 7 diagrams



(2)

where the last diagram contains the δ_3 counterterm that cancels the UV divergences of the other six diagrams.

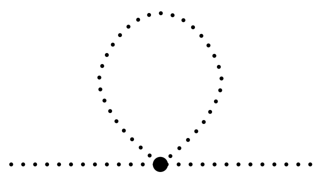
Seven diagrams sounds like a lot of work, but fortunately the loop diagrams come in pairs related by the charge conjugation symmetry $A \leftrightarrow B$, $V \to -V$. Thanks to this symmetry,

$$\mathcal{M}_1 = \mathcal{M}_2\,, \quad \mathcal{M}_3 = \mathcal{M}_4\,, \quad \mathcal{M}_5 = \mathcal{M}_6\,, \tag{3}$$

so we have only 3 independent loop diagrams to evaluate.

Let's start with the diagrams #5 and #6. The loop in each of these diagrams comprises a single propagator belonging to a chiral superfield, hence 4 spinor derivatives in the loop. This gives us

$$i\mathcal{M}_5 = i\mathcal{M}_6 =$$



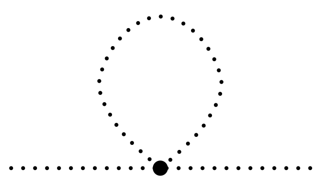
$$\times \int d^4\theta\, V_1(\theta)V_2(\theta)\ \tfrac{1}{16} D^2\overline{D}^2\, \delta^{(4)}(\theta-\theta')\Big|_{\theta=\theta'} \tag{4}$$

where the superfield factor evaluates to simply

$$\mathcal{S}_{5,6} = \int d^4\theta\, V_1(\theta)V_2(\theta). \tag{5}$$

As to the ordinary-graph factor, it yields

$$i\mathcal{G}_{5,6} =$$



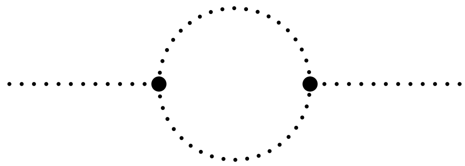
$$= \int\limits_{\text{reg}} \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - mm^* + i0} \times i(\pm 2g)^2, \tag{6}$$

where 'reg' indicates the UV regulation needed by the momentum integral — otherwise, it would diverge quadratically. Altogether

$$\mathcal{M}_5 = \mathcal{M}_6 = \mathcal{G}_{5,6} \times \mathcal{S}_{5,6} = 4ig^2 \times \int\limits_{\text{reg}} \frac{d^4p}{(2\pi)^4}\, \frac{1}{p^2 - mm^* + i0} \times \int d^4\theta\, V_1(\theta)V_2(\theta). \tag{7}$$

Next, consider the diagrams #3 and #4. This time, the loop in each of these diagrams comprises 2 scalar propagators, one of the AB type which carries $\overline{D}^2$, and the other of the

$\overline{B}A$ type which carries D^2. Altogether, there are 4 spinor derivatives in the loop, and that's precisely what we need to take care of the fermionic δ functions. Indeed,

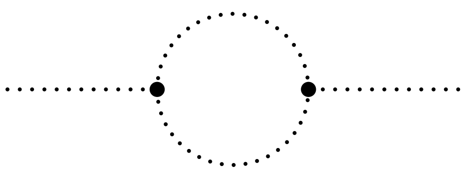
$\mathcal{M}_3 = \mathcal{M}_4 =$  $\times$

$$\times \int d^4\theta_1 \int d^4\theta_2\, V_1(\theta_1) V_2(\theta_2) \times \frac{mD^2}{-4}\, \delta^{(4)}(\theta_1 - \theta_2) \times \frac{m^* \overline{D}^2}{-4}\, \delta^{(4)}(\theta_2 - \theta_1) \tag{8}$$

where the superfield factors becomes (after integrating by parts)

$$\begin{aligned} \mathcal{S}_{3,4} &= mm^* \int d^4\theta_1 \int d^4\theta_2\, V_1(\theta_1) V_2(\theta_2) \times \delta^{(4)}(\theta_1 - \theta_2) \times \tfrac{1}{16} D^2 \overline{D}^2 \delta^{(4)}(\theta_2 - \theta_1) \\ &= mm^* \int d^4\theta_1\, V_1(\theta_1) V_2(\theta_1) \times \tfrac{1}{16} D^2 \overline{D}^2\, \delta^{(4)}(\theta_2 - \theta_1) \Big|_{\theta_2 = \theta_1} \\ &= mm^* \int d^4\theta_1\, V_1(\theta_1) V_2(\theta_1). \end{aligned} \tag{9}$$

As to the ordinary-graph part,

$i\mathcal{G}_{3,4}$ 

$$= \int\limits_{\text{reg}} \frac{d^4p_1}{(2\pi)^4} \frac{i}{p_1^2 - mm^* + i0} \times (+2ig) \times \frac{i}{(p_2 = p_1 + k)^2 - mm^* + i0} \times (-2ig) \tag{10}$$

where again we need to regulate the UV divergence of the momentum integral, although this time the divergence is logarithmic rather than quadratic. Altogether,

$$\begin{aligned} \mathcal{M}_3 = \mathcal{M}_4 = \mathcal{G}_{3,4} \times \mathcal{S}_{3,4} = 4ig^2 \times \int\limits_{\text{reg}} \frac{d^4p_1}{(2\pi)^4} \frac{mm^*}{[p_1^2 - mm^* + i0] \times [(p_1 + k)^2 - mm^* + i0]} \times \\ \times \int d^4\theta\, V_1(\theta) V_2(\theta). \end{aligned} \tag{11}$$

Note that the amplitudes (7) and (11) have similar superfield factors, so it would be convenient to combine their momentum integral parts together. Since this would involve identifying the momentum integration variables of two different divergent integrals, we have to be careful about implicitly shifting those momenta — which may be allowed or forbidden, depending on the precise UV regulation scheme we are going to use.

In these notes I am going to use DR — the *dimensional reduction* — as the UV regulator. This regulator is manifestly supersymmetric; moreover, it leaves all the superfields, the vertices, the propagators, and the derivative algebra precisely as in the un-regulated theory, so we do not need to redo any calculations. The only things affected by the DR are the momentum integrals — the dimension is reduced from $d = 4$ down to $d = 4 - 2\epsilon$. But even here, the DR allows us arbitrary constant shifts of the loop momentum variables, $\int d^d p \to \int d^d p'$ for $p'^\mu = p^\mu + (\text{const})^\mu$.

In particular, the DR allows us to identify the loop momentum p in eq. (7) — defined as the momentum of the only scalar propagator in the diagram #5 or #6 — with the loop momentum p_1 in eq. (11), defined as the momentum of the top scalar propagator in diagrams #3 or #4. Consequently, we may combine the DR-regulated momentum integrals as

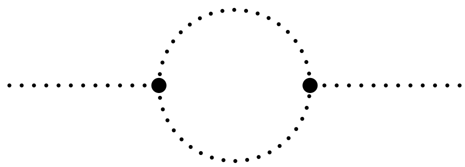
$$\begin{aligned}
\int\limits_{\text{DR}} \frac{d^4 p}{(2\pi)^4}\, \frac{1}{p^2 - mm^* + i0} \;+ \int\limits_{\text{DR}} \frac{d^4 p_1}{(2\pi)^4}\, \frac{mm^*}{[p_1^2 - mm^* + i0] \times [(p_1+k)^2 - mm^* + i0]} \\
= \int\limits_{\text{DR}} \frac{d^4 p_1}{(2\pi)^4} \left(\begin{matrix} \dfrac{1}{p_1^2 - mm^* + i0} \;+ \\ + \;\dfrac{mm^*}{[p_1^2 - mm^* + i0] \times [(p_1+k)^2 - mm^* + i0]} \end{matrix} \right) \\
= \int\limits_{\text{DR}} \frac{d^4 p_1}{(2\pi)^4}\, \frac{1}{p_1^2 - mm^* + i0} \times \left(1 + \frac{mm^*}{(p_1+k)^2 - mm^* + i0} \right) \\
= \int\limits_{\text{DR}} \frac{d^4 p_1}{(2\pi)^4}\, \frac{1}{p_1^2 - mm^* + i0} \times \frac{(p_1+k)^2}{(p_1+k)^2 - mm^* + i0}\,,
\end{aligned} \tag{12}$$

so the net contribution of the diagrams 3,4,5,6 to the two-photon amplitude is

$$\begin{aligned}\mathcal{M}_{3456} &\equiv \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 \\ &= 2 \times 4ig^2 \int\limits_{\rm DR} \frac{d^4 p_1}{(2\pi)^4} \frac{(p_1+k)^2}{[p_1^2 - mm^* + i0] \times [(p_1+k)^2 - mm^* + i0]} \times \\ &\qquad \times \int d^4\theta\, V_1(\theta) V_2(\theta)\end{aligned} \tag{13}$$

where the overall factor 2 comes from adding $\mathcal{M}_3 + \mathcal{M}_4$ as well as $\mathcal{M}_5 + \mathcal{M}_6$.

Finally, consider the diagrams #1 and #2. This time we get

$$\mathcal{M}_1 = \mathcal{M}_2 =$$  $$\times$$

$$\times \int d^4\theta_1 \int d^4\theta_2\, V_1(\theta_1) V_2(\theta_2) \times \tfrac{1}{16} D^2 \overline{D}^2 \delta^{(4)}(\theta_1 - \theta_2) \times \tfrac{1}{16} D^2 \overline{D}^2 \delta^{(4)}(\theta_2 - \theta_1) \tag{14}$$

where the superfield factor has 8 spinor derivatives — 4 more than we need to take care of the fermionic δ-functions. This situation is similar to what we had in the homework set#3, problem 2, so we proceed in a similar manner: First, we integrate by parts to move the derivatives from the first δ-function to the second δ-function or to the external field V_2, hence

$$\begin{aligned}\mathcal{S}_{1,2} &= \int d^4\theta_1 \int d^4\theta_2\, V_1(\theta_1) \delta^{(4)}(\theta_1 - \theta_2) \times \tfrac{1}{16} D^2 \overline{D}^2 \Big(V_2(\theta_2) \times \tfrac{1}{16} D^2 \overline{D}^2 \delta^{(4)}(\theta_2 - \theta_1) \Big) \\ &= \int d^4\theta_1\, V_1(\theta_1)\, \tfrac{1}{16} D^2 \overline{D}^2 V_2(\theta_1)\, \tfrac{1}{16} D^2 \overline{D}^2\, \delta^{(4)}(\theta_2 - \theta_1) \Big|_{\theta_1 = \theta_2}\end{aligned} \tag{15}$$

where on the last line all the derivatives are WRT θ_1 and each derivative acts on everything to its right. Second, we crank the Leibniz rule for the D^α and $\overline{D}^{\dot\alpha}$ derivatives to obtain

$$\begin{aligned}D^2 \overline{D}^2 \Big(V_2 \times D^2 \overline{D}^2 \delta \Big) &= \big(D^2 \overline{D}^2 V_2 \big) \times D^2 \overline{D}^2 \delta \\ &\quad + 4 \big(D_\alpha \overline{D}_{\dot\alpha} V_2 \big) \times D^\alpha \overline{D}^{\dot\alpha} D^2 \overline{D}^2 \delta \\ &\quad + V_2 \times D^2 \overline{D}^2 D^2 \overline{D}^2 \delta \\ &\quad + \text{ 6 more terms which vanish for } \theta_1 = \theta_2\end{aligned} \tag{16}$$

because they have different numbers of D and $\overline{D}$ derivatives acting on the $\delta \equiv \delta^{(4)}(\theta_2 - \theta_1)$.

Third, we use the anticommutation relation $\{D_\alpha, \overline{D}_{\dot\alpha}\} = 2q_{\alpha\dot\alpha}$ to simplify

$$D^\alpha \overline{D}^{\dot\alpha} D^2 \overline{D}^2 \delta = \ 2q^{\dot\alpha\alpha} \times D^2\overline{D}^2\delta \quad \text{and} \quad D^2\overline{D}^2 D^2\overline{D}^2\delta \ = \ 16q^2 \times D^2\overline{D}^2\delta. \tag{17}$$

Note: the momentum q^μ in these formulae belongs to the propagator that gave rise to the $\delta^{(4)}(\theta_2 - \theta_1)$, namely the bottom propagator in the diagram #1 or #2, thus $q \equiv p_2$. In the momentum integral associated with these diagrams we shall use the top propagators' momentum p_1 as the integration variable, hence $q = p_2 = p_1 + k$.

Finally, we make use of

$$D^2\overline{D}^2\delta^{(4)}(\theta_2 - \theta_1) \ = \ +16 \quad \text{for } \theta_1 = \theta_2\,. \tag{18}$$

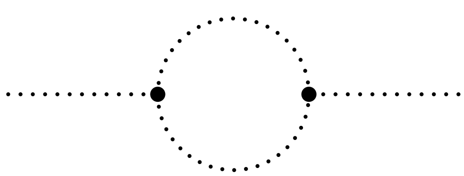
Plugging this formula and eqs. (17) into eq. (16), we obtain

$$\begin{aligned} D^2\overline{D}^2\left(V_2 \times D^2\overline{D}^2\delta\right)_{\theta_1=\theta_2} \ &= \ 16 D^2\overline{D}^2 V_2 \\ &+ \ 128(p_1+k)^\mu \bar\sigma_\mu^{\dot\alpha\alpha} \times D_\alpha \overline{D}_{\dot\alpha} V_2 \\ &+ \ 256(p_1+k)^2 \times V_2\,. \end{aligned} \tag{19}$$

and hence the superfield factor of the diagram #1 or #2:

$$\mathcal{S}_{1,2} \ = \int\! d^4\theta\, V_1 \times \left(\tfrac{1}{16} D^2\overline{D}^2 \ + \ \tfrac{1}{2}(p_1+k)^\mu \bar\sigma_\mu^{\dot\alpha\alpha} \times D_\alpha\overline{D}_{\dot\alpha} \ + \ (p_1+k)^2\right) V_2. \tag{20}$$

Note: the powers of momentum $(p_1{+}k)^\mu$ in this formula acts as numerators in the momentum integral stemming from the ordinary-graph part of the amplitude,

$i\mathcal{G}_{1,2}$ 

$$= \int\limits_{\text{DR}} \frac{d^4p_1}{(2\pi)^4}\, \frac{i}{p_1^2 - mm^* + i0} \times (\pm 2ig) \times \frac{i}{(p_2 = p_1 + k)^2 - mm^* + i0} \times (\pm 2ig) \tag{21}$$

Thus, altogether the first two diagrams yield

$$
\begin{aligned}
\mathcal{M}_{12} \;&\equiv\; \mathcal{M}_1 \;+\; \mathcal{M}_2 \;=\; 2\times M_1 \\
&=\; 2\times -4ig^2 \int\limits_{\rm DR} \frac{d^4p_1}{(2\pi)^4}\, \frac{1}{[p_1^2 - mm^* + i0]\times[(p_1+k)^2 - mm^* + i0]} \times \\
&\qquad \times \int d^4\theta\, V_1 \left(\tfrac{1}{16}D^2\overline{D}^2 V_2 \;+\; \tfrac{1}{2}(p_1+k)^\mu \bar\sigma_\mu^{\dot\alpha\alpha} \times D_\alpha \overline{D}_{\dot\alpha} V_2 \;+\; (p_1+k)^2\times V_2\right).
\end{aligned}
\tag{22}
$$

Now let us combine all six one-loop diagrams. In the DR regulation scheme for the momentum integrals we may identify the integration variable p_1 of eq. (13) with the similar p_1 variable of eq. (22). This gives us similar momentum integrals, or rather similar denominators of the momentum integrals for all the diagrams — the numerators hiding inside the $\int d^4\theta$ are quite different. Taking into account the opposite overall signs of eqs. (13) and (22), we obtain

$$
\begin{aligned}
\mathcal{M}^{\rm net}_{\rm loops} \;&=\; -8ig^2 \times \int\limits_{\rm DR} \frac{d^4p_1}{(2\pi)^4}\, \frac{1}{[p_1^2 - mm^* + i0]\times[(p_1+k)^2 - mm^* + i0]} \times \\
&\qquad \times \int d^4\theta\, V_1 \left(\begin{matrix} \tfrac{1}{16}D^2\overline{D}^2 V_2 \;+\; \tfrac{1}{2}(p_1+k)^\mu \bar\sigma_\mu^{\dot\alpha\alpha} \times D_\alpha \overline{D}_{\dot\alpha} V_2 \;+\; \cancel{(p_1+k)^2\times V_2} \\ -\;\cancel{(p_1+k)^2\times V_2} \end{matrix}\right)
\end{aligned}
\tag{23}
$$

Note that inside the superfield integral, the terms quadratic in $(p_2 = p_1 + k)$ cancel each other when we total up all the diagram. Consequently, the net momentum integral diverges linearly rather than quadratically before we apply the DR regularization.

Moreover, the linear divergence can be taken care of by averaging between p_1 and $-p_2 = -p_1 - k$. Indeed, both the integration measure and the net denominator are symmetric with respect to the variable change $p_1 \leftrightarrow -p_1 - k$, and they remain symmetric after the DR regulation. Consequently, for any numerator $f(p_1)$ we have

$$
\begin{aligned}
&\int\limits_{\rm DR}\frac{d^4p_1}{(2\pi)^4}\,\frac{f(p_1)}{[p_1^2-mm^*+i0]\times[(p_1+k)^2-mm^*+i0]}\\
&\qquad=\int\limits_{\rm DR}\frac{d^4p_1}{(2\pi)^4}\,\frac{f(-p_1-k)}{[(p_1+k)^2-mm^*+i0]\times[p_1^2-mm^*+i0]}\\
&\qquad=\int\limits_{\rm DR}\frac{d^4p_1}{(2\pi)^4}\,\frac{\frac12 f(p_1)+\frac12 f(-p_1-k)}{[p_1^2-mm^*+i0]\times[(p_1+k)^2-mm^*+i0]}\,,
\end{aligned}
\tag{24}
$$

or in other words, in the context of the numerator of the momentum integral, we may replace any

$$
f(p_1)\ \to \tfrac12 f(p_1)\ +\ \tfrac12 f(-p_1-k). \tag{25}
$$

In particular, for the linear terms in the numerator we may replace

$$
p_1^\mu\ \to \tfrac12 p_1^\mu\ +\ \tfrac12(-p_1-k)^\mu\ =\ -\tfrac12 k^\mu \tag{26}
$$

and hence

$$
(p_1+k)^\mu\ \to\ +\tfrac12 k^\mu. \tag{27}
$$

In the context of the amplitude (23), this replacement gives us

$$
\begin{aligned}
\mathcal{M}^{\rm net}_{\rm loops}\ =\ -8ig^2\times\int\limits_{\rm DR}\frac{d^4p_1}{(2\pi)^4}\,\frac{1}{[p_1^2-mm^*+i0]\times[(p_1+k)^2-mm^*+i0]}\times\\
\times\int d^4\theta\, V_1\left(\tfrac{1}{16}D^2\overline{D}^2V_2\ +\ \tfrac14 k^\mu\bar\sigma_\mu^{\dot\alpha\alpha}\times D_\alpha\overline{D}_{\dot\alpha}V_2\right)
\end{aligned}
\tag{28}
$$

Note that the second line here depends on the external photons' momentum k^μ but not on the loop momentum p_1^μ. Consequently, the momentum integral on the first line has only a logarithmic UV divergence.

Now let's go through one last round of simplification for the superfield factor of the amplitude (28). Assigning momenta to the external fields, we have

$$
\mathcal{S}^{\rm net}\ =\int d^4\theta\, V_1(+k,\theta)\times\left(\tfrac{1}{16}D^2\overline{D}^2\ +\ \tfrac14 k^\mu\bar\sigma_\mu^{\dot\alpha\alpha}D_\alpha\overline{D}_{\dot\alpha}\right)V_2(-k,\theta). \tag{29}
$$

Since all the fermionic derivatives here act on the $V_2(-k,\theta)$ superfield, they anticommute to

$\{D_\alpha, \overline{D}_{\dot\alpha}\} = -2k_\mu \sigma^\mu_{\alpha\dot\alpha}$, note the minus sign. Consequently

$$\left[D_\alpha, \overline{D}^2\right] \;=\; -4k_\mu \sigma^\mu_{\alpha\dot\alpha} \overline{D}^{\dot\alpha} \tag{30}$$

and hence

$$D^\alpha \overline{D}^2 D_\alpha \;=\; D^2 \overline{D}^2 \;-\; D^\alpha \left[D_\alpha, \overline{D}^2\right] \;=\; D^2 \overline{D}^2 \;+\; 4k_\mu \sigma^\mu_{\alpha\dot\alpha} D^\alpha \overline{D}^{\dot\alpha}. \tag{31}$$

Comparing this expression to the derivative operator inside () in eq. (29), we immediately obtain

$$\mathcal{S}^{\text{net}} \;=\; \int d^4\theta\, V_1(+k,\theta)\, \tfrac{1}{16} D^\alpha \overline{D}^2 D_\alpha V_2(-k,\theta). \tag{32}$$

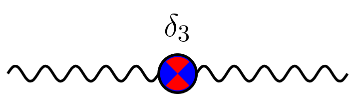
— which looks exactly like the kinetic-energy Lagrangian for the vector superfields. In other words, the whole two-photon amplitude amounts to the wave-function renormalization of the vector superfield!

Renormalization calls for the counterterms, in particular

$$\mathcal{L}_{\text{CT}} \;\supset\; \delta_3 \times \int d^4\theta\, V \tfrac{1}{8} D^\alpha \overline{D}^2 D_\alpha V. \tag{33}$$

At the one-loop level $\delta^3 = O(g^2)$ and we include the tree diagram containing this counterterm, thus

$i\mathcal{M}_7 \;=$



$$=\; 2i\delta_3 \times \int d^4\theta\, V_1(+k,\theta)\, \tfrac{1}{8} D^\alpha \overline{D}^2 D_\alpha V_2(-k,\theta). \tag{34}$$

Note the factor of $2 = 2!$ here, it stems from permutations of the two vector superfields.

Combining the six one-loop diagrams with this counterterm diagram, we obtain the net two-photon amplitude

$$\begin{aligned} \mathcal{M}^{\text{net}} \;&=\; \mathcal{M}_1 \;+\; \mathcal{M}_2 \;+\; \mathcal{M}_3 \;+\; \mathcal{M}_4 \;+\; \mathcal{M}_5 \;+\; \mathcal{M}_6 \;+\; \mathcal{M}_7 \\ &=\; \Big(\Pi^{\text{net}}(k^2) \;=\; \Pi^{\text{loop}}(k^2) + \delta^3\Big) \times 2 \int d^4\theta\, V_1(+k,\theta)\, \tfrac{1}{8} D^\alpha \overline{D}^2 D_\alpha V_2(-k,\theta). \end{aligned} \tag{35}$$

The $\Pi^{\text{loop}}(k^2)$ here is the momentum integral from eq. (28), including the pre-integral factor $-8ig^2$ and another factor 1/4 to account for different normalizations of the superfield factors in eqs. (32) and (35), thus

$$\Pi^{\rm loop}(k^2) \;=\; \frac{-8ig^2}{4}\times\int\limits_{\rm DR}\!\frac{d^4p_1}{(2\pi)^4}\,\frac{1}{[p_1^2-mm^*+i0]\times[(p_1+k)^2-mm^*+i0]}\,. \tag{36}$$

As far as this integral is concerned, the dimensional reduction DR acts exactly as the dimensional regularization, so we may evaluate this integral exactly as we have learned back in the QFT2 class. First, we use the Feynman parameter trick to simplify the denominator,

$$\frac{1}{[p_1^2-mm^*+i0]\times[(p_1+k)^2-mm^*+i0]} \;=\int\limits_0^1\!dx\,\frac{1}{[q^2-\Delta(x)+i0]^2} \tag{37}$$

for

$$\begin{aligned} q^2\,-\,\Delta(x)\,+i0 \;&=\; (1-x)\times[p_1^2-mm^*+i0]\;+\;x\times[(p_1+k)^2-mm^*+i0]\\ &\Downarrow\\ q^\mu \;=\; p_1^\mu\,+\,xk^\mu \quad &\text{and}\quad \Delta(x)\;=\;mm^*\,-\,x(1-x)k^2. \end{aligned} \tag{38}$$

Second, we shift the loop momentum from p to q and then rotate q to the Euclidean space, thus

$$\Pi^{\rm loop}(k^2) \;=\; +2g^2\times\int\limits_0^1\!dx\int\limits_{\rm DR}\!\frac{d^4q_e}{(2\pi)^4}\,\frac{1}{[q_e^2+\Delta(x)]^2}\,. \tag{39}$$

Third, we reduce the dimension of the Euclidean integral from 4 to $4-2\epsilon$, which gives us

$$\begin{aligned} \int\limits_{\rm DR}\!\frac{d^4q_e}{(2\pi)^4}\,\frac{1}{[q_e^2+\Delta(x)]^2} \;&=\int\!\frac{d^dq}{(2\pi)^d}\,\frac{\mu^{2\epsilon}}{[q^2+\Delta]^2}\\ &=\;\frac{1}{16\pi^2}\,\Gamma(\epsilon)\times\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon}\\ &\underset{\epsilon\to0}{\longrightarrow}\;\frac{1}{16\pi^2}\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{\Delta(x)}\right). \end{aligned} \tag{40}$$

Finally, we integrate over the Feynman parameter x to obtain

$$\Pi^{\rm loop}(k^2) \;=\; +\frac{2g^2}{16\pi^2}\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{m^2}\,+\,\mathcal{J}(-k^2/m^2)\right) \tag{41}$$

where

$$\mathcal{J}(-k^2/m^2) \;=\; \int_0^1 dx\,\log\frac{m^2}{m^2-k^2\times x(1-x)} \;=\; \begin{cases} O(k^2/m^2) & \text{for } k^2\ll m^2,\\ 2-\log(-k^2/m^2) & \text{for } k^2\gg m^2. \end{cases} \tag{42}$$

The $1/\epsilon$ pole in Π^{loop} corresponds in DR to the logarithmic UV divergence of the momentum integral. To cancel this divergence from

$$\Pi^{\text{net}}(k^2) \;=\; \Pi^{\text{loop}}(k^2) \;+\; \delta_3 \tag{43}$$

we should set the δ_3 counterterm to

$$\delta_3^{(1\,\text{loop})} \;=\; -\frac{2g^2}{16\pi^2}\times\left(\frac{1}{\epsilon}+\text{finite}\right) \tag{44}$$

where the finite part depends on our renormalization scheme. As we have learned in the QFT2 class, the negative coefficient of the $1/\epsilon$ pole corresponds to the positive anomalous dimension

$$\gamma_v \;=\; +\frac{2g^2}{16\pi^2} \;+\; O(g^4) \tag{45}$$

of the vector superfields V — and hence of the component EM and photino fields. Also, the SQED one-loop beta-function follows from this anomalous dimension as

$$\beta(g) \;=\; g\times\gamma_v(g) \;=\; +\frac{2g^3}{16\pi^2} \;+\; O(g^5). \tag{46}$$

Comparing to the ordinary QED.

In the component fields formalism, SQED is a QED-like theory of a massless EM field coupled to a charged Dirac fermion (the electron) and two species of charged scalars (the selectrons). There is also a neutral Majorana fermion (the photino) which has Yukawa coupling to the electron and the selectrons.

At the one-loop level — and only at the one-loop level, — the beta function for the gauge coupling does not care about any neutral fields and any non-gauge interactions. All it depends on is the spectrum of the charged fields, thus

$$\beta(e) \;=\; b_1 \times e^3 \;+\; O(e^5) \tag{47}$$

where

$$b_1 \;=\; \frac{1}{12\pi^2}[\text{for the electron}] \;+\; 2 \times \frac{1}{48\pi^2}[\text{for each selectron}] \;=\; \frac{1}{8\pi^2}\,. \tag{48}$$

By inspection, this formula agrees with eq. (46) when we identify $e = g$.

GENERALIZATION.

Finally, let's generalize eq. (46) to the one-loop beta-function of SQED coupled to any number of charged chiral superfields $A_1, \ldots, A_n$ of respective charges $q_1 g, \ldots, q_n g$. For such a theory,

$$\beta(g) \;=\; \frac{g^3}{16\pi^2} \times \sum_i q_i^2 \;+\; O(g^5). \tag{49}$$

Note: the sum here is over all the charged chiral superfields, so the electron and its superpartners count give rise to 2 terms in this sum: One for the A superfield containing the left-handed electron and one selectron, and one for the B superfield containing the left-handed positron and the other selectron.