SUSY NOTATIONS AND CONVENTIONS

In these notes I briefly summarize the nonations and the conventions I shall use in my SUSY class in Fall 2025. Let's start with the sign conventions:

- Metric signature $(+,-,-,-)$ as in Peskin & Schroeder.
- Sigma matrices for the Weyl spinors are $\sigma^\mu=(1,+\vec{\sigma})$ and $\bar{\sigma}^\mu=(1,-\vec{\sigma})$ as in Peskin & Schroeder.
  - ∗ Trace relation: $\operatorname{tr}\bigl(\sigma^\mu\bar{\sigma}^\nu\bigr)=2g^{\mu\nu}$. Note that one sigma inside the trace should be barred, the other un-barred. Here is another useful trace formula:

$$\begin{aligned}\operatorname{tr}\bigl(\sigma^\kappa\bar{\sigma}^\lambda\sigma^\mu\bar{\sigma}^\nu\bigr) &= \operatorname{tr}_{\text{Dirac}}\left(\frac{1-\gamma^5}{2}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right)\\ &= 2g^{\kappa\lambda}g^{\mu\nu}\;-\;2g^{\kappa\mu}g^{\lambda\nu}\;-\;2g^{\kappa^\nu}g^{\lambda^\mu}\;+\;2i\epsilon^{\kappa\lambda\mu\nu}.\end{aligned}\tag{1}$$

- The Levi–Civita tensors for the two-component Weyl spinors are as in Wess & Bagger: For the left-handed Weyl spinors,

$$\epsilon^{\alpha\beta}=\begin{cases}+1 & \text{for } \alpha=1,\,\beta=2,\\ -1 & \text{for } \alpha=2,\,\beta=1,\end{cases}\qquad \text{but}\quad \epsilon_{\alpha\beta}=\begin{cases}-1 & \text{for } \alpha=1,\,\beta=2,\\ +1 & \text{for } \alpha=2,\,\beta=1,\end{cases}\tag{2}$$

  while for the right-handed Weyl spinors

$$\epsilon_{\dot{\alpha}\dot{\beta}}=\begin{cases}+1 & \text{for } \dot{\alpha}=1,\,\dot{\beta}=2,\\ -1 & \text{for } \dot{\alpha}=2,\,\dot{\beta}=1,\end{cases}\qquad \text{but}\quad \epsilon^{\dot{\alpha}\dot{\beta}}=\begin{cases}-1 & \text{for } \dot{\alpha}=1,\,\dot{\beta}=2,\\ +1 & \text{for } \dot{\alpha}=2,\,\dot{\beta}=1.\end{cases}\tag{3}$$

  This way, we may consistently use these tensors to raise and lower the spinor indices,

$$\psi^\alpha=\epsilon^{\alpha\beta}\psi_\beta\,,\quad \psi_\beta=\epsilon_{\beta\gamma}\psi^\gamma,\quad \overline{\psi}_{\dot{\alpha}}=\epsilon_{\dot{\alpha}\dot{\beta}}\overline{\psi}^{\dot{\beta}},\quad \overline{\psi}^{\dot{\beta}}=\epsilon^{\dot{\beta}\dot{\gamma}}\overline{\psi}_{\dot{\gamma}}\,.\tag{4}$$

- The sigma matrices σ^μ and $\bar{\sigma}^\mu$ are related to each other by transposing their spinor indices and also raising/lowering the indices according to eqs. (4): their

$$\bar{\sigma}^{\mu,\dot{\alpha}\alpha}=\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\,\sigma^\mu_{\beta\dot{\beta}}\,,\qquad \sigma^\mu_{\alpha\dot{\alpha}}=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\sigma}^{\mu,\dot{\beta}\beta}\,.\tag{5}$$

- Weyl spinor products as in Wess & Bagger:

$$\psi\chi\;\stackrel{\text{def}}{=}\;\psi^\alpha\chi_\alpha\;=\;\epsilon_{\alpha\beta}\psi^\alpha\chi^\beta\;=\;-\psi_\alpha\chi^\alpha,\tag{6}$$

$$\overline{\psi}\overline{\chi} \stackrel{\text{def}}{=} \overline{\psi}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\alpha}}\overline{\chi}_{\dot{\beta}} = -\overline{\psi}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}}\,. \tag{7}$$

Note that the fermionic field anticommute with each other, hence

$$\psi\chi = +\chi\psi \quad \text{and} \quad \overline{\psi}\overline{\chi} = +\overline{\chi}\overline{\psi}. \tag{8}$$

In the $\mathcal{N}=1$ superspace, these definitions lead to:

- $\theta^2 = \theta^\alpha\theta_\alpha = 2\theta^1\theta^2$ while $\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = 2\bar{\theta}_1\bar{\theta}_2$. Consequently, the Berezin integrals $\int d^2\theta$ and $\int d^2\overline{\theta}$ should extract the coefficients of the corresponding θ^2 and $\bar{\theta}^2$ terms.
- $(\theta\psi)(\theta\chi) = +\frac{1}{2}\theta^2(\psi\chi)$ and $(\overline{\theta\psi})(\bar{\theta}\overline{\chi}) = +\frac{1}{2}\bar{\theta}^2(\overline{\psi}\overline{\chi})$.
- Combining θ's and $\bar{\theta}$'s with sigma matrices, we get

$$\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = -\bar{\theta}_{\dot{\beta}}\bar{\sigma}^{\mu,\dot{\beta}\beta}\theta_\beta \tag{9}$$

while

$$\left(\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right) \times \left(\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right) = \tfrac{1}{2}\theta^2\bar{\theta}^2 \times g^{\mu\nu}. \tag{10}$$

- The superspace derivatives D_α and $\overline{D}_{\dot{\alpha}}$ obey anti-commutation relations

$$\left\{D_\alpha, D_\beta\right\} = 0, \qquad \left\{\overline{D}_{\dot{\alpha}}, \overline{D}_{\dot{\beta}}\right\} = 0, \tag{11}$$

$$\left\{D_\alpha, \overline{D}_{\dot{\beta}}\right\} = 2i\sigma^\mu_{\alpha\dot{\beta}} \times \partial_\mu\,, \qquad \left\{\overline{D}^{\dot{\alpha}}, D^\beta\right\} = 2i\sigma^{\mu,\dot{\alpha}\beta} \times \partial_\mu\,. \tag{12}$$

- The projection operators onto chiral, antichiral, and linear superfields are

$$\text{chiral} \quad \Pi_C = \frac{-1}{16\partial^2}\,\overline{D}^2 D^2, \tag{13}$$

$$\text{antichiral} \quad \Pi_A = \frac{-1}{16\partial^2}\,D^2\overline{D}^2, \tag{14}$$

$$\text{linear} \quad \Pi_L = \frac{+1}{8\partial^2}\,D^\alpha\overline{D}^2 D_\alpha. \tag{15}$$

- In components, a chiral superfield is

$$\begin{aligned}\Phi(x,\theta,\bar\theta) \;&=\; \phi(y) \;+\; \sqrt{2}\theta^\alpha\psi_\alpha(y) \;+\; \theta^2 F(y)\\ &\qquad \text{where } y^\mu \;=\; x^\mu \;-\; i\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}\\ &=\; \phi(x) \;-\; i(\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta})\times\partial_\mu\phi(x) \;-\; \tfrac14\theta^2\bar\theta^2\times\partial^2\phi(x)\\ &\qquad +\; \sqrt{2}\theta^\alpha\psi_\alpha(x) \;+\; \frac{i}{\sqrt2}\theta^2\bar\theta_{\dot\alpha}\times\bar\sigma^{\mu,\dot\alpha\alpha}\partial_\mu\psi_\alpha(x) \;+\; \theta^2\times F(x).\end{aligned} \tag{16}$$

Likewise, an anitichiral superfield expands to

$$\begin{aligned}\overline\Phi(x,\theta,\bar\theta) \;&=\; \phi^*(\bar y) \;+\; \sqrt{2}\bar\theta_{\dot\alpha}\overline\psi^{\dot\alpha}(\bar y) \;+\; \theta^2 F(\bar y)\\ &\qquad \text{where } \bar y^\mu \;=\; x^\mu \;+\; i\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta}\\ &=\; \phi^*(x) \;+\; i(\theta^\beta\sigma^\mu_{\beta\dot\beta}\bar\theta^{\dot\beta})\times\partial_\mu\phi^*(x) \;-\; \tfrac14\theta^2\bar\theta^2\times\partial^2\phi^*(x)\\ &\qquad +\; \sqrt{2}\bar\theta_{\dot\alpha}\overline\psi^{\dot\alpha}(x) \;-\; \frac{i}{\sqrt2}\bar\theta^2\theta_\alpha\times\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline\psi^{\dot\alpha}(x) \;+\; \theta^2\times F^*(x).\end{aligned} \tag{17}$$

And the free Lagrangian for these superfields is

$$\begin{aligned}\mathcal{L}_{\text{free}} \;&=\; \int d^2\theta\, d^2\bar\theta\,\overline\Phi\,\Phi\\ &=\; \partial_\mu\phi^*\partial^\mu\phi \;+\; i\overline\psi_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\psi^\alpha \;+\; F^*F\\ &\qquad +\; \text{total derivatives.}\end{aligned} \tag{18}$$

- An abelian vector superfield $V(x,\theta,\bar\theta)$ is a Hermitian but otherwise unrestricted superfield. Its free Lagrangian is

$$\mathcal{L} \;=\; +\frac18\int d^4\theta\, V D^\alpha\overline{D}^2 D_\alpha V \;=\; -\frac12\int d^2\theta\, W^\alpha W_\alpha \;=\; \frac{-1}{2}\int d^2\bar\theta\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha} \tag{19}$$

where

$$W^\alpha \;\stackrel{\text{def}}{=}\; -\tfrac14\overline{D}^2 D^\alpha V, \qquad \overline{W}^{\dot\alpha} \;\stackrel{\text{def}}{=}\; -\tfrac14 Dr\overline{D}^{\dot\alpha} V \tag{20}$$

are the (abelian) tension superfields. They are invariant under the supersymmetric gauge transforms

$$V'(x,\theta,\bar\theta) \;=\; V(x,\theta,\bar\theta) \;-\; i\Lambda(y,\theta) \;+\; i\overline\Lambda(\bar y,\bar\theta) \tag{21}$$

for any chiral superfield $\Lambda(y,\theta)$ and its hermitian conjugate $\overline\Lambda(\bar y,\bar\theta)$.

• In components, the tension fields become

$$
\begin{aligned}
W_\alpha(y,\theta) \;&=\; \lambda_\alpha(y) \;+\; \theta_\alpha \times \mathcal{D}(y) \;+\; \tfrac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta}\theta_\beta \times F_{\mu\nu}(y) \;-\; i\theta^2\sigma^\mu_{\alpha\dot\alpha} \times \partial_\mu\bar\lambda^{\dot\alpha}(y), \qquad (22)\\
\overline{W}^{\dot\alpha}(\bar y,\bar\theta) \;&=\; \overline{\lambda}^{\dot\alpha}(\bar y) \;+\; \bar\theta^{\dot\alpha} \times \mathcal{D}(y) \;-\; \tfrac{i}{2}(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}_{\ \dot\beta}\bar\theta^{\dot\beta} \times F_{\mu\nu}(\bar y) \;-\; i\theta^2\bar\sigma^{\mu,\dot\alpha\alpha} \times \partial_\mu\lambda_\alpha(\bar y)\,. \qquad (23)
\end{aligned}
$$

As to the potential field V, *in the Wess–Zumino gauge* it becomes

$$
V(x,\theta,\bar\theta) \;=\; \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \times A_\mu(x) \;+\; \bar\theta^2\,\theta^\alpha\lambda_\alpha(x) \;+\; \theta^2\,\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) \;+\; \tfrac{1}{2}\theta^2\bar\theta^2\,\mathcal{D}(x), \qquad (24)
$$

while in a general gauge there would be a bunch of extra terms.