SUPERSYMMETRIC HIGGS MECHANISM

Caveat: These notes are about spontaneous gauge symmetry breaking in a supersymmetric theory — specifically, $\mathcal{N}=1$ SUSY in $d=4$ dimensions, — rather than spontaneous breaking of the supersymmetry itself.

Without SUSY, when a gauge symmetry is spontaneously broken, the Higgs mechanism 'eats' a real scalar for every vector field that becomes massive. The simplest explanation for that is basic polarization counting: A massless gauge boson has two transverse polarization states but no longitudinal polarization. In helicity terms, this means it may have $\lambda=+1$ or $\lambda=-1$ but not $\lambda=0$. On the other hand, a massive gauge boson must have a complete spin $=1$ multiplet of polarization states, which means helicities $\lambda=+1$, $\lambda=-1$ and also $\lambda=0$. That $\lambda=0$ polarization corresponds to a degree of freedom the gauge boson did not have when it was massless, so it must come from someplace else. In other words, the Higgs mechanism which makes the gauge boson massive must 'eat' a $\lambda=0$ degree of freedom that used to belong to some other kind of particle. And since the eaten degree of freedom has only one $\lambda=0$ polarization, it must have been a scalar (or pseudoscalar).

In supersymmetric theories we should consider SUSY multiplets rather than mere spin / helicity multiplets, and that makes for a much larger difference between the massless and massive vector multiplets. Specifically,

$$\begin{pmatrix}\text{massive}\\ \text{vector}\\ \text{multiplet}\end{pmatrix} - \begin{pmatrix}\text{massless}\\ \text{vector}\\ \text{multiplet}\end{pmatrix} = \begin{pmatrix}\text{complex}\\ \text{scalar}\\ \text{multiplet}\end{pmatrix}, \tag{1}$$

so *the supersymmetric Higgs mechanism eats a whole complex scalar multiplet for every gauge boson that becomes massive.*

Indeed, consider the $\mathcal{N}=1$ supermultiplets in 4D. The SUSY algebra is generated by 4 operators $\hat{Q}_{1,2}$ and $\hat{Q}^\dagger_{1,2}$ that commute with energy and momentum and obey anticommutation relations

$$\{\hat{Q}_1,\hat{Q}_2\} \;=\; \{\hat{Q}^\dagger_1,\hat{Q}^\dagger_2\} \;=\; 0, \tag{2}$$

$$\{\hat{Q}_\alpha,\hat{Q}^\dagger_{\dot\alpha}\} \;=\; 2P_\mu\sigma^\mu_{\alpha\dot\alpha}\,. \tag{3}$$

When acting on 1-particle states of definite *timelike* momentum P^μ, the matrix $P_\mu \sigma^\mu$ is positive-definite, so the algebra of $\hat{Q}$ and $\hat{Q}^\dagger$ operators is equivalent to the Clifford algebra of 2 pairs of fermionic creation and annihilation operators, so its basic multiplet has 4 states, 2 bosons and 2 fermions. Taking into account that both $\hat{Q}_\alpha$ and $\hat{Q}^\dagger_{\dot\alpha}$ operators have spin $= \frac{1}{2}$, we get the following diagrams how the operators act on the 4 states of this multiplet:

$$\boxed{\text{spin} = 0} \; \underset{\hat{Q}}{\overset{\hat{Q}^\dagger}{\rightleftarrows}} \; \boxed{\text{spin} = \tfrac{1}{2}} \; \underset{\hat{Q}}{\overset{\hat{Q}^\dagger}{\rightleftarrows}} \; \boxed{\text{spin} = 0} \tag{4}$$

This is the massive scalar multiplet, comprising 2 real scalars and one real (Majorana) spin $= \frac{1}{2}$ fermion. It obtains when the state annihilated by both $\hat{Q}_\alpha$ (leftmost on the above diagram) has spin $= 0$. Replacing that state with a spin $= \frac{1}{2}$ doublet of states, we construct the massive vector multiplet

$$\begin{tikzcd}
 & \boxed{\text{spin} = 1} \arrow[dr, shift left, "\hat{Q}^\dagger"] \arrow[dl, shift left, "\hat{Q}"] & \\
\boxed{\text{spin} = \frac{1}{2}} \arrow[ur, shift left, "\hat{Q}^\dagger"] \arrow[dr, shift left, "\hat{Q}^\dagger"] & & \boxed{\text{spin} = \frac{1}{2}} \arrow[ul, shift left, "\hat{Q}"] \arrow[dl, shift left, "\hat{Q}"] \\
 & \boxed{\text{spin} = 0} \arrow[ul, shift left, "\hat{Q}"] \arrow[ur, shift left, "\hat{Q}^\dagger"] &
\end{tikzcd} \qquad (5)$$

This multiplet comprises a massive spin $= 1$ vector, a real scalar, and two real spin $= \frac{1}{2}$ fermions. In terms of helicities, this multiplet has 8 states with helicities

$$\lambda \in \left(-1, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, +\tfrac{1}{2}, +\tfrac{1}{2}, +1\right). \tag{6}$$

Now consider the massless supermultiplets. For a lightlike momentum P^μ of a massless particle, the matrix $P_\mu \sigma^\mu$ in the anticommutation relation (3) has a zero eigenvalue, so one pair of the $\hat{Q}$ and $\hat{Q}^\dagger$ operators become null — they kill all the states — and only the other pair turn states into their superpartners. Thus, a basic massless supermultiplet comprises 2 states — one boson and one fermion — and their helicities differ by $\Delta\lambda = \pm\frac{1}{2}$ (because that's the helicities of the non-null $\hat{Q}$ and $\hat{Q}^\dagger$ operators). For example, $(\lambda = 0, \lambda = +\frac{1}{2})$

or $(\lambda = 0, \lambda = -\frac{1}{2})$ for a chiral scalar multiplet. Together, 2 such multiplets related by Hermitian conjugation form a complex scalar multiplet

$$\lambda \in \left(-\tfrac{1}{2}, 0, 0, +\tfrac{1}{2}\right). \tag{7}$$

Physically, it comprises a complex scalar, a Weyl fermion, and their respective antiparticles. Or in terms of real (neutral) particles, two real scalars and one Majorana fermion.

Likewise, a massless vector multiplet comprises two helicity pairs $(\lambda = +\frac{1}{2}, \lambda = +1)$ and $(\lambda = -\frac{1}{2}, \lambda = -1)$ related by CP symmetry. Altogether, it has 4 states

$$\lambda \in \left(-1, -\tfrac{1}{2}, +\tfrac{1}{2}, +1\right), \tag{8}$$

which physically comprise a real massless vector and a massless Majorana fermion.

Comparing the massive and massless vector multiplets (6) and (8), we immediately see a big difference: The massive multiplet has 8 helicity states while the massless multiplet has only 4. The missing helicity state comprise

$$\left(-1, -\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, +\tfrac{1}{2}, +\tfrac{1}{2}, +1\right) \;-\; \left(-1, -\tfrac{1}{2}, +\tfrac{1}{2}, +1\right) \;=\; \left(-\tfrac{1}{2}, 0, 0, +\tfrac{1}{2}\right), \tag{9}$$

which is precisely the content of a complex scalar multiplet (7). Thus, to make a massless vector multiplet into a massive vector multiplet, the supersymmetric Higgs mechanism must 'eat' a whole complex scalar multiplet.

SQED Example

Let me illustrate the supersymmetric Higgs mechanism — in the ordinary field formalism and in the superfield formalism — using SQED as an example. SQED stands for supersymmetric QED, and its particle spectrum has just 3 supermultiplets: the massless vector supermultiplet V comprising the photon and the photino, and two scalar multiplets A and B comprising the Dirac electron and two selectrons a and b. Specifically,

$$\Psi_{\text{Dirac}} = \begin{pmatrix} \psi^\alpha \\ \overline{\chi}_{\dot\alpha} \end{pmatrix} \quad \text{where} \quad a, \psi^\alpha \in A \quad \text{while} \quad b, \chi^\beta \in B, \tag{10}$$

so A has electric charge $-e$ while B has electric charge $+e$. In terms of the corresponding superfields — chiral A and B and vector V — the SQED Lagrangian is

$$\begin{aligned}\mathcal{L} \;=&\int d^4\theta\left(\tfrac{1}{8}VD^\alpha\overline{D}^2D_\alpha V\;+\;\overline{A}\exp(-2eV)A\;+\;\overline{B}\exp(+2eV)B\right)\\&+\int d^2\theta\, mAB\;+\int d^2\bar\theta\, m\overline{AB}.\end{aligned}\tag{11}$$

To get the component-field Lagrangian, we should first impose the Wess–Zumino gauge condition

$$\text{at } \theta=\bar\theta=0,\quad V\;=\;\frac{\partial V}{\partial\theta}\;=\;\frac{\partial V}{\partial\bar\theta}\;=\;\frac{\partial^2 V}{\partial\theta^2}\;=\;\frac{\partial^2 V}{\partial\bar\theta^2}\;=\;0,\tag{12}$$

and only then expand all superfields into components. The result is

$$\begin{aligned}\mathcal{L} \;=\;& -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\;+\;\overline{\lambda}\bar\sigma^\mu\partial_\mu\lambda\;+\;\tfrac{1}{2}\mathcal{D}^2\\&+\;(D_\mu a^*)(D^\mu a)\;+\;(D_\mu b^*)(D^\mu b)\;+\;F_a^*F_a\;+\;F_b^*F_b\\&+\;i\overline{\psi}\bar\sigma^\mu D_\mu\psi\;+\;i\overline{\chi}\bar\sigma^\mu D_\mu\chi\;+\;m\psi\chi\;+\;m\overline{\psi}\overline{\chi}\\&+\;\sqrt{2}e\Big(-a^*\lambda\psi\;+\;b^*\lambda\chi\;-\;a\overline{\lambda\psi}\;+\;b\overline{\lambda}\overline{\chi}\Big)\\&+\;e\mathcal{D}\times\big(-a^*a\;+\;b^*b\big)+\;m\big(aF_b\;+\;bF_a\;+\;a^*F_b^*\;+\;b^*F_a^*\big).\end{aligned}\tag{13}$$

where $\mathcal{D}$, F_a, and F_b are auxiliary fields. Eliminating them by their equations of motion, we get the scalar potential of the theory:

$$V\;=\;\frac{e^2}{2}\left(-a^*a\;+\;b^*b\right)^2\;+\;m^2\big(a^*a\;+\;b^*b\big).\tag{14}$$

For any non-zero mass m, this potential has a unique minimum at $a=b=0$, so there are no scalar VEVs, no Higgs mechanism, and all the electron ans selectron fields are massive. So from now on, we shall focus on the massless theory with $m=0$.

Component Field Formalism

For $m=0$ the scalar potential (14) vanishes whenever $|a|=|b|$. Moreover, we shall learn later in class that this locus of exactly zero potential is not affected by any quantum corrections. Thus, the theory has exactly degenerate family of vacua parametrized by 2 complex VEVs $\langle a\rangle$ and $\langle b\rangle$ subject to a real constraint $|\langle a\rangle|=|\langle b\rangle|$. This is a rather common phenomenon in supersymmetric theories, so it comes with a terminology: The parameters of the degenerate vacua are called *moduli*, the space of allowed values of these parameters is the *moduli space*, and the fields whose VEVs are moduli are called *moduli fields* or *moduli scalars* .

The moduli space of the massless SQED appears to have 3 real dimensions: it has 2 complex parameters subject to a real constraint. However, one of these 3 dimensions is an artefact of the residual gauge symmetry. Indeed, the Wess–Zumino gauge condition on the vector superfield V does not fix the ordinary gauge transforms of the component fields,

$$\begin{aligned}
a'(x) &= a(x)\times e^{-i\phi(x)}, & \psi'(x) &= \psi(x)\times e^{-i\phi(x)},\\
b'(x) &= b(x)\times e^{+i\phi(x)}, & \chi'(x) &= \chi(x)\times e^{+i\phi(x)},\\
\lambda'(x) &= \lambda(x), & A'_\mu(x) &= A_\mu(x)\;-\;\frac{1}{e}\partial_\mu\phi(x).
\end{aligned}\tag{15}$$

To fix this residual gauge redundancy, we need to impose another gauge condition at every point in space, and for the Higgsed down theory with $\langle a\rangle\neq 0$ and/or $\langle b\rangle\neq 0$, the simplest extra gauge condition would be the unitary gauge such as $\mathrm{phase}(a)=0$ or $\mathrm{phase}(b)=0$. However, for two scalar fields a and b of opposite electric charges $\mp e$, — and both fields having VEVs of similar magnitudes — the best unitary gauge condition is the combined condition

$$\mathrm{phase}(a)\;=\;\mathrm{phase}(b).\tag{16}$$

This condition is particularly convenient for the moduli space of the massless SQED. Indeed, combining the unitary gauge condition (16) with the zero-potential condition $|a|=|b|$, we get a complex constraint on the VEVs:

$$\langle a\rangle\;=\;\langle b\rangle\,.\tag{17}$$

Consequently, the moduli space of the gauge-fixed theory has only one complex dimension, or two real dimensions.

This is a general behavior of supersymmetric theories: Once the gauge redundancy is completely fixed, the moduli space of any theory with un-broken SUSY always has an even real dimension. Moreover, the moduli scalars can always be organized into complex fields that are lowest components of some massless chiral superfields, and the geometry of the moduli space is always a Kähler geometry. I shall return to this issue later in class.

Meanwhile, let's calculate the masses of all particle in a vacuum with $\langle a\rangle = \langle b\rangle \neq 0$. In light of the unitary gauge condition (16), let's parametrize the scalar fields as

$$a(x) \;=\; \tfrac{1}{2}\phi(x) \times e^{i\Theta(x)+\alpha(x)}, \qquad b(x) \;=\; \tfrac{1}{2}\phi(x) \times e^{i\Theta(x)-\alpha(x)}, \tag{18}$$

for real $\phi \neq 0$, Θ, and α. Consequently, the scalar kinetic terms become

$$\begin{aligned}
|D_\mu a|^2 \;+\; |D_\mu b|^2 \;&=\; |\partial_\mu a - ieA_\mu a|^2 \;\; |\partial_\mu b + ieA_\mu b|^2 \\
&=\; \frac{e^{+2\alpha}}{4}\Big((\partial_\mu\phi + \phi\partial_\mu\alpha)^2 \;+\; \phi^2(\partial_\mu\Theta - eA_\mu)^2\Big) \\
&\qquad +\; \frac{e^{-2\alpha}}{4}\Big((\partial_\mu\phi - \phi\partial_\mu\alpha)^2 \;+\; \phi^2(\partial_\mu\Theta + eA_\mu)^2\Big) \\
&=\; \frac{\cosh(2\alpha)}{2} \times \Big((\partial_\mu\phi)^2 \;+\; \phi^2(\partial_\mu\alpha)^2 \;+\; \phi^2(\partial_\mu\Theta)^2 \;+\; e^2\phi^2(A_\mu)^2\Big) \\
&\qquad +\; \sinh(2\alpha) \times \Big((\partial_\nu\phi) \times \phi(\partial_\mu\alpha) \;-\; (\partial_\mu\Theta) \times (e\phi A_\mu)\Big) \\
&=\; \frac{1}{2}(\partial_\mu\phi)^2 \;+\; \frac{\langle\phi\rangle^2}{2}(\partial_\mu\alpha)^2 \;+\; \frac{\langle\phi\rangle^2}{2}(\partial_\mu\alpha)^2 \;+\; \frac{e^2\langle\phi\rangle^2}{2}(A_\mu)^2 \\
&\qquad +\; \text{interaction terms},
\end{aligned} \tag{19}$$

which means the vector field A_μ gets mass $M = e\langle\phi\rangle$, while the 3 canonically normalized un-eaten scalar field correspond to

$$\delta\phi(x) \;=\; \phi(x) \;-\; \langle\phi\rangle\,, \qquad \langle\phi\rangle \times \Theta(x), \qquad \langle\phi\rangle \times \alpha(x). \tag{20}$$

The masses of these scalars follow from

$$a^*a \;-\; b^*b \;=\; \tfrac{1}{2}\phi^2 \times \sinh(2\alpha), \tag{21}$$

hence the scalar potential

$$V \;=\; \frac{e^2}{8}\,\phi^4 \times \sinh^2(2\alpha) \;=\; \frac{e^2\langle\phi\rangle^2}{2} \times \big(\langle\phi\rangle \times \alpha\big)^2 \;+\; \text{interaction terms}. \tag{22}$$

Thus, *the real scalar field vev$\phi \times \alpha(x)$ acquires exactly the same mass $M = e\langle\phi\rangle$ as the vector* *field*, while the other two real scalars remain exactly massless. Together, the two massless scalars comprise the complex modulus field

$$\Phi(x) \;=\; \sqrt{\tfrac{1}{2}}\phi(x)e^{i\Theta(x)}. \tag{23}$$

Next, consider the 3 LH Weyl fermions of the theory, λ, ψ, and χ. When the scalar fields a and b develop non-zero VEVs, the Yukawa couplings of the photino λ to the electron ψ and positron χ give rise to the fermionic mass terms,

$$\begin{aligned}\mathcal{L}_{\rm FM} \;&=\; -\sqrt{2}e\,\langle a\rangle^{*}\times\lambda\psi \;+\; -\sqrt{2}e\,\langle b\rangle^{*}\times\lambda\chi \;+\; \text{H. c.}\\ &=\; e\,\langle\phi\rangle\,e^{-i\langle\Theta\rangle}\times\lambda^{\alpha}\left(\frac{\chi-\psi}{\sqrt{2}}\right)_{\alpha} \;+\; \text{H. c.}\end{aligned} \tag{24}$$

This is a Dirac mass of magnitude $M = e\,\langle\phi\rangle$ — same as the photon's mass — connecting the photino λ and a combination $\frac{\chi-\psi}{\sqrt{2}}$ of the electron and positron fields. The other combination $\frac{\chi+\psi}{\sqrt{2}}$ remains massless.

Altogether, we end up with several particles of exactly the same mass $M = e\,\langle\phi\rangle$: the photon, the photino, another Weyl fermion, and a real scalar. From the SUSY point of view, these particles comprise a massive vector multiplet. At the same time, 2 real scalars and one Weyl fermion remain massless; together, they comprise a complex scalar multiplet. And this is it, there are no other particles.

So let us summarize: Before the Higgs mechanism, SQED had 1 massless vector multiplet and 2 complex scalar multiplets, which after the Higgs mechanism it has 1 massive vector multiplet and only 1 complex scalar multiplet. Thus, the supersymmetric Higgs mechanism has 'eaten' the whole complex scalar multiplet and turned it into the missing components of the massive vector multiplet.

In terms of the ordinary fields, different components of the 'eaten' scalar multiplet are 'swallowed' in different ways: One real scalar field is 'eaten' by the ordinary Higgs mechanism, another real scalar becomes massive via the D-term in the scalar potential, and a Weyl fermion acquires a Dirac mass connecting it to the photino by means of the Yukawa couplings. But both the Yukawa couplings and the D-term are related by SUSY to the gauge coupling e, so we end up with equal masses for all these 'eaten' fields. So while they seem to be 'swallowed' in different ways, they end up 'digested' together and all become missing components of the same massive vector supermultiplet.

Superfield Formalism

In the superfield formalism, SQED has one vector superfield $V(x,\theta,\bar{\theta})$ and two chiral superfields — $A(y,\theta)$ and $B(y,\theta)$ — together with their anti-chiral Hermitian conjugates $\overline{A}(\bar{y},\bar{\theta})$ and $\overline{B}(\bar{y},\bar{\theta})$. The massless Lagrangian

$$\mathcal{L} \;=\; \int d^4\theta \left(\tfrac{1}{8} V D^\alpha \overline{D}^2 D_\alpha V \;+\; \overline{A}\exp(-2eV)A \;+\; \overline{B}\exp(+2eV)B \right) \tag{25}$$

is not only gauge invariant, but it's invariant under the SUSY extension of the gauge symmetries

$$\begin{aligned} A'(y,\theta) \;&=\; A(y,\theta)\times e^{-i\Lambda(y,\theta)}, \qquad & B'(y,\theta) \;&=\; B(y,\theta)\times e^{+i\Lambda(y,\theta)},\\ \overline{A}'(\bar{y},\bar{\theta}) \;&=\; \overline{A}(\bar{y},\bar{\theta})\times e^{+i\overline{\Lambda}(\bar{y},\bar{\theta})}, \qquad & \overline{B}'(\bar{y},\bar{\theta}) \;&=\; \overline{B}(\bar{y},\bar{\theta})\times e^{-i\overline{\Lambda}(\bar{y},\bar{\theta})},\\ V'(x,\theta,\bar{\theta}) \;&=\; V(x,\theta,\bar{\theta}) \;-\; \frac{i}{2e}\Lambda(y,\theta) \;+\; \frac{i}{2e}\overline{\Lambda}(\bar{y},\bar{\theta}), \end{aligned} \tag{26}$$

parametrized by an arbitrary chiral superfield $\Lambda(y,\theta)$ and its hermitian conjugate $\overline{\Lambda}(\bar{y},\bar{\theta})$. Note that these gauge transforms may turn *any* non-zero chiral superfield $A(y,\theta)$ into any other non-zero $A'(y,\theta)$ we like, and ditto for their antichiral conjugates $\overline{A}(\bar{y},\bar{\theta}) \to \overline{A}'(\bar{y},\bar{\theta})$. However, while a gauge transform can turn any A into any A' and likewise any B into any B', it cannot do both at the same time. Instead, the product

$$\mathcal{M}(y,\theta) \;=\; A(y,\theta)\times B(y,\theta) \tag{27}$$

is a gauge-invariant chiral superfield, and we shall later use it as a modulus superfield.

Meanwhile, let's fix a gauge. Instead of imposing a non-supersymmetric Wess–Zumino gauge on the vector superfield, and then fix the residual ordinary gauge redundancy using unitary gauge condition for the component scalars, let's fix all the gauge redundancies in one go. Instead, let's impose *the supersymmetric unitary gauge* on the entire chiral parameter

$\Lambda(y,\theta)$ and its conjugate. An example of such a gauge would be

$$A(y,\theta) \;=\; \text{const}, \qquad \overline{A}(\bar{y},\bar{\theta}) \;=\; \text{const}, \tag{28}$$

or

$$B(y,\theta) \;=\; \text{const}, \qquad \overline{B}(\bar{y},\bar{\theta}) \;=\; \text{const}, \tag{29}$$

(but not both!). However, for our purposes, it's more convenient to demand

$$A(y,\theta) \;\equiv\; B(y,\theta), \qquad \overline{A}(\bar{y},\bar{\theta}) \;\equiv\; \overline{B}(\bar{y},\bar{\theta}). \tag{30}$$

So let us impose this particular unitary gauge condition and identify

$$A(y,\theta) \;=\; B(y,\theta) \;=\; \sqrt{\tfrac{1}{2}}\Phi(y,\theta) \tag{31}$$

and ditto for the conjugate field $\overline{\Phi}$. Then in terms of this field, the Lagrangian (25) becomes

$$\begin{aligned} \mathcal{L} \;&=\int\! d^4\theta \left(\tfrac{1}{8} V D^\alpha \overline{D}^2 D_\alpha V \;+\; \overline{\Phi}\Phi \times \cosh(2eV)\right) \\ &=\int\! d^4\theta \left(\overline{\Phi} \times \Phi \;+\; M^2 \times V^2 \;+\; \tfrac{1}{8} V D^\alpha \overline{D}^2 D_\alpha V \;+\; \text{interactions}\right) \end{aligned} \tag{32}$$

where

$$M^2 \;=\; 2e^2\left|\langle\Phi\rangle\right|^2 \;=\; e^2 \times \langle \text{real}\,\phi\rangle^2. \tag{33}$$

In the gauge-fixed Lagrangian (32), Φ is a massless chiral superfield — which we may use as a modulus — while V is a massive vector superfield. Indeed, in your first homework#1 (problem **3**) you shall see that the classical field equations for a vector superfield V with Lagrangian

$$\mathcal{L} \;=\int\! d^4\theta \left(M^2 \times V^2 \;+\; \tfrac{1}{8} V D^\alpha \overline{D}^2 D_\alpha V\right) \tag{34}$$

are $D^2V = \overline{D}^2 V = 0$ and $(\partial^2 + M^2)V = 0$, while the independent component fields of V correspond to a massive vector supermultiplet: they comprise a massive vector, a real scalar, and two Weyl spinors — or equivalently one Dirac spinor, — all of the same mass M.

Note that before the Higgs mechanism the theory had two chiral superfields A and B, while after the Higgs mechanism it has only one Φ. The other chiral superfield — together with its antichiral conjugate — got eaten by the Higgs mechanism. In the SUSY unitary gauge, this eaten chiral superfield is directly eliminated by the gauge condition (30), and then the massive vector superfield V gets extra degrees of freedom simply because it no longer has gauge redundancies. This is similar to how the ordinary (non-SUSY) Higgs mechanism eats a scalar field in the ordinary unitary gauge: the would-be Goldstone scalar is frozen by the gauge condition, while the vector field A_μ gets a third polarization simply because it's massive and no longer gauge-redundant. But in the SUSY unitary gauge, the gauge condition kills a whole chiral superfield and its conjugate, while the massive vector superfield gets a similar bunch of extra degrees of freedom rather than just one polarization.

In other gauges, the eating of scalar multiplets by the Higgs mechanism proceeds in a different manner, but the net result is always the same: each vector supermultiplet that becomes massive eats a whole complex scalar supermultiplet.

Moduli Spaces and Goldstone Fields

Let me conclude these notes with a few notes about the moduli space of SQED and the moduli space in general.

We saw that the scalar potential of SQED has a 3D space of exactly flat directions, and after modding out by the gauge symmetry it becomes a 2D moduli space. That is, it has 2 real parameters ϕ and Θ, or equivalently 1 parameter Φ which we may promote to a chiral modulus superfield $\Phi(y,\theta)$. The geometry of this moduli space follows from the gauge-fixed Lagrangian (32):

$$\mathcal{L} \supset \int d^4\theta\, \overline{\Phi} \times \Phi, \tag{35}$$

which corresponds to the Kähler function

$$K(\overline{\Phi}, \Phi) = \overline{\Phi} \times \Phi \tag{36}$$

and hence *flat* metric

$$g_{\overline{\Phi},\Phi} = \frac{\partial^2 K}{\partial\overline{\Phi}\,\partial_\Phi} \equiv 1. \tag{37}$$

So at first blush, the moduli space appears to be just the complex plane $\mathbf{C}$.

The actual geometry is a bit more complicated. Although the moduli space metric is indeed *locally flat* in the complex Φ coordinate, this coordinate is double-valued rather than single-valued. Note that the SUSY unitary gauge condition

$$A(y,\theta) \;\equiv\; B(y,\theta), \qquad \overline{A}(\bar{y},\bar{\theta}) \;\equiv\; \overline{B}(\bar{y},\bar{\theta}), \tag{38}$$

does not completely fix the (supersymmetrized) $U(1)$ gauge symmetry but leaves un-fixed a discrete subgroup $\mathbf{Z}_2 \in U(1)$ corresponding to $\Lambda = \overline{\Lambda} = \pi$. This subgroup acts as

$$A\ to\ -A, \quad B \;\to\; -B, \quad \overline{A} \;\to\; -\overline{A}, \quad \overline{B} \;\to\; -\overline{B}, \tag{39}$$

which indeed preserves the conditions (38), but it also act on the moduli fields as

$$\Phi(y,\theta) \;\to\; -\Phi(y,\theta), \quad \overline{\Phi}(\bar{y},\bar{\theta}) \;\to\; -\overline{\Phi}(\bar{y},\bar{\theta}). \tag{40}$$

Since this $\mathbf{Z}_2$ is a gauge redundancy rather than a physical symmetry, the complex coordinates $+\Phi$ and $-\Phi$ are redundant descriptions of the same point of the moduli space (rather than 2 similar points related by a physical symmetry). Consequently, *the moduli space geometry is* $\mathbf{C}/\mathbf{Z}_2$ *cone rather than the flat complex plane* $\mathbf{C}$.

A good single-valued gauge-invariant complex coordinate for this cone is

$$\mathcal{M} \;\stackrel{\rm def}{=}\; A\times B \;=\; \tfrac{1}{2}\Phi^2. \tag{41}$$

But in terms of this coordinate, the Kähler metric of the moduli space is no longer constant. Instead,

$$K(\overline{\mathcal{M}},\mathcal{M}) \;=\; K(\overline{\Phi},\Phi) \;=\; \overline{\Phi}\Phi \;=\; 2\sqrt{\overline{\mathcal{M}}\mathcal{M}}, \tag{42}$$

hence

$$g_{\overline{\mathcal{M}},\mathcal{M}} \;=\; \frac{\partial^2 K}{\partial\overline{\mathcal{M}}\,\partial\mathcal{M}} \;=\; \frac{1}{2\sqrt{\overline{\mathcal{M}}\mathcal{M}}}\,. \tag{43}$$

Now consider the moduli spaces of more general theories. In the non-supersymmetric field theories, continuous families of exactly degenerate vacua are always related by some symmetries realized in the Nambu–Goldstone mode. Indeed, without such a symmetry, the quantum corrections (perturbative or non-perturbative) would generate different vacuum energies for different vacua, which will spoil the degeneracy. Consequently, the massless moduli scalar whose VEVs parametrize different vacua within the degenerate family must be the Goldstone bosons of some spontaneously broken symmetry.

But in supersymmetric theories, there may be all kinds of moduli fields that are not Goldstone bosons. First of all, even if a theory does have a spontaneously broken symmetry and hence a Goldstone scalar, that Goldstone scalar would have superpartners — a fermion and another scalar — that are not themselves Goldstone particles. For example, the massless SQED has a global axial $U(1)$ symmetry which acts as

$$\begin{aligned} A(y,\theta) &\to e^{+i\alpha}A(y,\theta), \qquad & B(y,\theta) &\to e^{+i\alpha}B(y,\theta), \\ \overline{A}(\bar{y},\bar{\theta}) &\to e^{-i\alpha}\overline{A}(\bar{y},\bar{\theta}), \qquad & \overline{B}(\bar{y},\bar{\theta}) &\to e^{-i\alpha}\overline{B}(\bar{y},\bar{\theta}). \end{aligned} \tag{44}$$

When

$$\langle A\rangle \;=\; \langle B\rangle \;=\; \tfrac{1}{2}\phi e^{i\Theta} \;\neq\; 0 \tag{45}$$

this axial $U(1)$ is spontaneously broken, with the Goldstone boson of this SSB being the real scalar field $\delta\Theta(x)$; that's why this real scalar field is exactly massless. On the other hand, the radial scalar $\delta\phi(x)$ is NOT a Goldstone mode, but it is a superpartner of the Goldstone scalar $\delta\Theta$, so it is also exactly massless. And their fermionic superpartner $\widetilde{\psi} = (\psi_\chi)/\sqrt{2}$ is also exactly massless, because all members of the same supermultiplet must have exactly the same mass. Moreover, SUSY extends the exact flatness of the scalar potential WRT the Goldstone mode Θ into similarly exact flatness WRT to its superpartner ϕ. So the degenerate vacuum space has two real moduli Θ and ϕ even though only one of them — the Θ — is a Goldstone mode.

Similarly, for many other SUSY theories with some spontaneously broken continuous symmetries, each real Goldstone scalar becomes a part of an exactly massless modulus superfield. The other components of that chiral superfield — the fermion and the other scalar — are not Goldstone fields, but they are just as massless, and the effective potential for both scalars is similarly flat. Thus, the moduli space stemming from the SSB is always even-dimensional.

The moduli superfields are chiral but have no superpotential, $W(\Phi_{\rm moduli}) = 0$, that's how their effective low-energy Lagrangian has no scalar potential at all. Instead, the moduli have derivative interactions stemming from a non-linear Kähler metric

$$\begin{aligned} g_{i,\bar{\jmath}}(\Phi,\Phi^*) &= \frac{\partial^2 K(\Phi,\Phi^*)}{\partial\Phi^i\,\partial\Phi^{*\bar{\jmath}}}\,, \\ \mathcal{L}_{\rm moduli} &= \int d^4\theta\, K(\Phi,\overline{\Phi}) = g_{i,\bar{\jmath}}(\Phi,\Phi^*) \times \Big(\partial_\mu\overline{\Phi}^{\bar{\jmath}}\,\partial^\mu\Phi^i + i\overline{\psi}^{\bar{\jmath}}\bar{\sigma}^\mu\,\overleftrightarrow{\partial_\mu}\,\psi^i\Big). \end{aligned} \tag{46}$$

The moduli also have couplings to other kinds of fields, but I shall deal with them later in class.

Beside the Goldstone modes and their superpartners, supersymmetric theories may have connected vacua with $V = 0$ (exactly) without any spontaneously broken symmetries but simply because the superpotential $W(\Phi)$ happens to vanish along with all its first derivative $\partial W/\partial\Phi^i$ along some continuous locus in the field space. As we shall learn in the next few lectures, the superpotential does not receive any perturbative corrections at all, and sometimes even the non-perturbative corrections happen to vanish. Consequently, if the scalar potential has some classical flat directions, they will remain exactly flat to all orders of the perturbation theory, and sometimes even non-perturbatively, which makes them moduli. String-based models have many examples of moduli of this kind.